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Some remarks on oscillation inequalities

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Abstract. In this paper, we establish uniform oscillation estimates on $L^p(X)$ with $p \in (1, \infty)$ for the polynomial ergodic averages. This result contributes to a certain problem about uniform oscillation bounds for ergodic averages formulated by Rosenblatt and Wierdl in the early 1990s [Pointwise ergodic theorems via harmonic analysis. *Proceedings of Conference on Ergodic Theory (Alexandria, Egypt, 1993) (London Mathematical Society Lecture Notes, 205)*. Eds. K. Petersen and I. Salama. Cambridge University Press, Cambridge, 1995, pp. 3–151]. We also give a slightly different proof of the uniform oscillation inequality of Jones, Kaufman, Rosenblatt, and Wierdl for bounded martingales [Oscillation in ergodic theory. *Ergod. Th. & Dynam. Sys.* **18**(4) (1998), 889–935]. Finally, we show that oscillations, in contrast to jump inequalities, cannot be seen as an endpoint for r-variation inequalities.

Key words: classic ergodic theory, oscillation seminorm, jump inequalities

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1. *Introduction*

1.1. *Statement of the main results.*

For $d, k \in \mathbb{Z}_+$, let us consider a polynomial mapping

$$P := (P_1, \dots, P_d) \colon \mathbb{Z}^k \to \mathbb{Z}^d, \tag{1.1}$$

where each $P_j\colon \mathbb{Z}^k \to \mathbb{Z}$ is a k-variate polynomial with integer coefficients such that $P_j(0) = 0$.

Let Ω be a non-empty convex body (not necessarily symmetric) in $\mathbb{R}^k$, which simply means that Ω is a bounded convex open subset of $\mathbb{R}^k$. For $t > 0$, we define its dilates

$$\Omega_t := \{x \in \mathbb{R}^k : t^{-1}x \in \Omega\}. \tag{1.2}$$

We will additionally assume that $B(0, c_\Omega) \subseteq \Omega \subseteq B(0, 1) \subset \mathbb{R}^k$ for some $c_\Omega \in (0, 1)$, where $B(x, t)$ denotes an open Euclidean ball in $\mathbb{R}^k$ centered at $x \in \mathbb{R}^k$ with radius $t > 0$. This ensures that $\Omega_t \cap \mathbb{Z}^k = \{0\}$ for all $t \in (0, 1)$. A typical choice of Ω_t is a ball of radius t associated with some norm on $\mathbb{R}^k$.

For $t \in \mathbb{X} := [1, \infty)$, $x \in X$, and $f \in L^0(X)$ (see §2 for appropriate definitions), we can define the corresponding ergodic polynomial averaging operator by

$$A_t^P f(x) := \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{m \in \Omega_t \cap \mathbb{Z}^k} f(T_1^{P_1(m)} \dots T_d^{P_d(m)} x). \tag{1.3}$$

The first main result of this note is the following uniform oscillation ergodic theorem.

THEOREM 1.1. *Let $d, k \in \mathbb{Z}_+$, a polynomial mapping P as in equation (1.1), and a set Ω_t as in equation (1.2) be given. Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space endowed with a family of commuting invertible measure preserving transformations $T_1, \dots, T_d : X \to X$. Then for every $p \in (1, \infty)$, there exists a constant $C_{d,k,p,\deg P} > 0$ such that for every $f \in L^p(X)$, we have*

$$\sup_{J \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_J(\mathbb{X})} \|O_{I,J}^2(A_t^P f : t \in \mathbb{X})\|_{L^p(X)} \le C_{d,k,p,\deg P} \|f\|_{L^p(X)}; \tag{1.4}$$

we refer to equation (2.3) for the definition of oscillations. Moreover, the implied constant in equation (1.4) is independent of the coefficients of the polynomial mapping P.

We now give some remarks about Theorem 1.1.

(1) Theorem 1.1 is a contribution to a problem from the early 1990s of Rosenblatt and Wierdl [**21**, Problem 4.12, p. 80] about uniform estimates of oscillation inequalities for ergodic averages. This problem has a long and interesting history, which we briefly describe below.

(2) Inequality (1.4) is a useful tool, as it was shown by Bourgain [**1–3**], in establishing pointwise convergence for operators in equation (1.3). Inequality (1.4) also implies, in view of equation (2.5), that for all $p \in (1, \infty]$, there exists a constant $C_{d,k,p,\deg P} > 0$ (with $C_{d,k,\infty,\deg P} = 1$ for $p = \infty$), such that for every $f \in L^p(X)$, we have

$$\Big\| \sup_{t \in \mathbb{X}} |A_t^P f| \Big\|_{L^p(X)} \le C_{d,k,p,\deg P} \|f\|_{L^p(X)}. \tag{1.5}$$

The constant in equation (1.5) is also independent of the coefficients of the polynomial mapping P.

(3) A non-uniform variant of inequality (1.4) for one dimensional averages in equation (1.3) with $d = k = 1$ was established by Bourgain in [**1–3**]. More precisely, Bourgain

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proved that for any $\tau > 1$, any sequence of integers $I = (I_j : j \in \mathbb{N})$ such that $I_{j+1} > 2I_j$ for all $j \in \mathbb{N}$, and any $f \in L^2(X)$, one has

$$\|O^2_{I,J}(A^P_{\tau^n} f : n \in \mathbb{N})\|_{L^2(X)} \leq C_{I,\tau}(J)\|f\|_{L^2(X)}, \quad J \in \mathbb{Z}_+, \tag{1.6}$$

where $C_{I,\tau}(J)$ is a constant depending on I and τ, and such that $\lim_{J\to\infty} J^{-1/2}C_{I,\tau}(J) = 0$. Interestingly, this non-uniform inequality (1.6) suffices to establish pointwise convergence of the averaging operators from equation (1.3) for any $f \in L^2(X)$, see [1–3].

(4) Not long afterward, Lacey refined Bourgain's argument [21, Theorem 4.23, p. 95] and showed that for every $\tau > 1$, there is a constant $C_\tau > 0$ such that for any $f \in L^2(X)$, one has

$$\sup_{J\in\mathbb{Z}_+} \sup_{I\in\mathfrak{S}_J(\mathbb{L}_\tau)} \|O^2_{I,J}(A^P_t f : t \in \mathbb{L}_\tau)\|_{L^2(X)} \leq C_\tau\|f\|_{L^2(X)}, \tag{1.7}$$

where $\mathbb{L}_\tau := \{\tau^n : n \in \mathbb{N}\}$. This was the first uniform oscillation result in the class of τ-lacunary sequences. Lacey's observation naturally motivated a question about uniform estimates, independent of $\tau > 1$, of oscillation inequalities in equation (1.7), which, for the Birkhoff averages, was explicitly formulated in [21, Problem 4.12, p. 80].

(5) In the groundbreaking paper of Jones *et al* [8], the authors established Theorem 1.1 for the classical Birkhoff averages with $d = k = 1$ and $P_1(n) = n$ giving affirmative answer to [21, Problem 4.12, p. 80]. Here, our aim will be to show that [21, Problem 4.12, p. 80] remains true for Bourgain's polynomial ergodic averages even in the multidimensional setting as in equation (1.3).

(6) Finally, we mention that a non-uniform variant of Theorem 1.1 was in fact established in [15] (see also [18]). Specifically, Hölder's inequality and inequality (2.13) and r-variational estimates for $r > 2$ (see definition in equation (2.11)), established in [15, 18], yield that for every $p \in (1, \infty)$, there is a constant $C_p > 0$ such that for any $r > 2$ and every $f \in L^p(X)$, one has

$$\sup_{I\in\mathfrak{S}_J(\mathbb{X})} \|O^2_{I,J}(A^P_t f : n \in \mathbb{X})\|_{L^p(X)} \leq C_p \frac{r}{r-2} J^{1/2-1/r}\|f\|_{L^p(X)}, \quad J \in \mathbb{Z}_+.$$

(7) The proof of Theorem 1.1 will be an elaboration of methods developed in [15, 18]. The main tools are the Hardy–Littlewood circle method (major arcs estimates in Proposition 3.7, lattice points estimates in Proposition 3.5 and Weyl's inequality from Theorem 3.6), the Ionescu–Wainger multiplier theory (Theorem 3.3, see also [7]), the Rademacher–Menshov argument (inequality (2.14), see also [17]), and the sampling principle of Magyar–Stein–Wainger (Proposition 3.4, see also [13]). The details are presented in §3, where we closely follow the exposition from [18]. Another important ingredient of the proof of Theorem 1.1 is a uniform oscillation inequality for martingales. Although this inequality was originally proved in [8, Theorem 6.4, p. 930], a slightly different proof is presented in §2, see Proposition 2.2 for the details.

The second main theorem of this paper is the following counterexample.

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THEOREM 1.2. *Let $1 \le p < \infty$ and $1 < \rho \le r < \infty$ be fixed. It is not true that the estimate*

$$\sup_{\lambda>0} \|\lambda N_\lambda(f(\cdot,t) : t \in \mathbb{N})^{1/r}\|_{\ell^{p,\infty}(\mathbb{Z})} \le C_{p,\rho,r} \sup_{I\in\mathfrak{S}_\infty(\mathbb{N})} \|O^\rho_{I,\infty}(f(\cdot,t) : t \in \mathbb{N})\|_{\ell^p(\mathbb{Z})} \tag{1.8}$$

holds uniformly for every measurable function $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$.

As a consequence of equation (1.8), the following estimate

$$\|V^r(f(\cdot,t) : t \in \mathbb{N})\|_{\ell^{p,\infty}(\mathbb{Z})} \le C_{p,\rho,r} \sup_{I\in\mathfrak{S}_\infty(\mathbb{N})} \|O^\rho_{I,\infty}(f(\cdot,t) : t \in \mathbb{N})\|_{\ell^p(\mathbb{Z})} \tag{1.9}$$

cannot hold uniformly for all measurable functions $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$. We refer to §2 for definitions of ρ-oscillations (equation (2.3)), r-variations (equation (2.11)), and λ-jumps (equation (2.16)).

Theorem 1.2 states, in particular, that ρ-oscillation inequalities in equation (2.3) cannot be seen (at least in a straightforward way) as endpoint estimates for r-variations in equation (2.11). It also shows that ρ-oscillations are incomparable with uniform λ-jumps in equation (2.16). Our motivation to study inequalities (1.8) and (1.9) from Theorem 1.2 arose from the desire of better understanding relations between ρ-oscillations, r-variations, and λ-jumps. We now use martingales to illustrate these relations.

The r-variations in equation (2.11) for a family of bounded martingales $\mathfrak{f} = (\mathfrak{f}_n : X \to \mathbb{C} : n \in \mathbb{Z}_+)$ were studied by Lépingle [**12**] who showed that for all $r \in (2,\infty)$ and $p \in (1,\infty)$, there is a constant $C_{p,r} > 0$ such that the following inequality:

$$\|V^r(\mathfrak{f}_n : n \in \mathbb{Z}_+)\|_{L^p(X)} \le C_{p,r} \sup_{n\in\mathbb{Z}_+} \|\mathfrak{f}_n\|_{L^p(X)} \tag{1.10}$$

holds with sharp ranges of exponents, see also [**11**] for a counterexample at $r = 2$. In [**12**], a weak type (1, 1) variant of the inequality (1.10) was proved as well. Inequality (1.10) is an extension of Doob's maximal inequality for martingales and gives a quantitative form of the martingale convergence theorem. We also refer to [**3**, **16**, **20**] for generalizations and different proofs of equation (1.10).

Bourgain rediscovered inequality (1.10) in his seminal paper [**3**], where it was used to address the issue of pointwise convergence of ergodic-theoretic averages along polynomial orbits. This initiated systematic studies of r-variations in harmonic analysis and ergodic theory, which resulted in a vast literature [**8–11**, **15–18**]. For applications in analysis and ergodic theory, only $r > 2$ and $p > 1$ matter, and in fact this is the best that we can expect due to the Lépingle inequality.

It is not difficult to see that for any sequence of measurable functions $(\mathfrak{a}_n : n \in \mathbb{Z}_+) \subseteq \mathbb{C}$, one has

$$\sup_{\lambda>0} \|\lambda N_\lambda(\mathfrak{a}_n : n \in \mathbb{Z}_+)^{1/r}\|_{L^p(X)} \le \|V^r(\mathfrak{a}_n : n \in \mathbb{Z}_+)\|_{L^p(X)}, \tag{1.11}$$

see equation (2.17). Therefore, equation (1.11) combined with equation (1.10) imply jump inequalities for martingales for any $r > 2$. However, as was first shown by Pisier and Xu [**20**] on $L^2(X)$ and by Bourgain [**3**, inequality (3.5)] on $L^p(X)$ with $p \in (1,\infty)$, endpoint

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estimates for $r = 2$ are also true. More precisely, for every $p \in (1, \infty)$, there exists a constant $C_p > 0$ such that

$$\sup_{\lambda>0} \|\lambda N_\lambda(\mathfrak{f}_n : n \in \mathbb{Z}_+)^{1/2}\|_{L^p(X)} \leq C_p \sup_{n\in\mathbb{Z}_+} \|\mathfrak{f}_n\|_{L^p(X)}. \tag{1.12}$$

A remarkable feature of Bourgain's [**3**] approach was based on the observation that inequality (1.11) can be reversed in the sense that for every $p \in [1, \infty]$ and $1 \leq \rho < r \leq \infty$, one has

$$\|V^r(\mathfrak{a}_n : n \in \mathbb{Z}_+)\|_{L^{p,\infty}(X)} \lesssim_{p,\rho,r} \sup_{\lambda>0} \|\lambda N_\lambda(\mathfrak{a}_n : n \in \mathbb{Z}_+)^{1/\rho}\|_{L^{p,\infty}(X)}. \tag{1.13}$$

In Lemma 2.5, it is shown that one cannot replace $L^{p,\infty}(X)$ with $L^p(X)$ in equation (1.13). In fact, equations (1.12) and (1.13) allowed Bourgain [**3**] to recover Lépingle's inequality (1.10). Inequality (1.13) is a very striking result (see also equation (2.18)) which states that *a priori* uniform λ-jump estimates corresponding to some $\rho \in [1, \infty)$ and $p \in [1, \infty]$ imply weak type r-variational estimates for the same range of p and for any $r \in (\rho, \infty]$. Therefore, uniform λ-jump estimates can be thought of as endpoint estimates for r-variations, even though the r-variations may be unbounded at the endpoint in question, as we have seen in the context of the Lépingle inequality (1.10) with $r = 2$.

This gives us a fairly complete picture of relations between r-variations and λ-jumps, which immediately lead to a question about similar phenomena between ρ-oscillations and r-variations as well as λ-jumps. This problem has been undertaken in Theorem 1.2 and arose from the following two observations. On the one hand, for any $r \geq 1$ and any sequence of measurable functions $(\mathfrak{a}_n : n \in \mathbb{Z}_+) \subseteq \mathbb{C}$, one has

$$\sup_{J\in\mathbb{Z}_+} \sup_{I\in\mathfrak{S}_J(\mathbb{Z}_+)} \|O^r_{I,J}(\mathfrak{a}_n : n \in \mathbb{Z}_+)\|_{L^p(X)} \leq \|V^r(\mathfrak{a}_n : n \in \mathbb{Z}_+)\|_{L^p(X)}, \tag{1.14}$$

which follows from equation (2.13). Thus, equation (1.10) combined with equation (1.14) gives bounds of r-oscillations for martingales on $L^p(X)$ for all $r \in (2, \infty)$ and $p \in (1, \infty)$. On the other hand, it was shown by Jones *et al* [**8**, Theorem 6.4, p. 930] that for every $p \in (1, \infty)$, there is a constant $C_p > 0$ such that

$$\sup_{J\in\mathbb{Z}_+} \sup_{I\in\mathfrak{S}_J(\mathbb{Z}_+)} \|O^2_{I,J}(\mathfrak{f}_n : n \in \mathbb{Z}_+)\|_{L^p(X)} \leq C_p \sup_{n\in\mathbb{Z}_+} \|\mathfrak{f}_n\|_{L^p(X)}. \tag{1.15}$$

A slightly different proof of this inequality is given in Proposition 2.2.

Inequalities (1.14) and (1.15) exhibit a similar phenomenon to the one that we have seen above in the case of λ-jumps (see equations (1.11) and (1.12)), where 2-variations for martingales explode on $L^p(X)$, but corresponding λ-jumps (see inequality (1.12)) and oscillations (see inequality (1.15)) are bounded.

This observation gave rise to a natural question whether 2-oscillation can be interpreted as an endpoint for r-variations for any $r > 2$ in the sense of inequality (1.13). Theorem 1.2 provides an answer in the negative. A detailed proof of Theorem 1.2 is given in §4, where a concept of the sequential jump counting function has been introduced, see equation (4.1). The sequential jumps can be thought of as analogs of classical jumps in equation (2.16) adjusted to ρ-oscillations, see for instance equation (4.3) and Lemma 4.2. Theorem 1.2

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also shows that the space induced by ρ-oscillations is different from the spaces induced by r-variations and λ jumps corresponding to the parameter $\rho \leq r$.

Even though Theorem 1.2 shows that ρ-oscillation inequalities cannot be seen (at least in a straightforward way understood in the sense of inequality (1.13)) as endpoint estimates for r-variations, it is still natural to ask whether *a priori* bounds for 2-oscillations imply bounds for r-variations for any $r > 2$. It is an intriguing question from the point of view of pointwise convergence problems. If it were true, it would reduce pointwise convergence problems to study 2-oscillations, which in certain cases are simpler as they are closer to square functions.

The paper is organized as follows. In §2, we set notation and collect some important facts about oscillations, variations, and jumps, as well as prove Proposition 2.2. In §3, we give a proof of Theorem 1.1. Finally in §4, we prove our counterexamples from Theorem 1.2.

2. *Notation and basic tools*

We now set up notation and terminology that will be used throughout the paper. We also gather basic properties of jumps, as well as oscillation and variation seminorms that will be used later.

2.1. *Basic notation.* We denote $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For $d \in \mathbb{Z}_+$, the sets $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ have a standard meaning. For any $x \in \mathbb{R}$, we will use the floor function

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}.$$

We denote $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{X} := [1, \infty)$, and for every $N \in \mathbb{R}_+$, we set

$$[N] := (0, N] \cap \mathbb{Z} = \{1, \ldots, \lfloor N \rfloor\},$$

and we will also write

$$\mathbb{N}_{\leq N} := [0, N] \cap \mathbb{N} \quad \text{and} \quad \mathbb{N}_{<N} := [0, N) \cap \mathbb{N},$$
$$\mathbb{N}_{\geq N} := [N, \infty) \cap \mathbb{N} \quad \text{and} \quad \mathbb{N}_{>N} := (N, \infty) \cap \mathbb{N}.$$

For $\tau \in (0, 1)$ and $u \in \mathbb{Z}_+$, we define sets

$$\mathbb{D}_\tau := \{2^{n^\tau} : n \in \mathbb{N}\} \quad \text{and} \quad 2^{u\mathbb{Z}_+} := \{2^{un} : n \in \mathbb{Z}_+\}.$$

We use $\mathbb{1}_A$ to denote the indicator function of a set A. If S is a statement, we write $\mathbb{1}_S$ to denote its indicator, equal to 1 if S is true and 0 if S is false. For instance, $\mathbb{1}_A(x) = \mathbb{1}_{x \in A}$.

For two non-negative quantities A, B, we write $A \lesssim B$ if there is an absolute constant $C > 0$ such that $A \leq CB$. However, the constant C may change from line to line. If $A \lesssim B \lesssim A$, then we write $A \simeq B$. We will write $\lesssim_\delta$ or $\simeq_\delta$ to indicate that the implicit constant depends on δ. For two functions $f : X \to \mathbb{C}$ and $g : X \to [0, \infty)$, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in X$. We will write $f = \mathcal{O}_\delta(g)$ if the implicit constant depends on δ.

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2.2. *Euclidean spaces.* The standard inner product, the corresponding Euclidean norm, and the maximum norm on $\mathbb{R}^d$ are denoted respectively, for any $x = (x_1, \dots, x_d)$, $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, by

$$x \cdot \xi := \sum_{k=1}^{d} x_k \xi_k, \quad |x| := |x|_2 := \sqrt{x \cdot x}, \quad \text{and} \quad |x|_\infty := \max_{k \in [d]} |x_k|.$$

For any multi-index $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}^k$, by abuse of notation, we will write $|\gamma| := \gamma_1 + \cdots + \gamma_k$. This will never cause confusion since the multi-indices will always be denoted by Greek letters.

Throughout the paper, the d-dimensional torus $\mathbb{T}^d$ is *a priori* endowed with the periodic norm

$$\|\xi\| := \Big(\sum_{k=1}^{d} \|\xi_k\|^2 \Big)^{1/2} \quad \text{for } \xi = (\xi_1, \dots, \xi_d) \in \mathbb{T}^d, \tag{2.1}$$

where $\|\xi_k\| = \operatorname{dist}(\xi_k, \mathbb{Z})$ for all $\xi_k \in \mathbb{T}$ and $k \in [d]$. Identifying $\mathbb{T}^d$ with $[-1/2, 1/2)^d$, we see that the norm $\|\cdot\|$ coincides with the Euclidean norm $|\cdot|$ restricted to $[-1/2, 1/2)^d$.

2.3. *Function spaces.* In this paper, all vector spaces will be defined over $\mathbb{C}$. The triple $(X, \mathcal{B}(X), \mu)$ denotes a measure space X with a σ-algebra $\mathcal{B}(X)$ and a σ-finite measure μ. The space of all μ-measurable functions $f : X \to \mathbb{C}$ will be denoted by $L^0(X)$. The space of all functions in $L^0(X)$ whose modulus is integrable with pth power is denoted by $L^p(X)$ for $p \in (0, \infty)$, whereas $L^\infty(X)$ denotes the space of all essentially bounded functions in $L^0(X)$. These notions can be extended to functions taking values in a finite dimensional normed vector space $(B, \|\cdot\|_B)$, for instance,

$$L^p(X; B) := \{F \in L^0(X; B) : \|F\|_{L^p(X;B)} := \|\|F\|_B\|_{L^p(X)} < \infty\},$$

where $L^0(X; B)$ denotes the space of measurable functions from X to B (up to almost everywhere equivalence). If B is separable, these notions can be extended to infinite-dimensional B. However, in this paper, by appealing to standard approximation arguments, we will always be able to work in finite-dimensional settings.

For any $p \in [1, \infty]$, we define a weak-L^p space of measurable functions on X by setting

$$L^{p,\infty}(X) := \{f : X \to \mathbb{C} \colon \|f\|_{L^{p,\infty}(X)} < \infty\},$$

where for any $p \in [1, \infty)$, we have

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \quad \text{and} \quad \|f\|_{L^{\infty,\infty}(X)} := \|f\|_{L^\infty(X)}.$$

In our case, we will mainly have $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$ equipped with the Lebesgue measure, and $X = \mathbb{Z}^d$ endowed with the counting measure. If X is endowed with a counting measure, we will abbreviate $L^p(X)$ to $\ell^p(X)$, $L^p(X; B)$ to $\ell^p(X; B)$, and $L^{p,\infty}(X)$ to $\ell^{p,\infty}(X)$.

If $T : B_1 \to B_2$ is a continuous linear map between two normed vector spaces B_1 and B_2, we use $\|T\|_{B_1 \to B_2}$ to denote its operator norm.

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2.4. *Fourier transform.* We will use convention that $\boldsymbol{e}(z) = e^{2\pi \boldsymbol{i} z}$ for every $z \in \mathbb{C}$, where $\boldsymbol{i}^2 = -1$. Let $\mathcal{F}_{\mathbb{R}^d}$ denote the Fourier transform on $\mathbb{R}^d$ defined for any $f \in L^1(\mathbb{R}^d)$ and for any $\xi \in \mathbb{R}^d$ as

$$\mathcal{F}_{\mathbb{R}^d} f(\xi) := \int_{\mathbb{R}^d} f(x) \boldsymbol{e}(x \cdot \xi)\, dx.$$

If $f \in \ell^1(\mathbb{Z}^d)$, we define the discrete Fourier transform (Fourier series) $\mathcal{F}_{\mathbb{Z}^d}$, for any $\xi \in \mathbb{T}^d$, by setting

$$\mathcal{F}_{\mathbb{Z}^d} f(\xi) := \sum_{x \in \mathbb{Z}^d} f(x) \boldsymbol{e}(x \cdot \xi).$$

Sometimes, we shall abbreviate $\mathcal{F}_{\mathbb{Z}^d} f$ to $\hat{f}$.

Let $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$. It is well known that their corresponding dual groups are $\mathbb{G}^* = (\mathbb{R}^d)^* = \mathbb{R}^d$ or $\mathbb{G}^* = (\mathbb{Z}^d)^* = \mathbb{T}^d$, respectively. For any bounded function $\mathfrak{m} : \mathbb{G}^* \to \mathbb{C}$ and a test function $f : \mathbb{G} \to \mathbb{C}$, we define the Fourier multiplier operator by

$$T_{\mathbb{G}}[\mathfrak{m}] f(x) := \int_{\mathbb{G}^*} \boldsymbol{e}(-\xi \cdot x) \mathfrak{m}(\xi) \mathcal{F}_{\mathbb{G}} f(\xi)\, d\xi \quad \text{for } x \in \mathbb{G}. \tag{2.2}$$

One may think that $f : \mathbb{G} \to \mathbb{C}$ is a compactly supported function on $\mathbb{G}$ (and smooth if $\mathbb{G} = \mathbb{R}^d$) or any other function for which equation (2.2) makes sense.

2.5. *Oscillation seminorms.* Let $\mathbb{I} \subseteq \mathbb{R}$ be such that $\#\mathbb{I} \geq 2$. For every $J \in \mathbb{Z}_+ \cup \{\infty\}$, define

$$\mathcal{S}_J(\mathbb{I}) := \{(t_i : i \in \mathbb{N}_{\leq J}) \subseteq \mathbb{I} \colon t_0 < t_1 < \ldots < t_J\},$$

where $\mathbb{N}_{\leq \infty} := \mathbb{N}$. In other words, $\mathcal{S}_J(\mathbb{I})$ is a family of all strictly increasing sequences of length $J + 1$ taking their values in the index set $\mathbb{I}$.

Let $(\mathfrak{a}_t(x) : t \in \mathbb{I}) \subseteq \mathbb{C}$ be a family of measurable functions defined on X. For any $r \in [1, \infty)$, $\mathbb{J} \subseteq \mathbb{I}$, and a sequence $I = (I_i : i \in \mathbb{N}_{\leq J}) \in \mathcal{S}_J(\mathbb{I})$, the oscillation seminorm is defined by

$$O^r_{I,J}(\mathfrak{a}_t(x) : t \in \mathbb{J}) := \Bigg(\sum_{j=0}^{J-1} \sup_{t \in [I_j, I_{j+1}) \cap \mathbb{J}} |\mathfrak{a}_t(x) - \mathfrak{a}_{I_j}(x)|^r \Bigg)^{1/r}. \tag{2.3}$$

There will be no problems with measurability in equation (2.3) since we will always assume that $\mathbb{I} \ni t \mapsto \mathfrak{a}_t(x) \in \mathbb{C}$ is continuous for μ-almost every $x \in X$, or $\mathbb{J}$ is countable. We also use the convention that the supremum taken over the empty set is zero.

Remark 2.1. Some remarks concerning the definition in equation (2.3) are in order.

(1) It is not difficult to see that $O^r_{I,J}(\mathfrak{a}_t : t \in \mathbb{J})$ defines a seminorm.

(2) Let $\mathbb{I} \subseteq \mathbb{R}$ be an index set such that $\#\mathbb{I} \geq 2$, and let $\mathbb{J}_1, \mathbb{J}_2 \subseteq \mathbb{I}$ be disjoint. Then for any family $(\mathfrak{a}_t : t \in \mathbb{I}) \subseteq \mathbb{C}$, any $J \in \mathbb{Z}_+$, and any $I \in \mathcal{S}_J(\mathbb{I})$, one has

$$O^r_{I,J}(\mathfrak{a}_t : t \in \mathbb{J}_1 \cup \mathbb{J}_2) \leq O^r_{I,J}(\mathfrak{a}_t : t \in \mathbb{J}_1) + O^r_{I,J}(\mathfrak{a}_t : t \in \mathbb{J}_2). \tag{2.4}$$

(3) Let $(\mathfrak{a}_t(x) : t \in \mathbb{R}) \subseteq \mathbb{C}$ be a family of measurable functions on a σ-finite measure space $(X, \mathcal{B}(X), \mu)$. Let $\mathbb{I} \subseteq \mathbb{R}$ and $\#\mathbb{I} \geq 2$, then for every $p \in [1, \infty)$ and

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$r \in [1, \infty)$, we have

$$\Big\| \sup_{t \in \mathbb{I} \setminus \sup \mathbb{I}} |\mathfrak{a}_t| \Big\|_{L^p(X)} \leq \sup_{t \in \mathbb{I}} \|\mathfrak{a}_t\|_{L^p(X)} + \sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \|O^r_{I,N}(\mathfrak{a}_t : t \in \mathbb{I})\|_{L^p(X)}. \tag{2.5}$$

The inequality (2.5) states that oscillations always dominate maximal functions.

(4) Let $(\mathfrak{a}_t(x) : t \in \mathbb{R}) \subseteq \mathbb{C}$ be a family of measurable functions on a σ-finite measure space $(X, \mathcal{B}(X), \mu)$. Suppose that there are $p \in [1, \infty)$, $r \in [1, \infty)$, and $0 < C_{p,r} < \infty$ such that

$$\sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \|O^r_{I,N}(\mathfrak{a}_t : t \in \mathbb{R}_+)\|_{L^p(X)} \leq C_{p,r}. \tag{2.6}$$

Then, the limit $\lim_{t \to \infty} \mathfrak{a}_t(x)$ exists for μ-almost every $x \in X$. In other words, the condition in equation (2.6) implies pointwise almost everywhere convergence of $\mathfrak{a}_t(x)$ as $t \to \infty$.

We recall some notation from [**6**, §3, p. 165]. Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space and let $\mathbb{I}$ be a totally ordered set. A sequence of sub-σ-algebras $(\mathcal{F}_t : t \in \mathbb{I})$ is called *filtration* if it is increasing and the measure μ is σ-finite on each $\mathcal{F}_t$. Recall that a *martingale* adapted to a filtration $(\mathcal{F}_t : t \in \mathbb{I})$ is a family of functions $\mathfrak{f} = (\mathfrak{f}_t : t \in \mathbb{I}) \subseteq L^1(X, \mathcal{B}(X), \mu)$ such that $\mathfrak{f}_s = \mathbb{E}[\mathfrak{f}_t | \mathcal{F}_s]$ for every $s, t \in \mathbb{I}$ so that $s \leq t$, where $\mathbb{E}[\cdot|\mathcal{F}]$ denotes the the conditional expectation with respect to a sub-σ-algebra $\mathcal{F} \subseteq \mathcal{B}(X)$. We say that a martingale $\mathfrak{f} = (\mathfrak{f}_t : t \in \mathbb{I}) \subseteq L^p(X, \mathcal{B}(X), \mu)$ is bounded if

$$\sup_{t \in \mathbb{I}} \|\mathfrak{f}_t\|_{L^p(X)} \lesssim_p 1.$$

We now establish oscillation inequalities for bounded martingales in $L^p(X, \mathcal{B}(X), \mu)$.

PROPOSITION 2.2. *For every $p \in (1, \infty)$, there exists a constant $C_p > 0$ such that for every bounded martingale $\mathfrak{f} = (\mathfrak{f}_n : n \in \mathbb{Z}) \subseteq L^p(X, \mathcal{B}(X), \mu)$ corresponding to a filtration $(\mathcal{F}_n : n \in \mathbb{Z})$, one has*

$$\sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{Z})} \|O^2_{I,N}(\mathfrak{f}_n : n \in \mathbb{Z})\|_{L^p(X)} \leq C_p \sup_{n \in \mathbb{Z}} \|\mathfrak{f}_n\|_{L^p(X)}. \tag{2.7}$$

This proposition was established in [**8**, Theorem 6.4, p. 930]. The authors first established equation (2.7) for $p = 2$, then proved weak type (1, 1) as well as $L^\infty \to$ BMO variants of equation (2.7), and consequently, by a simple interpolation, derived equation (2.7) for all $p \in (1, \infty)$. Here, we give a slightly different and simplified proof based on a weighted Doob's inequality, which avoids $L^\infty \to$ BMO estimates and, in fact, is very much in the spirit of the estimates for $p = 2$ from [**8**, Theorem 6.1, p. 927].

Proof of Proposition 2.2. We fix $N \in \mathbb{Z}_+$ and a sequence $I \in \mathfrak{S}_N(\mathbb{Z})$. We first prove equation (2.7) for $p \geq 2$. Since $r = p/2 \geq 1$, we take a non-negative $w \in L^{r'}(X)$ such that $\|w\|_{L^{r'}(X)} \leq 1$ and

$$\|O^2_{I,N}(\mathfrak{f}_n : n \in \mathbb{Z})\|^2_{L^p(X)} = \sum_{i=0}^{N-1} \int_X \sup_{I_i \leq n < I_{i+1}} |\mathfrak{f}_n - \mathfrak{f}_{I_i}|^2 w \, d\mu.$$

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To estimate the last sum, we will use a weighted version of Doob's maximal inequality, see [6, Theorem 3.2.3, p. 175], which asserts that for every $p \in (1,\infty)$, every function $f \in L^p(X)$, and a non-negative measurable weight w, and for any $n \in \mathbb{Z}$, we have

$$\Big(\int_X \sup_{m\leq n} |\mathfrak{f}_m|^p w \ d\mu \Big)^{1/p} \leq p' \Big(\int_X |\mathfrak{f}_n|^p \sup_{m\in\mathbb{Z}} |\mathbb{E}[w|\mathcal{F}_m]| \ d\mu \Big)^{1/p}. \tag{2.8}$$

We will also use an unweighted Doob's inequality, see [6, Theorem 3.2.2, p. 175], which yields that for every $p \in (1,\infty)$, we have

$$\Big\| \sup_{m\in\mathbb{Z}} |\mathfrak{f}_m| \Big\|_{L^p(X)} \leq p' \sup_{m\in\mathbb{Z}} \|\mathfrak{f}_m\|_{L^p(X)}. \tag{2.9}$$

Finally, we will use for every $p \in (1,\infty)$, the following bound

$$\sup_{(\omega_n:n\in\mathbb{Z})\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{k\in\mathbb{Z}} \omega_k (\mathfrak{f}_k - \mathfrak{f}_{k-1}) \Big\|_{L^p(X)} \lesssim_p \sup_{m\in\mathbb{Z}} \|\mathfrak{f}_m\|_{L^p(X)},$$

which, by Khintchine's inequality, ensures

$$\Big\| \Big(\sum_{i=0}^{N-1} \Big| \sum_{k=I_i+1}^{I_{i+1}} (\mathfrak{f}_k - \mathfrak{f}_{k-1}) \Big|^2 \Big)^{1/2} \Big\|_{L^p(X)} \lesssim_p \sup_{m\in\mathbb{Z}} \|\mathfrak{f}_m\|_{L^p(X)}. \tag{2.10}$$

Then we conclude

$$\begin{aligned}
&\sum_{i=0}^{N-1} \int_X \sup_{I_i\leq n<I_{i+1}} |\mathfrak{f}_n - \mathfrak{f}_{I_i}|^2 w \, d\mu \\
&= \sum_{i=0}^{N-1} \int_X \sup_{I_i\leq n<I_{i+1}} |\mathbb{E}[(\mathfrak{f}_{I_{i+1}} - \mathfrak{f}_{I_i})|\mathcal{F}_n]|^2 w \, d\mu \\
&\leq 4 \sum_{i=0}^{N-1} \int_X |\mathbb{E}[(\mathfrak{f}_{I_{i+1}} - \mathfrak{f}_{I_i})|\mathcal{F}_{I_{i+1}}]|^2 \sup_{n\in\mathbb{Z}} |\mathbb{E}[w|\mathcal{F}_n]| d\mu \quad \text{by (2.8)} \\
&= 4 \int_X \sum_{i=0}^{N-1} \Big| \sum_{k=I_i+1}^{I_{i+1}} (\mathfrak{f}_k - \mathfrak{f}_{k-1}) \Big|^2 \sup_{n\in\mathbb{Z}} |\mathbb{E}[w|\mathcal{F}_n]| d\mu \\
&\leq 4 \Big\| \Big(\sum_{i=0}^{N-1} \Big| \sum_{k=I_i+1}^{I_{i+1}} (\mathfrak{f}_k - \mathfrak{f}_{k-1}) \Big|^2 \Big)^{1/2} \Big\|_{L^p(X)}^2 \Big\| \sup_{n\in\mathbb{Z}} |\mathbb{E}[w|\mathcal{F}_n]| \Big\|_{L^{r'}(X)} \\
&\qquad\qquad \text{by Hölder's inequality} \\
&\lesssim_{p,r} \sup_{m\in\mathbb{Z}} \|\mathfrak{f}_m\|_{L^p(X)}^2 \|w\|_{L^{r'}(X)} \quad \text{by equations (2.10) and (2.9).}
\end{aligned}$$

This proves equation (2.7) for all $p \in [2,\infty)$. To prove equation (2.7) for $p \in (1,2)$, it suffices to show the corresponding weak type $(1,1)$ estimate. This follows from [8, Theorem 6.2, p. 928], so we omit the details. □

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2.6. *Variation seminorms.* We also recall the definition of r-variations. For any $\mathbb{I} \subseteq \mathbb{R}$, any family $(\mathfrak{a}_t : t \in \mathbb{I}) \subseteq \mathbb{C}$, and any exponent $1 \le r < \infty$, the r-variation seminorm is defined to be

$$V^r(\mathfrak{a}_t : t \in \mathbb{I}) := \sup_{J \in \mathbb{Z}_+} \sup_{\substack{t_0 < \cdots < t_J \\ t_j \in \mathbb{I}}} \Big(\sum_{j=0}^{J-1} |\mathfrak{a}_{t_{j+1}} - \mathfrak{a}_{t_j}|^r \Big)^{1/r}, \tag{2.11}$$

where the latter supremum is taken over all finite increasing sequences in $\mathbb{I}$.

Remark 2.3. Some remarks about the definition in equation (2.11) are in order.

(1) Clearly, $V^r(\mathfrak{a}_t : t \in \mathbb{I})$ defines a seminorm.

(2) The function $r \mapsto V^r(\mathfrak{a}_t : t \in \mathbb{I})$ is non-increasing. Moreover, if $\mathbb{I}_1 \subseteq \mathbb{I}_2$, then

$$V^r(\mathfrak{a}_t : t \in \mathbb{I}_1) \le V^r(\mathfrak{a}_t : t \in \mathbb{I}_2).$$

(3) Let $\mathbb{I} \subseteq \mathbb{R}$ be such that $\#\mathbb{I} \ge 2$. Let $(\mathfrak{a}_t : t \in \mathbb{R}) \subseteq \mathbb{C}$ be given, and let $r \in [1, \infty)$. If $V^r(\mathfrak{a}_t : t \in \mathbb{R}_+) < \infty$, then $\lim_{t \to \infty} \mathfrak{a}_t$ exists. Moreover, for any $t_0 \in \mathbb{I}$, one has

$$\sup_{t \in \mathbb{I}} |\mathfrak{a}_t| \le |\mathfrak{a}_{t_0}| + V^r(\mathfrak{a}_t : t \in \mathbb{I}). \tag{2.12}$$

(4) Let $\mathbb{I} \subseteq \mathbb{R}$ be a countable index set such that $\#\mathbb{I} \ge 2$. Then for any $r \ge 1$, and any family $(\mathfrak{a}_t : t \in \mathbb{I}) \subseteq \mathbb{C}$, any $J \in \mathbb{Z}_+ \cup \{\infty\}$, and any $I \in \mathfrak{S}_J(\mathbb{I})$, one has

$$O^r_{I,J}(\mathfrak{a}_t : t \in \mathbb{I}) \le V^r(\mathfrak{a}_t : t \in \mathbb{I}) \le 2 \Big(\sum_{t \in \mathbb{I}} |\mathfrak{a}_t|^r \Big)^{1/r}. \tag{2.13}$$

(5) The first inequality in equation (2.13) allows us to deduce the Rademacher–Menshov inequality for oscillations, which asserts that for any $j_0, m \in \mathbb{N}$ so that $j_0 < 2^m$ and any sequence of complex numbers $(\mathfrak{a}_k : k \in \mathbb{N})$, any $J \in [2^m]$, and any $I \in \mathfrak{S}_J([j_0, 2^m])$, we have

$$\begin{aligned} O^2_{I,J}(\mathfrak{a}_j : j_0 \le j \le 2^m) &\le V^2(\mathfrak{a}_j : j_0 \le j \le 2^m) \\ &\le \sqrt{2} \sum_{i=0}^{m} \Big(\sum_{j=0}^{2^{m-i}-1} \Big| \sum_{\substack{k \in U^i_j \\ U^i_j \subseteq [j_0, 2^m)}} (\mathfrak{a}_{k+1} - \mathfrak{a}_k) \Big|^2 \Big)^{1/2}, \end{aligned} \tag{2.14}$$

where $U^i_j := [j2^i, (j+1)2^i)$ for any $i, j \in \mathbb{N}$. The latter inequality in equation (2.14) immediately follows from the proof of [**17**, Lemma 2.5, p. 534]. The inequality (2.14) will be used in §3.

(6) Let $(\mathfrak{a}_t(x) : t \in \mathbb{I}) \subseteq \mathbb{C}$ be a family of measurable functions on a σ-finite measure space $(X, \mathcal{B}(X), \mu)$. Then for any $p \ge 1$ and $\tau > 0$, we have

$$\begin{aligned} \sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \|_{L^p(X)} &\lesssim \sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{D}_\tau)} \| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{D}_\tau) \|_{L^p(X)} \\ &+ \Big\| \Big(\sum_{n \in \mathbb{Z}} V^2(\mathfrak{a}_t : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})^2 \Big)^{1/2} \Big\|_{L^p(X)}. \end{aligned} \tag{2.15}$$

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The inequality (2.15) is an analog of [**10**, Lemma 1.3, p. 6716] for oscillation seminorms.

2.7. *Jumps.* The r-variation is closely related to the λ-jump counting function. Recall that for any $\lambda > 0$, the λ-jump counting function of a function $f : \mathbb{I} \to \mathbb{C}$ is defined by

$$N_\lambda f := N_\lambda(f(t) : t \in \mathbb{I}) := \sup\Big\{J \in \mathbb{N} : \exists_{\substack{t_0<\ldots<t_J \\ t_j\in\mathbb{I}}} : \min_{0\leq j\leq J-1} |f(t_{j+1}) - f(t_j)| \geq \lambda\Big\}. \tag{2.16}$$

Remark 2.4. Some remarks about the definition in equation (2.16) are in order.

(1) For any $\lambda > 0$ and a function $f : \mathbb{I} \to \mathbb{C}$, let us also define the following quantity:

$$\begin{aligned}\mathcal{N}_\lambda f &:= \mathcal{N}_\lambda(f(t) : t \in \mathbb{I}) \\ &:= \sup\Big\{J \in \mathbb{N} : \exists_{\substack{s_1<t_1\leq\ldots\leq s_J<t_J \\ s_j,t_j\in\mathbb{I}}} : \min_{1\leq j\leq J} |f(t_j) - f(s_j)| \geq \lambda\Big\}.\end{aligned}$$

Then one has $N_\lambda f \leq \mathcal{N}_\lambda f \leq N_{\lambda/2} f$.

(2) Let $(\mathfrak{a}_t(x) : t \in \mathbb{R}) \subseteq \mathbb{C}$ be a family of measurable functions on a σ-finite measure space $(X, \mathcal{B}(X), \mu)$. Let $\mathbb{I} \subseteq \mathbb{R}$ and $\#\mathbb{I} \geq 2$, then for every $p \in [1, \infty]$ and $r \in [1, \infty)$, we have

$$\sup_{\lambda>0} \|\lambda N_\lambda(\mathfrak{a}_t : t \in \mathbb{I})^{1/r}\|_{L^p(X)} \leq \|V^r(\mathfrak{a}_t : t \in \mathbb{I})\|_{L^p(X)}, \tag{2.17}$$

since for all $\lambda > 0$, we have the following pointwise estimate:

$$\lambda N_\lambda(\mathfrak{a}_t(x) : t \in \mathbb{I})^{1/r} \leq V^r(\mathfrak{a}_t(x) : t \in \mathbb{I}).$$

(3) Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space and $\mathbb{I} \subseteq \mathbb{R}$. Fix $p \in [1, \infty]$, and $1 \leq \rho < r \leq \infty$. Then for every measurable function $f : X \times \mathbb{I} \to \mathbb{C}$, we have the estimate

$$\|V^r(f(\cdot, t) : t \in \mathbb{I})\|_{L^{p,\infty}(X)} \lesssim_{p,\rho,r} \sup_{\lambda>0} \|\lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I})^{1/\rho}\|_{L^{p,\infty}(X)}. \tag{2.18}$$

The inequality (2.18) can be thought of as an inversion of the inequality (2.17). A detailed proof of equation (2.18) can be found in [**16**, Lemma 2.3, p. 805].

(4) For every $p \in (1, \infty)$, and $\rho \in (1, \infty)$, there exists a constant $0 < C < \infty$ such that for every measure space $(X, \mathcal{B}(X), \mu)$, and $\mathbb{I} \subseteq \mathbb{R}$, there exists a (subadditive) seminorm $|||\cdot|||$ such that

$$C^{-1}|||f||| \leq \sup_{\lambda>0} \|\lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I})^{1/\rho}\|_{L^p(X)} \leq C|||f||| \tag{2.19}$$

holds for all measurable functions $f : X \times \mathbb{I} \to \mathbb{C}$. Inequalities in equation (2.19) were established in [**16**, Corollary 2.2, p. 805]. This shows that jumps are very close to seminorms.

We close this discussion by showing that we cannot replace $L^{p,\infty}(X)$ with $L^p(X)$ in equation (2.18).

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LEMMA 2.5. *For a fixed* $1 \le p < \infty$, *there exists a function* $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ *such that*

$$\|V^r(f(\cdot, n) : n \in \mathbb{Z}_+)\|_{\ell^p(\mathbb{Z}_+)} = \infty, \quad 2 \le r \le \infty$$

and

$$\sup_{\lambda>0} \|\lambda N_\lambda(f(\cdot, n) : n \in \mathbb{Z}_+)^{1/2}\|_{\ell^p(\mathbb{Z}_+)} < \infty.$$

Proof. Take a sequence $(a_x : x \in \mathbb{Z}_+) \subseteq \mathbb{C}$, to be specified later, such that $a_1 > a_2 > \ldots > 0$ and such that $a_x \to 0$ as $x \to \infty$ (actually, $a_x = x^{-1/p}$ will work). We define a function f as

$$f(x, 1) = a_x, \quad f(x, n) = 0, \quad n \in \mathbb{N}_{\ge 2}, \quad x \in \mathbb{Z}_+.$$

Then, for every $2 \le r \le \infty$, we have $V^r f(x) = a_x$ and, consequently, we have

$$\|V^r(f(\cdot, n) : n \in \mathbb{Z}_+)\|^p_{\ell^p(\mathbb{Z}_+)} = \sum_{j \in \mathbb{Z}_+} a_j^p, \quad 2 \le r \le \infty.$$

We now compute $N_\lambda f$. Observe that

$$N_\lambda f(x) = \begin{cases} 0, & \lambda > a_x, \\ 1, & \lambda \le a_x. \end{cases}$$

Using this, we see that $N_\lambda f(x) = 0$ for all $x \in \mathbb{Z}_+$ if $\lambda > a_1$. Otherwise, if $a_j \ge \lambda > a_{j+1}$ for some $j \ge 1$, then $N_\lambda f(x) = \mathbb{1}_{[j]}(x)$ and, consequently, we get

$$\|(N_\lambda f)^{1/2}\|^p_{\ell^p(\mathbb{Z}_+)} = \begin{cases} 0, & \lambda > a_1, \\ j, & a_j \ge \lambda > a_{j+1}, \quad j \in \mathbb{Z}_+. \end{cases}$$

Therefore, we obtain

$$\begin{aligned} \sup_{\lambda>0} \|\lambda(N_\lambda f)^{1/2}\|^p_{\ell^p(\mathbb{Z}_+)} &= \sup_{\lambda>0} \lambda^p \begin{cases} 0, & \lambda > a_1, \\ j, & a_j \ge \lambda > a_{j+1}, \quad j \in \mathbb{Z}_+. \end{cases} \\ &= \sup_{j \in \mathbb{Z}_+} j a_j^p. \end{aligned}$$

Taking $a_j = j^{-1/p}$, the desired conclusion follows and the proof of Lemma 2.5 is finished. □

Remark 2.6. Note that the same example can be used to show that λ-jumps with the parameter $r = 2$ may not imply r-oscillations with $r \ge 2$ in the ℓ^p sense. To be more precise, there exists a function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that

$$\sup_{N \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_N(\mathbb{Z}_+)} \|O^r_{I,N}(f(\cdot, n) : n \in \mathbb{Z}_+)\|_{\ell^p(\mathbb{Z}_+)} = \infty, \quad 2 \le r \le \infty$$

and

$$\sup_{\lambda>0} \|\lambda N_\lambda(f(\cdot, n) : n \in \mathbb{Z}_+)^{1/2}\|_{\ell^p(\mathbb{Z}_+)} < \infty.$$

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3. *Uniform oscillation inequalities: Proof of Theorem 1.1*

In this section, we prove the main uniform oscillation inequality from Theorem 1.1. We will closely follow the ideas and notation from [**18**]. We begin with standard reductions which allow us to simplify our arguments. By a standard lifting argument, it suffices to prove equation (1.4) for canonical polynomial mappings. Fix a non-empty $\Gamma \subset \mathbb{N}^k \setminus \{0\}$ and recall that the canonical polynomial mapping is defined by

$$\mathbb{R}^k \ni x = (x_1, \dots, x_k) \mapsto (x)^\Gamma := (x^\gamma : \gamma \in \Gamma) \in \mathbb{R}^\Gamma,$$

where $x^\gamma := x_1^{\gamma_1} \dots x_k^{\gamma_k}$ for $\gamma \in \Gamma$. Here, $\mathbb{R}^\Gamma$ denotes the space of tuples of real numbers labeled by multi-indices $\gamma = (\gamma_1, \dots, \gamma_k)$, so that $\mathbb{R}^\Gamma \cong \mathbb{R}^{|\Gamma|}$, and similarly for $\mathbb{Z}^\Gamma \cong \mathbb{Z}^{|\Gamma|}$.

For $t \in \mathbb{X}$, $x \in X$, and $f \in L^0(X)$, define the averaging operator

$$A_t f(x) = \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{m \in \Omega_t \cap \mathbb{Z}^k} f\Big(\prod_{\gamma \in \Gamma} T_\gamma^{m^\gamma} x \Big), \tag{3.1}$$

where $T_\gamma : X \to X$, $\gamma \in \Gamma$, is a family of commuting invertible measure preserving transformations. Our aim will be to prove the following ergodic theorem along the canonical polynomial mappings.

THEOREM 3.1. *Let $k \in \mathbb{Z}_+$ and a finite non-empty set $\Gamma \subset \mathbb{N}^k \setminus \{0\}$ be given. Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space endowed with a family of commuting invertible measure preserving transformations $\{T_\gamma : X \to X : \gamma \in \Gamma\}$. Then for every $p \in (1, \infty)$, there exists a constant $C_{\Gamma,k,p} > 0$ such that for every $f \in L^p(X)$, we have*

$$\sup_{J \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_J(\mathbb{X})} \| O_{I,J}^2(A_t f : t \in \mathbb{X}) \|_{L^p(X)} \le C_{\Gamma,k,p} \| f \|_{L^p(X)}. \tag{3.2}$$

We immediately see that Theorem 3.1 is a special case of Theorem 1.1. However, for every polynomial mapping $P = (P_1, \dots, P_d) : \mathbb{Z}^k \to \mathbb{Z}^d$ as in equation (1.1), there exists a set of multi-indices $\Gamma = \mathbb{N}^k_{\le d_0} \setminus \{0\}$ for some $d_0 \in \mathbb{Z}_+$ such that for any $j \in [d]$, each P_j can be written as

$$P_j(m) = \sum_{\gamma \in \Gamma} a_{j,\gamma} m^\gamma, \quad m \in \mathbb{Z}^k$$

for some coefficients $a_{j,\gamma} \in \mathbb{Z}$. Setting

$$T_\gamma = \prod_{j=1}^d T_j^{a_{j,\gamma}}, \quad \gamma \in \Gamma, \tag{3.3}$$

where $T_1, \dots, T_d : X \to X$ is a family of commuting invertible measure preserving transformations, we see that $A_t^P = A_t$, since

$$\prod_{j=1}^d T_j^{P_j(m)} = \prod_{j=1}^d \prod_{\gamma \in \Gamma} T_j^{a_{j,\gamma} m^\gamma} = \prod_{\gamma \in \Gamma} T_\gamma^{m^\gamma}.$$

This shows that the inequality (3.2) from Theorem 3.1 implies the corresponding inequality in Theorem 1.1 and thus it suffices to prove Theorem 3.1. For this purpose, a further

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reduction to the set of integers is in order. By invoking Calderón's transference [4] principle, the matter is reduced to proving the following theorem.

THEOREM 3.2. *Let $k \in \mathbb{Z}_+$ and a finite non-empty set $\Gamma \subset \mathbb{N}^k \setminus \{0\}$ be given. Let*

$$M_t f(x) := \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(x - (y)^\Gamma), \quad x \in \mathbb{Z}^\Gamma \tag{3.4}$$

be an integer counterpart of A_t defined in equation (3.1). Then for every $p \in (1, \infty)$, there exists a constant $C_{\Gamma,k,p} > 0$ such that for every $f \in L^p(X)$, we have

$$\sup_{J \in \mathbb{Z}_+} \sup_{I \in \mathfrak{S}_J(\mathbb{X})} \|O^2_{I,J}(M_t f : t \in \mathbb{X})\|_{\ell^p(\mathbb{Z}^\Gamma)} \leq C_{\Gamma,k,p} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.5}$$

This reduction will allows us to use Fourier methods which are not available in the abstract measure spaces setting. The operators from equation (3.4) are sometimes called discrete averaging Radon operators. From now on, we will closely follow the ideas from [18] to establish equation (3.5). Its proof will be illustrated in the next few subsections.

3.1. *Ionescu–Wainger multiplier theorem.* The key tool, which will be used in the proof of Theorem 3.2, is the Ionescu–Wainger theorem [7]. We now recall the d-dimensional vector-valued Ionescu–Wainger multiplier theorem from [18, §2].

THEOREM 3.3. *For every $\varrho > 0$, there exists a family $(P_{\leq N})_{N \in \mathbb{Z}_+}$ of subsets of $\mathbb{Z}_+$ satisfying the following properties.*

(i) *One has $[N] \subseteq P_{\leq N} \subseteq [\max\{N, e^{N^\varrho}\}]$.*
(ii) *If $N_1 \leq N_2$, then $P_{\leq N_1} \subseteq P_{\leq N_2}$.*
(iii) *If $q \in P_{\leq N}$, then all factors of q also lie in $P_{\leq N}$.*
(iv) *One has $\operatorname{lcm}(P_N) \leq 3^N$.*

Furthermore, for every $p \in (1, \infty)$, there exists $0 < C_{p,\varrho,d} < \infty$ such that, for every $N \in \mathbb{Z}_+$, the following holds.

Let $0 < \varepsilon_N \leq e^{-N^{2\varrho}}$, and let $\Theta : \mathbb{R}^d \to L(H_0, H_1)$ be a measurable function supported on $\varepsilon_N \mathbf{Q}$, where $\mathbf{Q} := [-1/2, 1/2)^d$ is a unit cube, with values in the space $L(H_0, H_1)$ of bounded linear operators between separable Hilbert spaces H_0 and H_1. Let $0 \leq \mathbf{A}_p \leq \infty$ denote the smallest constant such that, for every function $f \in L^2(\mathbb{R}^d; H_0) \cap L^p(\mathbb{R}^d; H_0)$, we have

$$\|T_{\mathbb{R}^d}[\Theta] f\|_{L^p(\mathbb{R}^d; H_1)} \leq \mathbf{A}_p \|f\|_{L^p(\mathbb{R}^d; H_0)}. \tag{3.6}$$

Then the multiplier

$$\Delta_N(\xi) := \sum_{b \in \Sigma_{\leq N}} \Theta(\xi - b), \tag{3.7}$$

where $\Sigma_{\leq N}$ is 1-periodic subsets of $\mathbb{T}^d$ defined by

$$\Sigma_{\leq N} := \left\{ \frac{a}{q} \in \mathbb{Q}^d \cap \mathbb{T}^d : q \in P_{\leq N} \text{ and } \gcd(a, q) = 1 \right\}, \tag{3.8}$$

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satisfies, for every $f \in L^p(\mathbb{Z}^d; H_0)$*, the following inequality:*

$$\|T_{\mathbb{Z}^d}[\Delta_N]f\|_{\ell^p(\mathbb{Z}^d;H_1)} \leq C_{p,\varrho,d}(\log N)\mathbf{A}_p\|f\|_{\ell^p(\mathbb{Z}^d;H_0)}. \tag{3.9}$$

An important feature of Theorem 3.3 is that one can directly transfer square function estimates from the continuous to the discrete setting. The constant $\mathbf{A}_p$ in equation (3.6) remains unchanged when Θ is replaced by $\Theta(A\cdot)$ for any invertible linear transformation $A : \mathbb{R}^d \to \mathbb{R}^d$. By property i, we see

$$|\Sigma_{\leq N}| \lesssim e^{(d+1)N^{\varrho}}. \tag{3.10}$$

A scalar-valued version of Theorem 3.3 was originally established by Ionescu and Wainger [7] with the factor $(\log N)^D$ in place of $\log N$ in equation (3.9). Their proof was based on a delicate inductive argument that exploited super-orthogonality phenomena arising from disjoint supports in the definition of equation (3.7). A somewhat different proof with factor $\log N$ in equation (3.9) was given in [14]. The latter proof, instead of induction as in [7], used certain recursive arguments, which clarified the role of the underlying square functions and strong-orthogonalities that lie behind of the proof of the Ionescu–Wainger multiplier theorem. A much broader context of the super-orthogonality phenomena was recently discussed in the survey article of Pierce [19]. A detailed proof of Theorem 3.3 (in the spirit of [14]) can be found in [18, §2]. We also refer to the recent remarkable paper of Tao [23], where the factor $\log N$ was removed from equation (3.9).

An important ingredient in the proof of Theorem 3.3 is the sampling principle of Magyar–Stein–Wainger from [13]. We recall this principle as it will play an essential role in our further discussion.

PROPOSITION 3.4. *Let* $d \in \mathbb{Z}_+$ *be fixed. There exists an absolute constant* $C > 0$ *such that the following holds. Let* $p \in [1, \infty]$ *and* $q \in \mathbb{Z}_+$*, and let* B_1*,* B_2 *be finite-dimensional Banach spaces. Let* $\mathfrak{m} : \mathbb{R}^d \to L(B_1, B_2)$ *be a bounded operator-valued function supported on* $q^{-1}\boldsymbol{Q}$ *and denote the associated Fourier multiplier operator over* $\mathbb{R}^d$ *by* $T_{\mathbb{R}^d}[\mathfrak{m}]$*. Let* $\mathfrak{m}^q_{\mathrm{per}}$ *be the periodic multiplier*

$$\mathfrak{m}^q_{\mathrm{per}}(\xi) := \sum_{n\in\mathbb{Z}^d} \mathfrak{m}(\xi - n/q), \quad \xi \in \mathbb{T}^d.$$

Then,

$$\|T_{\mathbb{Z}^d}[\mathfrak{m}^q_{\mathrm{per}}]\|_{\ell^p(\mathbb{Z}^d;B_1)\to\ell^p(\mathbb{Z}^d;B_2)} \leq C\|T_{\mathbb{R}^d}[\mathfrak{m}]\|_{L^p(\mathbb{R}^d;B_1)\to L^p(\mathbb{R}^d;B_2)}.$$

The proof can be found in [13, Corollary 2.1, p. 196]. We also refer to [16] for generalization of Proposition 3.4 to real interpolation spaces, which in particular covers the case of jump inequalities.

3.2. *Proof of inequality* (3.5) *from Theorem 3.2.* Fix $p \in (1, \infty)$ and let $f \in \ell^p(\mathbb{Z}^\Gamma)$ be a finitely supported function. We fix $p_0 > 1$ close to 1 such that $p \in (p_0, p_0')$. We take $\tau \in (0, 1)$ such that

$$\tau < \tfrac{1}{2}\min\{p_0 - 1, 1\}. \tag{3.11}$$

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Now by equation (2.15), one can split equation (3.5) into long oscillations and short variations respectively by

$$\begin{aligned}&\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{X})}\|O^2_{I,N}(M_t f : t\in\mathbb{X})\|_{\ell^p(\mathbb{Z}^\Gamma)}\\&\lesssim \sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau)}\|O^2_{I,N}(M_t f : t\in\mathbb{D}_\tau)\|_{\ell^p(\mathbb{Z}^\Gamma)}\\&\quad+\Big\|\Big(\sum_{n=0}^{\infty}V^2(M_t f : t\in[2^{n^\tau},2^{(n+1)^\tau}))^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}.\end{aligned}\tag{3.12}$$

To handle the short variations, we may proceed as in [18] (see also [24]) and conclude that

$$\begin{aligned}&\Big\|\Big(\sum_{n=0}^{\infty}V^2(M_{2^t} f : t\in[n^\tau,(n+1)^\tau))^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\&\lesssim\Big(\sum_{n=0}^{\infty}n^{-q(1-\tau)}\Big)^{1/q}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim\|f\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}\tag{3.13}$$

since $q(1-\tau)>1$ by equation (3.11); here, $q=\min\{p,2\}$. The first inequality in equation (3.13) follows from a simple observation that for any finite sequence $t_0<t_1<\ldots<t_J$ contained in $[n^\tau,(n+1)^\tau)$, one has

$$\sum_{j=1}^{J}\|(M_{2^{t_j}}-M_{2^{t_{j-1}}})\mathbb{1}_{\{0\}}\|_{\ell^1(\mathbb{Z}^\Gamma)}\lesssim 2^{-kn^\tau}|\mathbb{Z}^k\cap(\Omega_{2^{(n+1)^\tau}}\setminus\Omega_{2^{n^\tau}})|.$$

The latter quantity is controlled by

$$2^{-kn^\tau}|\mathbb{Z}^k\cap(\Omega_{2^{(n+1)^\tau}}\setminus\Omega_{2^{n^\tau}})|\lesssim n^{\tau-1},$$

which immediately follows by invoking the following proposition.

PROPOSITION 3.5. ([18, Proposition 4.16, p. 45], see also [17, Lemma A.1, p. 554]) *Let $\Omega\subset\mathbb{R}^k$ be a bounded and convex set and let $1\le s\le\operatorname{diam}(\Omega)$. Then,*

$$\#\{x\in\mathbb{Z}^k:\operatorname{dist}(x,\partial\Omega)<s\}\lesssim_k s\operatorname{diam}(\Omega)^{k-1}.\tag{3.14}$$

The implicit constant depends only on the dimension k, but not on the convex set Ω.

Now we have to bound long oscillations from equation (3.12). By Davenport's result [5], we know that

$$\#(\Omega_{2^{n^\tau}}\cap\mathbb{Z}^k)=|\Omega_{2^{n^\tau}}|+\mathcal{O}(2^{n^\tau(k-1)}).$$

Consequently, one has the following estimate:

$$\|M_{2^{n^\tau}}f-T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}]f\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim 2^{-n^\tau}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

where

$$m_t(\xi):=\frac{1}{|\Omega_t|}\sum_{y\in\Omega_t\cap\mathbb{Z}^k}\boldsymbol{e}(\xi\cdot(y)^\Gamma),\quad\xi\in\mathbb{T}^\Gamma,\quad t\in\mathbb{R}_+.$$

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Thus, we are reduced to prove

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau)}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_t]f : t\in\mathbb{D}_\tau)\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.15}$$

3.3. *Proof of inequality* (3.15). Now, let $\chi\in(0,1/10)$. The proof of equation (3.15) will require several appropriately chosen parameters. We choose $\alpha>0$ such that

$$\alpha>\left(\frac{1}{p_0}-\frac{1}{2}\right)\left(\frac{1}{p_0}-\frac{1}{\min\{p,p'\}}\right)^{-1}.$$

Let $u\in\mathbb{Z}_+$ be a large natural number to be specified later. We set

$$\varrho:=\min\left\{\frac{1}{10u},\frac{\delta}{8\alpha}\right\}, \tag{3.16}$$

where $\delta>0$ is the exponent from the Gauss sum estimates in equation (3.28). Let $S_0:=\max(2^{u\mathbb{Z}_+}\cap[1,n^{\tau u}])$. We shall, by convenient abuse of notation, write

$$\Sigma_{\leq n^{\tau u}}:=\Sigma_{\leq S_0}.$$

Next, for dyadic integers $S\in 2^{u\mathbb{Z}_+}$, we define

$$\Sigma_S:=\begin{cases}\Sigma_{\leq S} & \text{if } S=2^u,\\ \Sigma_{\leq S}\setminus\Sigma_{\leq S/2^u} & \text{if } S>2^u.\end{cases}$$

Then, it is easy to see that

$$\Sigma_{\leq n^{\tau u}}=\bigcup_{\substack{S\leq S_0,\\ S\in 2^{u\mathbb{Z}_+}}}\Sigma_S. \tag{3.17}$$

Now we define the Ionescu–Wainger projection multipliers. For this purpose, we introduce a diagonal matrix A of size $|\Gamma|\times|\Gamma|$ such that $(Av)_\gamma:=|\gamma|\,v_\gamma$ for any $\gamma\in\Gamma$ and $v\in\mathbb{R}^\Gamma$, and for any $t>0$, we also define corresponding dilations by setting $t^Ax=(t^{|\gamma|}x_\gamma:\gamma\in\Gamma)$ for every $x\in\mathbb{R}^\Gamma$. Let $\eta:\mathbb{R}^\Gamma\to[0,1]$ be a smooth function such that

$$\eta(x)=\begin{cases}1, & |x|\leq 1/(32|\Gamma|),\\ 0, & |x|\geq 1/(16|\Gamma|).\end{cases}$$

For any $n\in\mathbb{Z}_+$, we define

$$\Pi_{\leq n^\tau,\,n^\tau(A-\chi\mathrm{Id})}(\xi):=\sum_{a/q\in\Sigma_{\leq n^{\tau u}}}\eta(2^{n^\tau(A-\chi\mathrm{Id})}(\xi-a/q)),\quad \xi\in\mathbb{T}^\Gamma, \tag{3.18}$$

as well as

$$\Pi_{S,\,n^\tau(A-\chi\mathrm{Id})}(\xi):=\sum_{a/q\in\Sigma_S}\eta(2^{n^\tau(A-\chi\mathrm{Id})}(\xi-a/q)),\quad \xi\in\mathbb{T}^\Gamma,\quad S\in 2^{u\mathbb{Z}_+}. \tag{3.19}$$

Note that Theorem 3.3 applies for equations (3.18) and (3.19) since $2^{-n^\tau(|\gamma|-\chi)}\leq e^{-n^{\tau/10}}\leq e^{-S_0^{2\varrho}}$ for sufficiently large $n\in\mathbb{Z}_+$. Using equation (3.18), one sees that

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$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau)}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_t]f : t\in\mathbb{D}_\tau)\|_{\ell^p(\mathbb{Z}^\Gamma)}$$
$$\lesssim \sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}(1-\Pi_{\leq n^\tau,\, n^\tau(A-\chi\mathrm{Id})})]f : n\in\mathbb{N})\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.20}$$
$$+\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}\Pi_{\leq n^\tau,\, n^\tau(A-\chi\mathrm{Id})}]f : n\in\mathbb{N})\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.21}$$

The first and second terms in the above inequality correspond to minor and major arcs, respectively.

3.4. *Minor arcs: estimates for equation* (3.20). A key ingredient in estimating equation (3.20) will be a multidimensional Weyl's inequality.

THEOREM 3.6. [**18**, Theorem A.1, p. 49] *For every $k\in\mathbb{Z}_+$ and a finite non-empty set $\Gamma\subset\mathbb{N}^k\setminus\{0\}$, there exists $\epsilon>0$ such that, for every polynomial*

$$P(n)=\sum_{\gamma\in\Gamma}\xi_\gamma n^\gamma,$$

every $N>1$, convex set $\Omega\subseteq B(0,N)\subset\mathbb{R}^k$, function $\phi:\Omega\cap\mathbb{Z}^k\to\mathbb{C}$, multi-index $\gamma_0\in\Gamma$, and integers $0\leq a<q$ with $\gcd(a,q)=1$ and

$$\left|\xi_{\gamma_0}-\frac{a}{q}\right|\leq\frac{1}{q^2},$$

we have

$$\left|\sum_{n\in\Omega\cap\mathbb{Z}^k}\boldsymbol{e}(P(n))\phi(n)\right|\lesssim_{\Gamma,k} N^k\kappa^{-\epsilon}\log(N+1)\|\phi\|_{L^\infty(\Omega)}+N^k\sup_{|x-y|\leq N\kappa^{-\epsilon}}|\phi(x)-\phi(y)|, \tag{3.22}$$

where

$$\kappa:=\min\{q,N^{|\gamma_0|}/q\}.$$

The implicit constant in equation (3.22) is independent on the coefficients of P and the numbers a, q, and N.

To estimate equation (3.20), we first observe, using equation (2.13), that

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}(1-\Pi_{\leq n^\tau,\, n^\tau(A-\chi\mathrm{Id})})]f : n\in\mathbb{N})\|_{\ell^p(\mathbb{Z}^\Gamma)}$$
$$\lesssim\sum_{n=0}^{\infty}\|T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}(1-\Pi_{\leq n^\tau,\, n^\tau(A-\chi\mathrm{Id})})]f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.23}$$

Using Theorem 3.6 and proceeding as in [**18**, Lemma 3.29, p. 34], we may conclude that

$$\|T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}(1-\Pi_{\leq n^\tau,\, n^\tau(A-\chi\mathrm{Id})})]f\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim(n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.24}$$

provided that u is large. Combining equation (3.23) with equation (3.24), we obtain the desired bound for equation (3.20).

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3.5. *Major arcs: estimates for equation* (3.21). Our aim is to prove

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}\Pi_{\leq n^\tau,\,n^\tau(A-\chi\mathrm{Id})}]f:n\in\mathbb{N})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.25}$$

For $n\in\mathbb{N}$ and $\xi\in\mathbb{T}^\Gamma$, define a new multiplier

$$\boldsymbol{m}_n(\xi):=\sum_{a/q\in\Sigma_{\leq n^{\tau u}}}G(a/q)\Phi_{2^{n^\tau}}(\xi-a/q)\eta(2^{n^\tau(A-\chi\mathrm{Id})}(\xi-a/q)), \tag{3.26}$$

where Φ_N is a continuous version of the multiplier m_N given by

$$\Phi_N(\xi):=\frac{1}{|\Omega_N|}\int_{\Omega_N}\boldsymbol{e}(\xi\cdot(t)^\Gamma)\,dt,$$

and $G(a/q)$ is the Gauss sum defined by

$$G(a/q):=\frac{1}{q^k}\sum_{r\in[q]^k}\boldsymbol{e}((a/q)\cdot(r)^\Gamma).$$

It is well known from the multidimensional Van der Corput lemma [22, Proposition 5, p. 342] that

$$|\Phi_N(\xi)|\lesssim|N^A\xi|_\infty^{-1/|\Gamma|}\quad\text{and}\quad|\Phi_N(\xi)-1|\lesssim|N^A\xi|_\infty. \tag{3.27}$$

By Theorem 3.6 (see also [18, Lemma 4.14, p. 44]), one can easily find $\delta\in(0,1)$ such that

$$|G(a/q)|\lesssim_k q^{-\delta} \tag{3.28}$$

for every $q\in\mathbb{Z}_+$ and $a=(a_\gamma:\gamma\in\Gamma)\in[q]^\Gamma$ such that $\gcd(a,q)=1$.

We claim that

$$\|T_{\mathbb{Z}^\Gamma}[m_{2^{n^\tau}}\Pi_{\leq n^\tau,\,n^\tau(A-\chi\mathrm{Id})}-\boldsymbol{m}_n]f\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim 2^{-n^{\tau/2}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.29}$$

To establish equation (3.29), one can proceed as in [18, Lemma 3.38, p. 36] by appealing to the proposition stated below and the estimate in equation (3.10), Theorem 3.3, and Proposition 3.4.

PROPOSITION 3.7. [18, Proposition 4.18, p. 47] *Let $\Omega\subseteq B(0,N)\subset\mathbb{R}^k$ be a convex set and $\mathcal{K}\colon\Omega\to\mathbb{C}$ be a continuous function. Then for any $q\in\mathbb{Z}_+$, $a=(a_\gamma:\gamma\in\Gamma)\in[q]^\Gamma$ such that $\gcd(a,q)=1$ and $\xi=a/q+\theta\in\mathbb{R}^\Gamma$, we have*

$$\begin{aligned}\Big|\sum_{y\in\Omega\cap\mathbb{Z}^k}\boldsymbol{e}(\xi\cdot(y)^\Gamma)\mathcal{K}(y)-G(a/q)\int_\Omega\boldsymbol{e}(\theta\cdot(t)^\Gamma)\mathcal{K}(t)\,dt\Big|&\\ \lesssim_k\frac{q}{N}N^k\|\mathcal{K}\|_{L^\infty(\Omega)}+N^k\|\mathcal{K}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma}(q|\theta_\gamma|N^{|\gamma|-1})^{\varepsilon_\gamma}&\\ +N^k\sup_{x,y\in\Omega:\,|x-y|\leq q}|\mathcal{K}(x)-\mathcal{K}(y)|,&\end{aligned}$$

for any sequence $(\varepsilon_\gamma:\gamma\in\Gamma)\subseteq[0,1]$. The implicit constant is independent of a, q, N, θ and the kernel $\mathcal{K}$.

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Using equations (3.29) and (2.13), the proof of the inequality (3.25) is reduced to showing

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\boldsymbol{m}_n]f : n\in\mathbb{N})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.30}$$

3.6. *Major arcs: estimates for equation* (3.30). For $n\in\mathbb{Z}_+$, $S\in 2^{u\mathbb{Z}_+}$, and $\xi\in\mathbb{T}^\Gamma$, we define a new multiplier

$$\boldsymbol{m}^S_n(\xi):=\sum_{a/q\in\Sigma_S}G(a/q)\Phi_{2^{n^\tau}}(\xi-a/q)\eta(2^{n^\tau(A-\chi\mathrm{Id})}(\xi-a/q)). \tag{3.31}$$

Then using equations (3.17) and (3.31), we see that to prove equation (3.30), it suffices to show that

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_{\geq S^{1/(\tau u)}})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\boldsymbol{m}^S_n]f : n\in\mathbb{N}_{\geq S^{1/(\tau u)}})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim S^{-4\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.32}$$

since $S^{-4\varrho}$ is summable in $S\in 2^{u\mathbb{Z}_+}$.

To establish equation (3.32), we define $\tilde\eta(x):=\eta(x/2)$ and two new multipliers

$$\nu^S_n(\xi):=\sum_{a/q\in\Sigma_S}\Phi_{2^{n^\tau}}(\xi-a/q)\eta(2^{n^\tau(A-\chi\mathrm{Id})}(\xi-a/q)),$$
$$\mu_S(\xi):=\sum_{a/q\in\Sigma_S}G(a/q)\tilde\eta(2^{S^{1/u}(A-\chi\mathrm{Id})}(\xi-a/q)).$$

Obviously, we have $\boldsymbol{m}^S_n=\nu^S_n\mu_S$ and we see that the estimate in equation (3.32) will follow if we show that

$$\|T_{\mathbb{Z}^\Gamma}[\mu_S]f\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim S^{-7\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.33}$$

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_{\geq S^{1/(\tau u)}})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\nu^S_n]f : n\in\mathbb{N}_{\geq S^{1/(\tau u)}})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim S^{3\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.34}$$

Using Proposition 3.7, Theorem 3.3, and the Gauss sum estimates in equation (3.28), we can easily establish equation (3.33), and we refer to [**18**, Lemma 3.47 and estimate (3.49) pp. 38–39] for more details. Now we return to equation (3.34). We define $\kappa_S:=\lceil S^{2\varrho}\rceil$ and split equation (3.34) into small and large scales respectively by

$$\begin{aligned}&\text{Left-hand side of equation (3.34)}\\ &\lesssim\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\leq S})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\nu^S_n]f : n\in\mathbb{D}^\tau_{\leq S})\|_{\ell^p(\mathbb{Z}^\Gamma)}\\ &\quad+\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\nu^S_n]f : n\in\mathbb{D}^\tau_{\geq S})\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where $\mathbb{D}^\tau_{\leq S}:=\{n\in\mathbb{Z}_+ : n\in[S^{1/(\tau u)},2^{\kappa_S+1}]\}$ and $\mathbb{D}^\tau_{\geq S}:=\{n\in\mathbb{Z}_+ : n\geq 2^{\kappa_S}\}$.

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3.7. *Major arcs: small scales estimates.* Our aim is to prove

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^{\tau}_{\leq S})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\nu^S_n]f : n\in\mathbb{D}^{\tau}_{\leq S})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim \kappa_S\log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.35}$$

Using the Rademacher–Menschov inequality from equation (2.14), one has

$$\text{Left-hand side of equation (3.35)}\lesssim \sum_{i=0}^{\kappa_S+1}\Big\|\Big(\sum_j\Big|\sum_{n\in U^i_j}T_{\mathbb{Z}^\Gamma}[\nu^S_{n+1}-\nu^S_n]f\Big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

where $j\in\mathbb{Z}_+$ runs over the set of integers such that $U^i_j=[j2^i,(j+1)2^i)\subseteq[S^{1/(\tau u)},2^{\kappa_S+1}]$. By Theorem 3.3, it suffices to justify that uniformly in $0\leq i\leq\kappa_S+1$, we have

$$\Big\|\Big(\sum_j\Big|\sum_{n\in U^i_j}T_{\mathbb{R}^d}[\Phi_{2^{(n+1)^\tau}}\eta(2^{(n+1)^\tau(A-\chi\mathrm{Id})}\cdot)-\Phi_{2^{n^\tau}}\eta(2^{n^\tau(A-\chi\mathrm{Id})}\cdot)]f\Big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}$$
$$\lesssim\|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.36}$$

This, in turn, is a fairly straightforward matter by appealing to equation (3.27) and standard arguments from the Littlewood–Paley theory. We refer to [17] for more details, see also discussion below [18, Theorem 4.3, p. 42].

3.8. *Major arcs: large scales estimates.* Finally, our aim is to prove

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^{\tau}_{\geq S})}\|O^2_{I,N}(T_{\mathbb{Z}^\Gamma}[\nu^S_n]f : n\in\mathbb{D}^{\tau}_{\geq S})\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.37}$$

Proceeding as in [18, §3.6, pp. 40–41] and invoking Proposition 3.4, the estimate in equation (3.37) is reduced to showing for every $p\in(1,\infty)$ and $f\in L^p(\mathbb{R}^\Gamma)$, the following uniform oscillation inequality:

$$\sup_{N\in\mathbb{Z}_+}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(T_{\mathbb{R}^d}[\Phi_t]f : t\in\mathbb{R}_+)\|_{L^p(\mathbb{R}^d)}\lesssim\|f\|_{L^p(\mathbb{R}^d)}. \tag{3.38}$$

The inequality (3.38) can be reduced to the martingale setting from Proposition 2.2 by invoking square function arguments [10, Lemma 3.2, p. 6722] and the standard Littlewood–Paley theory. The details may be found in [17]. This completes the proof of Theorem 3.2.

4. *Main counterexample: Proof of Theorem 1.2*

In this section, we prove our main counterexample. We introduce the definition of the sequential λ-jump counting function. Let $\mathbb{I}\subseteq\mathbb{R}$, and for a given increasing sequence $I=(I_j : j\in\mathbb{N})\subseteq\mathbb{I}$ and any $\lambda>0$, the sequential λ-jump counting function of a function $f:\mathbb{I}\to\mathbb{C}$ is defined by

$$N_{\lambda,I}f:=N_{\lambda,I}(f(t) : t\in\mathbb{I}):=\#\Big\{k\in\mathbb{N} : \sup_{\substack{I_k\leq t<I_{k+1}\\ t\in\mathbb{I}}}|f(t)-f(I_k)|\geq\lambda\Big\}. \tag{4.1}$$

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Then it is easy to see that $N_{\lambda,I} f \leq \mathcal{N}_{\lambda,I} f \leq N_{\lambda/2,I} f$, where

$$\mathcal{N}_{\lambda,I} f := \mathcal{N}_{\lambda,I}(f(t) : t \in \mathbb{I}) := \#\Big\{k \in \mathbb{N} : \sup_{\substack{I_k \leq s,t < I_{k+1} \\ s,t \in \mathbb{I}}} |f(t) - f(s)| \geq \lambda\Big\}.$$

Let $\mathbb{I} \subseteq \mathbb{R}$, and an increasing sequence $I = (I_j : j \in \mathbb{N}) \subseteq \mathbb{I}$ be given. For any function $f : \mathbb{I} \to \mathbb{C}$ and $\lambda > 0$, we record the following inequality:

$$N_{\lambda,I} f \leq \mathcal{N}_{\lambda-\varepsilon} f, \quad 0 < \varepsilon < \lambda, \tag{4.2}$$

with $\mathcal{N}_\lambda f$ defined in Remark 2.4.

Remark 4.1. Note that the sequential λ-jump counting function in equation (4.1) corresponds to the r-oscillations in a similar way as the standard λ-jump counting function in equation (2.16) corresponds to the r-variations. Namely, for any $\lambda > 0$, and $I \in \mathfrak{S}_\infty(\mathbb{I})$ and $r \geq 1$, one has the following pointwise inequality:

$$\lambda N_{\lambda,I}(f(t) : t \in \mathbb{I})^{1/r} \leq O^r_{I,\infty}(f(t) : t \in \mathbb{I}), \tag{4.3}$$

which is a straightforward consequence of equation (2.3) and the definition of $N_{\lambda,I} f$.

We have the following counterpart of the inequality (2.18) but stated in terms of the sequential λ-jump counting function and the oscillation seminorm.

LEMMA 4.2. *Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space and $\mathbb{I} \subseteq \mathbb{R}$. Fix $p \in [1, \infty]$, and $1 \leq \rho < r \leq \infty$. Then for every measurable function $f : X \times \mathbb{I} \to \mathbb{C}$, we have the estimate*

$$\begin{aligned} &\sup_{I \in \mathfrak{S}_\infty(\mathbb{I})} \| O^r_{I,\infty}(f(\cdot,t) : t \in \mathbb{I}) \|_{L^{p,\infty}(X)} \\ &\quad \lesssim_{p,\rho,r} \sup_{I \in \mathfrak{S}_\infty(\mathbb{I})} \sup_{\lambda > 0} \| \lambda N_{\lambda,I}(f(\cdot,t) : t \in \mathbb{I})^{1/\rho} \|_{L^{p,\infty}(X)}. \end{aligned} \tag{4.4}$$

Proof. The proof is a repetition of the arguments from [**16**, Lemma 2.3, p. 805]. We omit the details. □

Now we can state the main result of this section.

LEMMA 4.3. *Let $1 \leq p < \infty$ and $1 < \rho \leq r < \infty$ be fixed. It is not true that the estimate*

$$\sup_{\lambda > 0} \| \lambda N_\lambda(f(\cdot,t) : t \in \mathbb{N})^{1/r} \|_{\ell^{p,\infty}(\mathbb{Z})} \leq C_{p,\rho,r} \sup_{I \in \mathfrak{S}_\infty(\mathbb{N})} \| O^\rho_{I,\infty}(f(\cdot,t) : t \in \mathbb{N}) \|_{\ell^p(\mathbb{Z})} \tag{4.5}$$

holds uniformly for every measurable function $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$.

Before we prove Lemma 4.3, let us state its consequences.

COROLLARY 4.4. *Let $1 \leq p < \infty$ and $1 < \rho \leq r < \infty$ be fixed. Then the following estimates are not true uniformly for all measurable function $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$:*

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$$\|V^r(f(\cdot,t):t\in\mathbb{N})\|_{\ell^{p,\infty}(\mathbb{Z})}\lesssim_{p,\rho,r}\sup_{I\in\mathfrak{S}_\infty(\mathbb{N})}\|O^\rho_{I,\infty}(f(\cdot,t):t\in\mathbb{N})\|_{\ell^p(\mathbb{Z})},$$

$$\|V^r(f(\cdot,t):t\in\mathbb{N})\|_{\ell^{p,\infty}(\mathbb{Z})}\lesssim_{p,\rho,r}\sup_{I\in\mathfrak{S}_\infty(\mathbb{N})}\sup_{\lambda>0}\|\lambda(N_{\lambda,I}(f(\cdot,t):t\in\mathbb{N})^{1/\rho}\|_{\ell^p(\mathbb{Z})},$$

$$\sup_{\lambda>0}\|\lambda(N_\lambda(f(\cdot,t):t\in\mathbb{N})^{1/r}\|_{\ell^{p,\infty}(\mathbb{Z})}$$
$$\lesssim_{p,\rho,r}\sup_{I\in\mathfrak{S}_\infty(\mathbb{N})}\sup_{\lambda>0}\|\lambda(N_{\lambda,I}(f(\cdot,t):t\in\mathbb{N})^{1/\rho}\|_{\ell^p(\mathbb{Z})}.$$

Proof. The result is a simple consequence of Lemma 4.3 and inequalities (2.17), (4.3). □

We will produce a counterexample to equation (4.5) on a slightly different space than $\mathbb{Z}$ equipped with the counting measure. However, our counterexample can be easily transferred to the set of integers $\mathbb{Z}$.

Let $X:=\{(2^k,n):k\in\mathbb{N},\ n\in\mathbb{Z}_+\}$ be equipped with the measure μ given by

$$\mu(\{(2^k,n)\}):=2^k,\quad k\in\mathbb{N},\quad n\in\mathbb{Z}_+.$$

For $j,M\in\mathbb{Z}_+$, we define a sequence $f_{j,M}\colon X\times\mathbb{N}\to[0,1]$ as follows. If $k\in\mathbb{N}_{\le M}$, $n\in[2^{M-k}]$, and $t\in\mathbb{N}$, we set

$$\begin{aligned}&f_{j,M}(2^k,n,t)\\&:=\sum_{0\le m<2^{j-1}}(t-2^k((n-1)2^j+2m))2^{-k}\mathbb{1}_{[2^k((n-1)2^j+2m),\ 2^k((n-1)2^j+2m+1))}(t)\\&\quad+\sum_{0\le m<2^{j-1}}(-t+2^k((n-1)2^j+(2m+2)))\\&\quad\times 2^{-k}\mathbb{1}_{[2^k((n-1)2^j+(2m+1)),\ 2^k((n-1)2^j+(2m+2)))}(t).\end{aligned}$$

Further, we put $f_{j,M}(2^k,n,t)=0$ if $k\in\mathbb{N}_{>M}$ or $k\in\mathbb{N}_{\le M}$ but $n\in\mathbb{N}_{>2^{M-k}}$.

Let us observe that for every $j,M\in\mathbb{Z}_+$ and any sequence $\bar I=(I_n:n\in\mathbb{N})$, we have

$$\operatorname{supp}N_{2^{-N}}f_{j,M}\subseteq\{(2^k,n):k\in\mathbb{N}_{\le M},\ n\in[2^{M-k}]\},\quad N\in\mathbb{N},\tag{4.6}$$

$$N_{2^{-N}}f_{j,M}(x)=N_{2^{-M}}f_{j,M}(x),\quad N\in\mathbb{N}_{\ge M},\quad x\in X,\tag{4.7}$$

$$N_{2^{-N},\bar I}f_{j,M}(x)=N_{2^{-M},\bar I}f_{j,M}(x),\quad N\in\mathbb{N}_{\ge M},\quad x\in X,\tag{4.8}$$

$$N_{2^{-N}}f_{j,M}(2^k,n)=\begin{cases}2^{j+k}, & k\in\mathbb{N}_{\le N},\\ 2^{j+N}, & k\in\mathbb{N}_{\le M}\setminus\mathbb{N}_{<N},\end{cases}\qquad N\in\mathbb{N}_{\le M},\quad n\in[2^{M-k}],\tag{4.9}$$

$$N_{2^{-N},\bar I}f_{j,M}(x)\le N_{2^{-N}}f_{j,M}(x)\le 2^{j+N},\quad N\in\mathbb{N},\quad x\in X.\tag{4.10}$$

The next lemma will be a key ingredient in the proof of Lemma 4.3, allowing to control the right-hand side of equation (4.5) from above. The latter estimates after appropriate choice of the parameters j, M will lead us to the desired conclusion in Lemma 4.3.

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LEMMA 4.5. *For $j, M \in \mathbb{Z}_+$, $N \in \mathbb{N}_{\le M}$, and $W \in \mathbb{N}_{\le j+N}$, one has*

$$\sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \mu(\{x \in X : N_{2^{-N},\bar{I}} f_{j,M}(x) \ge 2^W\}) \le 2^{M+j+2+N-W} + (N+1)2^M + (M+1)2^M \mathbb{1}_{\{0\}}(W). \tag{4.11}$$

Proof. Observe first that the case of $W = 0$ is trivial because by using equations (4.10) and (4.6), we see that the left-hand side of equation (4.11) is controlled by

$$\mu(\{(2^k, n) : k \in \mathbb{N}_{\le M}, n \in [2^{M-k}]\}) = (M+1)2^M. \tag{4.12}$$

Thus from now on, we may assume that $W \ge 1$ and consequently $2^W \ge 2$.

Let us fix $\bar{I} \in \mathfrak{S}_\infty(\mathbb{N})$. For $k \in \mathbb{N}_{\le M}$, let us define $A_k := \{(2^k, n) : N_{2^{-N},\bar{I}} f_{j,M}(2^k, n) \ge 2^W\}$ and $a_k := \#A_k$. The number a_k means that the sequence $\bar{I}$ detects at least 2^W jumps of height at least 2^{-N} on a_k elements in the set $\{(2^k, n) : n \in [2^{M-k}]\}$. Note that

$$\mu(\{x \in X : N_{2^{-N},\bar{I}} f_{j,M}(x) \ge 2^W\}) = \sum_{k\in\mathbb{N}_{\le M}} 2^k a_k. \tag{4.13}$$

Observe that $\operatorname{supp} f_{j,M}(2^k, n, \cdot) \subseteq [0, 2^{M+j}]$ and consequently, without any loss of generality, we may assume that $\bar{I}$ has a finite length with the first term equal to 0 and the last term equal to 2^{M+j}. This shows that

$$2^{M+j} = \sum_l (I_{l+1} - I_l). \tag{4.14}$$

Let us assume that $k \in \mathbb{N}_{\le M} \setminus \mathbb{N}_{<N}$. Observe that for every element from A_k, there exist at least 2^W jumps of height at least 2^{-N}, which are detected by the sequence $\bar{I}$. Hence, for each element of A_k, we can always find $2^W - 1 \ge 1$ distinct jumps. Furthermore, the existence of a jump implies the existence of two consecutive terms of $\bar{I}$, say I_l and I_{l+1}, satisfying $I_{l+1} - I_l \ge 2^{k-N}$. Consequently, we see that for every $k \in \mathbb{N}_{\le M} \setminus \mathbb{N}_{<N}$, there exist $(2^W - 1)a_k$ pairs of consecutive terms of $\bar{I}$ whose difference is at least 2^{k-N}. Setting $b_{M+1} = 0$ and $b_k = \max_{k\le m\le M}(2^W - 1)a_m$ for $N \le k \le M$, we see that the sequence b_k is non-increasing and there are b_k pairs of consecutive terms of $\bar{I}$ whose difference is at least 2^{k-N}. Thus, one obtains

$$\sum_l (I_{l+1} - I_l) \ge \sum_{k=N}^{M} 2^{k-N}(b_k - b_{k+1}),$$

as the pairs that were counted at levels $\ge k+1$ are not counted at level k. This, together with equation (4.14), implies

$$\begin{aligned} 2^{M+j} &\ge \sum_{k=N}^{M} 2^{k-N} b_k - \sum_{k=N+1}^{M} 2^{k-N-1} b_k \\ &= b_N + \sum_{k=N+1}^{M} 2^{k-N-1} b_k \ge 2^{W-1} \sum_{k=N+1}^{M} 2^{k-1-N} a_k, \end{aligned}$$

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where in the last inequality, we have used the fact that $b_k \geq (2^W - 1)a_k \geq 2^{W-1}a_k$. Combining this with equation (4.13) and the trivial bound $a_k \leq 2^{M-k}$, we obtain

$$\begin{aligned}&\mu(\{x \in X : N_{2^{-N},\bar{I}} f_{j,M}(x) \geq 2^W\})\\ &\quad \leq \sum_{k \in \mathbb{N}_{\leq M}} 2^k a_k \leq 2^{M+j+2+N-W} + (N+1)2^M, \quad W \geq 1.\end{aligned}$$

This completes the proof of Lemma 4.5. □

A simple consequence of Lemma 4.5 is the following useful estimate, which will be used in the proof of Lemma 4.3 only with $q \geq 1$.

LEMMA 4.6. *Let $q \in \mathbb{R}_+$ be fixed. Then*

$$\sup_{\bar{I} \in \mathfrak{S}_\infty(\mathbb{N})} \int_X (N_{2^{-N},\bar{I}} f_{j,M}(x))^q \, d\mu(x) \lesssim (M+1)2^M + \begin{cases} (N+1)2^{M+(j+N)q}, & q > 1,\\ (j+N)2^{M+j+N}, & q = 1,\\ 2^{M+j+N}, & 0 < q < 1,\end{cases}$$

uniformly in $j, M \in \mathbb{Z}_+$, and $N \in \mathbb{N}_{\leq M}$.

Proof. Using equation (4.10) and then Lemma 4.5, we see that

$$\begin{aligned}&\int_X (N_{2^{-N},\bar{I}} f_{j,M}(x))^q \, d\mu(x)\\ &\quad = \sum_{W \in \mathbb{N}_{\leq j+N}} \int_{\{x \in X : N_{2^{-N},\bar{I}} f_{j,M}(x) \in [2^W, 2^{W+1})\}} (N_{2^{-N},\bar{I}} f_{j,M}(x))^q \, d\mu(x)\\ &\quad \lesssim \sum_{W \in \mathbb{N}_{\leq j+N}} 2^{Wq} \mu(\{x \in X : N_{2^{-N},\bar{I}} f_{j,M}(x) \geq 2^W\})\\ &\quad \lesssim \sum_{W \in \mathbb{N}_{\leq j+N}} 2^{Wq} (2^{M+j+N-W} + (N+1)2^M + (M+1)2^M \mathbb{1}_{\{0\}}(W))\\ &\quad \simeq (M+1)2^M + (N+1)2^{M+(j+N)q} + 2^{M+j+N} \begin{cases} 2^{(j+N)(q-1)}, & q > 1,\\ j+N, & q = 1,\\ 1, & 0 < q < 1.\end{cases}\end{aligned}$$

This gives the desired estimates. □

Now we are able to prove Lemma 4.3.

Proof of Lemma 4.3. At first we deal with the left-hand side of equation (4.5). Let us denote

$$L_{j,M} := \sup_{\lambda > 0} \|\lambda (N_\lambda f_{j,M})^{1/r}\|_{L^{p,\infty}(X)}, \quad j, M \in \mathbb{Z}_+.$$

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By changing the variable $\alpha \mapsto \lambda a^{1/r}$, we obtain

$$\begin{aligned} L_{j,M} &= \sup_{\alpha,\lambda>0} \alpha\mu(\{(2^k,n) : \lambda(N_\lambda f_{j,M}(2^k,n))^{1/r} \geq \alpha\})^{1/p} \\ &= \sup_{a,\lambda>0} \lambda a^{1/r}\mu(\{(2^k,n) : N_\lambda f_{j,M}(2^k,n) \geq a\})^{1/p} \\ &\simeq \sup_{N\in\mathbb{N}} \sup_{a>0} 2^{-N} a^{1/r}\mu(\{(2^k,n) : N_{2^{-N}} f_{j,M}(2^k,n) \geq a\})^{1/p}. \end{aligned}$$

Further, using equations (4.7) and (4.9), we get

$$\begin{aligned} L_{j,M} &\simeq \sup_{N\in\mathbb{N}_{\leq M}} \sup_{a>0} 2^{-N} a^{1/r}\mu(\{(2^k,n) : N_{2^{-N}} f_{j,M}(2^k,n) \geq a\})^{1/p} \\ &= \sup_{N\in\mathbb{N}_{\leq M}} \sup_{l\in\mathbb{N}_{\leq N}} 2^{-N+(j+l)/r}\mu(\{(2^k,n) : N_{2^{-N}} f_{j,M}(2^k,n) \geq 2^{j+l}\})^{1/p} \\ &= \sup_{N\in\mathbb{N}_{\leq M}} \sup_{l\in\mathbb{N}_{\leq N}} 2^{-N+(j+l)/r}(2^M(M+1-l))^{1/p} \\ &= 2^{j/r+M/p}(M+1)^{1/p}. \end{aligned} \tag{4.15}$$

We now focus on the right-hand side of equation (4.5). Let

$$R_{j,M} := \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \|O^\rho_{\bar{I},\infty}(f_{j,M}(\cdot,t) : t\in\mathbb{N})\|_{L^p(X)}, \quad j, M \in \mathbb{Z}_+.$$

Since $|f_{j,M}(x,t) - f_{j,M}(x,s)| \geq 2^{-M}$, $x \in X$, $s,t \in \mathbb{N}$, provided that $f_{j,M}(x,t) \neq f_{j,M}(x,s)$, we obtain

$$\begin{aligned} (O^\rho_{\bar{I},\infty}(f_{j,M}(x,t) : t\in\mathbb{N}))^\rho &= \sum_{N\in[M+1]} \sum_{k=0}^{\infty} \sup_{I_k\leq t<I_{k+1}} |f_{j,M}(x,t) - f_{j,M}(x,I_k)|^\rho \\ &\quad \times \mathbb{1}_{\{\sup_{I_k\leq t<I_{k+1}} |f_{j,M}(x,t)-f_{j,M}(x,I_k)|\in(2^{-N},2^{-N+1}]\}} \\ &\leq \sum_{N\in[M+1]} 2^{-N\rho+\rho} N_{2^{-N},\bar{I}} f_{j,M}(x). \end{aligned}$$

Using equation (4.8), we see that the terms corresponding to $N = M+1$ and $N = M$ are comparable and therefore we get

$$O^\rho_{\bar{I},\infty}(f_{j,M}(x,t) : t\in\mathbb{N}) \lesssim \Big(\sum_{N\in[M]} 2^{-N\rho} N_{2^{-N},\bar{I}} f_{j,M}(x)\Big)^{1/\rho}, \tag{4.16}$$

uniformly in $j, M \in \mathbb{Z}_+$, $x \in X$, and any sequence $\bar{I} \in \mathfrak{S}_\infty(\mathbb{N})$.

In what follows, we distinguish three cases.

Case 1: $p = \rho$. Using equation (4.16) and then Lemma 4.6 (with $q = 1$), we arrive at

$$\begin{aligned} (R_{j,M})^p &\lesssim \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \sum_{N\in[M]} 2^{-N\rho} \int_X N_{2^{-N},\bar{I}} f_{j,M}(x)\, d\mu(x) \\ &\lesssim \sum_{N\in[M]} 2^{-N\rho}((M+1)2^M + (j+N)2^{M+j+N}) \\ &\simeq (M+1)2^M + j2^{M+j}. \end{aligned}$$

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This implies

$$R_{j,M} \lesssim (M+1)^{1/p}2^{M/p} + j^{1/p}2^{(M+j)/p}, \quad j, M \in \mathbb{Z}_+.$$

Consequently, if equation (4.5) were true, then combining the above estimate with equation (4.15), we would have

$$2^{j/r+M/p}(M+1)^{1/p} \lesssim (M+1)^{1/p}2^{M/p} + j^{1/p}2^{(M+j)/p}, \quad j, M \in \mathbb{Z}_+,$$

which is equivalent to

$$2^{j/r} \lesssim 1 + j^{1/p}2^{j/p}(M+1)^{-1/p}, \quad j, M \in \mathbb{Z}_+.$$

Taking $M = 2^{100j}$, we see that

$$2^{j/r} \lesssim 1 + j^{1/p}2^{-99j/p}, \quad j \in \mathbb{Z}_+.$$

Letting $j \to \infty$, we get the contradiction. This finishes the proof of Case 1.

Case 2: $p > \rho$. Using equation (4.16), the triangle inequality for the $L^{p/\rho}(X)$ norm, and Lemma 4.6 (with $q = p/\rho$), we obtain

$$\begin{aligned}
R_{j,M} &\lesssim \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \Big\| \sum_{N\in[M]} 2^{-N\rho} N_{2^{-N},\bar{I}} f_{j,M} \Big\|^{1/\rho}_{L^{p/\rho}(X)} \\
&\le \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \Big(\sum_{N\in[M]} 2^{-N\rho} \| N_{2^{-N},\bar{I}} f_{j,M} \|_{L^{p/\rho}(X)} \Big)^{1/\rho} \\
&\lesssim \Big(\sum_{N\in[M]} 2^{-N\rho} ((M+1)2^M + (N+1)2^{M+(j+N)p/\rho})^{\rho/p} \Big)^{1/\rho} \\
&\simeq ((M+1)^{\rho/p}2^{M\rho/p} + 2^{M\rho/p+j})^{1/\rho} \\
&\simeq (M+1)^{1/p}2^{M/p} + 2^{M/p+j/\rho}.
\end{aligned}$$

Therefore, if equation (4.5) were true, then the above estimate together with equation (4.15) would lead us to the estimate

$$2^{j/r+M/p}(M+1)^{1/p} \lesssim (M+1)^{1/p}2^{M/p} + 2^{M/p+j/\rho}, \quad j, M \in \mathbb{Z}_+,$$

which is equivalent to

$$2^{j/r} \lesssim 1 + 2^{j/\rho}(M+1)^{-1/p}, \quad j, M \in \mathbb{Z}_+.$$

Taking $M = \lfloor 2^{100jp/\rho} \rfloor$, we get

$$2^{j/r} \lesssim 1 + 2^{-99j/\rho}, \quad j \in \mathbb{Z}_+,$$

which leads to the contradiction if we let $j \to \infty$. This finishes the proof of Case 2.

Case 3: $p < \rho$. Using equation (4.16) and then applying the Hölder inequality (here we use equations (4.6) and (4.12) as well) and Lemma 4.6 (with $q = 1$), we infer that

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$$
\begin{aligned}
(R_{j,M})^p &\lesssim \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \int_X \Big(\sum_{N\in[M]} 2^{-N\rho} N_{2^{-N},\bar{I}} f_{j,M}(x) \Big)^{p/\rho} d\mu(x) \\
&\le \sup_{\bar{I}\in\mathfrak{S}_\infty(\mathbb{N})} \Big(\int_X \sum_{N\in[M]} 2^{-N\rho} N_{2^{-N},\bar{I}} f_{j,M}(x)\, d\mu(x) \Big)^{p/\rho} ((M+1)2^M)^{1-p/\rho} \\
&\lesssim ((M+1)2^M)^{1-p/\rho} \Big(\sum_{N\in[M]} 2^{-N\rho} ((M+1)2^M + (j+N)2^{M+j+N}) \Big)^{p/\rho} \\
&\simeq (M+1)2^M + ((M+1)2^M)^{1-p/\rho} (j2^{M+j})^{p/\rho}.
\end{aligned}
$$

This shows that

$$
R_{j,M} \lesssim (M+1)^{1/p} 2^{M/p} + 2^{M/p+j/\rho} j^{1/\rho} (M+1)^{1/p-1/\rho}.
$$

Consequently, if equation (4.5) were true, then combining the above estimate with equation (4.15), we would get

$$
2^{j/r+M/p}(M+1)^{1/p} \lesssim (M+1)^{1/p} 2^{M/p} + 2^{M/p+j/\rho} j^{1/\rho} (M+1)^{1/p-1/\rho}, \quad j, M \in \mathbb{Z}_+,
$$

which is equivalent to

$$
2^{j/r} \lesssim 1 + 2^{j/\rho} j^{1/\rho} (M+1)^{-1/\rho}, \quad j, M \in \mathbb{Z}_+.
$$

Taking $M = 2^{100j}$, we see that

$$
2^{j/r} \lesssim 1 + j^{1/\rho} 2^{-99j/\rho}, \quad j \in \mathbb{Z}_+.
$$

Letting $j \to \infty$, we get the contradiction. This finishes the proof of Case 3. Consequently, the proof of Lemma 4.3 is finished. □

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