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Subdivisions of Simplicial Complexes Preserving the Metric Topology

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Abstract. Let $|K|$ be the metric polyhedron of a simplicial complex K. In this paper, we characterize a simplicial subdivision K' of K preserving the metric topology for $|K|$ as the one such that the set $K'^{(0)}$ of vertices of K' is discrete in $|K|$. We also prove that two such subdivisions of K have such a common subdivision.

1 Introduction

For a simplicial complex K, the polyhedron $|K|$ has two topologies: the Whitehead (weak) topology and the metric topology. In the theory of infinite-dimensional manifolds, polyhedra with metric topology are important because of the triangulation theorem. For instance, let E be a linear metric space with density τ such that E is an absolute retract and homeomorphic to the countable power $E^{\mathbb{N}}$ or its subspace $E^{\mathbb{N}}_f = \{(x_i)_{i\in\mathbb{N}} \mid x_i = 0 \text{ except for finitely many } i\}$. Every manifold modeled on E is homeomorphic to the product of a polyhedron $|K|$ with the metric topology and the model space E, where K is a locally finite-dimensional simplicial complex such that the star at each vertex has at most τ many simplexes [4, Proposition 3.3]. Recently, it was shown that each open set in LF-spaces can be triangulated in the above sense [2, 3].

In this paper, we assign $|K|$ *the metric topology*. The metric topology has the disadvantage of the Whitehead topology in that a subdivision K' of K changes the metric topology in general, that is, $|K'| \neq |K|$ as spaces. A simplicial subdivision is said to be *admissible* if it preserves the metric topology.[1] The barycentric subdivision $\operatorname{Sd} K$ is admissible. D. W. Henderson established the following characterization [1, Lemma V.5] to prove the metric version of Whitehead's theorem on small subdivisions, that is, every simplicial complex has arbitrarily small admissible subdivisions [1, Lemma V.7].

Theorem 1.1 (D. W. Henderson) *A simplicial subdivision K' of K is admissible if and only if the open star $O(v, K')$ at each vertex $v \in K'^{(0)}$ is open in $|K|$.*

In this paper, we give another characterization which can be more easily checked than Theorem 1.1 above.

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[1]This naming comes from the fact that the metric defined by such a subdivision is admissible. Henderson called it a *proper* subdivision [1].

Theorem 1.2 *A simplicial subdivision K' of K is admissible if and only if the set $K'^{(0)}$ of vertices of K' is discrete in $|K|$.*

In the paper [3], this theorem was proved for a *derived* subdivision K' of a *locally finite-dimensional* simplicial complex K. Now any assumption is not necessary. Moreover, we can prove the following.

Theorem 1.3 *Any two admissible subdivisions of K have an admissible common subdivision.*

A homeomorphism $f\colon |K| \to |L|$ is *admissible PL* if f is simplicial with respect to admissible subdivisions K' and L' of K and L, respectively. See §3 for the existence of PL homeomorphisms which are not admissible PL, that is, there is a homeomorphism $f\colon |K| \to |L|$ with respect to the *metric topologies* of $|K|$ and $|L|$ which is simplicial with respect to some subdivisions K' and L' of K and L, respectively, but such subdivisions K' and L' are not admissible. Of course, f is also a homeomorphism with respect to the weak (Whitehead) topologies of $|K|$ and $|L|$. As a corollary of Theorem 1.3, we have the following.

Corollary 1.4 *The composition of admissible PL homeomorphisms is also an admissible PL homeomorphism.*

2 Proofs of Theorems 1.2 and 1.3

Let K be a simplicial complex. By $K^{(0)}$, we denote the 0-skeleton of K, that is, $K^{(0)}$ is the set of all vertices of K. The set of vertices and the interior of a simplex σ are denoted by $\sigma^{(0)}$ and σ°, respectively. When a simplex σ is spanned by vertices $v_0, \dots, v_n$, *i.e.*, $\sigma^{(0)} = \{v_0, \dots, v_n\}$, we write $\sigma = \langle v_0, \dots, v_n\rangle$. The notation $\sigma \leqslant \sigma'$ means that σ is a face of σ'.

For each point $x \in |K|$, let $\sigma_x \in K$ be the carrier of x, that is, $x \in \sigma_x^\circ$. Let $(\beta_v^K(x))_{v\in K^{(0)}} \in \mathbf{I}^{K^{(0)}}$ be the barycentric coordinate, that is, $\sum_{v\in K^{(0)}} \beta_v^K(x) = 1$ and $\{v \in K^{(0)} \mid \beta_v^K(x) > 0\} = \sigma_x^{(0)}$. Then we can write $x = \sum_{v\in K^{(0)}} \beta_v^K(x)v$. The *open star* at $v \in K^{(0)}$ is defined by $O(v, K) = \{x \in |K| \mid \beta_v^K(x) > 0\}$. The metric ρ_K for the polyhedron $|K|$ is defined as follows:

$$\rho_K(x, y) = \sum_{v\in K^{(0)}} |\beta_v^K(x) - \beta_v^K(y)|.$$

Identifying x with $(\beta_v^K(x))_{v\in K^{(0)}} \in \ell_1(K^{(0)})$, we can regard $|K|$ as a subspace of the Banach space $\ell_1(K^{(0)})$. Then the metric ρ_K is induced by the norm of $\ell_1(K^{(0)})$. A simplicial subdivision K' of K is admissible if and only if the metric $\rho_{K'}$ is admissible for $|K|$.

For each $x \in |K|$, define $O(x, K) = \bigcap_{v\in\sigma_x^{(0)}} O(v, K)$. Then $O(x, K)$ is open in $|K|$ with $\mathrm{cl}_{|K|}\, O(x, K) = |\operatorname{St}(\sigma_x, K)|$, where $\operatorname{St}(\sigma, K)$ is the star at $\sigma \in K$ which is the subcomplex of K defined as follows:

$$\operatorname{St}(\sigma, K) = \{\sigma' \in K \mid \exists \sigma'' \in K \text{ such that } \sigma, \sigma' \leqslant \sigma''\}.$$

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For each $x \in |K|$ and $0 < t < 1$, we can define $\varphi_t^x \colon |\operatorname{St}(\sigma_x, K)| \to |\operatorname{St}(\sigma_x, K)|$ by $\varphi_t^x(y) = (1-t)x + ty$, where $\sigma_x \in K$ is the carrier of x. The following is easy.

Lemma 2.1 *For each $x \in |K|$ and $0 < t \leqslant 1$, the image $\varphi_t^x(|\operatorname{St}(\sigma_x, K)|)$ is closed in $|K|$ and $\varphi_t^x(O(x, K))$ is open in $|K|$.*

For $A \subset |K|$, let $C(A, K) = \{\sigma \in K \mid \sigma \cap A = \varnothing\}$. Then $C(A, K)$ is a subcomplex of K. In case $A = \{x\}$, we write $C(x, K)$ instead of $C(\{x\}, K)$. Then $O(x, K) = |K| \setminus |C(x, K)|$. Observe that $K = \operatorname{St}(\sigma, K) \cup C(\sigma^\circ, K)$ for each simplex $\sigma \in K$. In particular, $K = \operatorname{St}(v, K) \cup C(v, K)$ for each vertex $v \in K^{(0)}$. Note that $K \neq \operatorname{St}(\sigma, K) \cup C(\sigma, K)$ in general.

For each $v \in |K| \setminus K^{(0)}$ and $\sigma \in \operatorname{St}(\sigma_v, K) \cap C(v, K)$, let $v\sigma \in K$ be the simplex spanned by $\{v\} \cup \sigma^{(0)}$, that is, $(v\sigma)^{(0)} = \{v\} \cup \sigma^{(0)}$. Then we can define the simplicial subdivision K_v of K as follows:

$$K_v = C(v, K) \cup \{v\sigma \mid \sigma \in \operatorname{St}(\sigma_v, K) \cap C(v, K)\}.$$

Observe that $K_v^{(0)} = \{v\} \cup K^{(0)}$, $C(v, K_v) = C(v, K)$, and $O(v, K_v) = O(v, K)$ for each $v \in |K| \setminus |K^{(0)}|$. The following was proved in [3].

Lemma 2.2 ([3, Lemma 9]) *For each $w \in |K| \setminus K^{(0)}$, K_w is an admissible subdivision of K.*

We shall prove the following lemma.

Lemma 2.3 *Let K' and K'' be simplicial subdivisions of K such that $K'^{(0)}$ and $K''^{(0)}$ are discrete in $|K|$. Then, K' and K'' have a common simplicial subdivision K''' such that $K'''^{(0)}$ is discrete in $|K|$.*

Proof Here we use the following admissible metric on $|K|$ defined as follows:

$$d(x, y) = \sqrt{\sum_{v \in K^{(0)}} \left(\beta_v^K(x) - \beta_v^K(y)\right)^2}.$$

Then each n-simplex $\sigma \in K$ with this metric is isometric to the standard n-simplex of Euclidean space $\mathbb{R}^{n+1}$ and $\operatorname{diam}_d \sigma = \sqrt{2}$ if $n \neq 0$.

The following is a cell complex which is a common subdivision of K' and K'':

$$L = \{\sigma' \cap \sigma'' \mid \sigma' \in K',\ \sigma'' \in K'' \text{ such that } \sigma' \cap \sigma'' \neq \varnothing\}.$$

Since L has a simplicial subdivision K''' such that $K'''^{(0)} = L^{(0)}$, it suffices to show that $L^{(0)}$ is discrete in $|K|$.

Let $x_0 \in |K|$ and let $\sigma_0 \in K$ be the carrier of x_0. Since $L^{(0)} \cap \sigma_0$ is finite and σ_0 is compact, we can find $0 < \delta < 1$ such that

$$B_d(x_0, \delta) \subset O(x_0, K),\ \delta < \operatorname{dist}_d\left(\sigma_0, (K'^{(0)} \cup K''^{(0)}) \setminus \sigma_0\right)$$

$$d(v, x) > \delta \text{ for each distinct two points } v, x \in (L^{(0)} \cap \sigma_0) \cup \{x_0\}.$$

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We show that $d(x_0, v) > \delta^2/4$ for every $v \in L^{(0)} \cap B_d(x_0, \delta) \setminus \sigma_0$. Then we would have $B_d(x_0, \delta^2/4) \cap (L^{(0)} \setminus \{x_0\}) = \varnothing$.

Let $\sigma \in K$ be the carrier of v. Then σ_0 is a proper face of σ, *i.e.*, $\sigma_0 < \sigma$, because $v \in O(x_0, K) \setminus \sigma_0$. Since σ is isometric to the standard simplex of Euclidean space, there exists the nearest point $u \in \sigma_0$ from v, that is, $d(v, u) = \operatorname{dist}_d(v, \sigma_0)$. Then the segment from u to v is upright on σ_0. Since $v \in L^{(0)} \setminus (K'^{(0)} \cup K''^{(0)})$, $\{v\} = \sigma' \cap \sigma''$ for some $\sigma' \in K' \setminus K'^{(0)}$ and $\sigma'' \in K'' \setminus K''^{(0)}$. Then $\sigma_0' = \sigma' \cap \sigma_0 \neq \varnothing$ and $\sigma_0'' = \sigma'' \cap \sigma_0 \neq \varnothing$. Otherwise,

$$\operatorname{dist}_d(x_0, v) \geqslant \operatorname{dist}_d\left(\sigma_0, (K'^{(0)} \cup K''^{(0)}) \setminus \sigma_0\right) \geqslant \delta,$$

which is a contradiction. Let σ_1' and σ_1'' be the faces of σ' and σ'' which are opposite to σ_0' and σ_0'', respectively. In other words, σ_1' and σ_1'' are the simplexes spanned by the vertices σ' and σ'' which do not belong to σ_0' and σ_0'', respectively. Then we can write

$$v = (1 - t')y' + t'z' = (1 - t'')y'' + t''z'',$$

where $y' \in \sigma_0'$, $z' \in \sigma_1'$, $y'' \in \sigma_0''$, $z'' \in \sigma_1''$, and $t', t'' \in (0, 1)$. Since $\sigma' \cap \sigma''$ is a singleton which is not contained in σ_0, we have $\sigma_0' \cap \sigma_0'' = \sigma' \cap \sigma'' \cap \sigma_0 = \varnothing$, hence

$$d(y', u) + d(y'', u) \geqslant d(y', y'') \geqslant \operatorname{dist}_d(\sigma_0', \sigma_0'') = \operatorname{dist}_d((\sigma_0')^{(0)}, (\sigma_0'')^{(0)}) \geqslant \delta.$$

Then $d(y', u) \geqslant \delta/2$ or $d(y'', u) \geqslant \delta/2$. We may assume that $d(y', u) \geqslant \delta/2$.

Similarly to u, let $x' \in \sigma_0$ be the nearest point from z', that is, $d(z', x') = \operatorname{dist}_d(z', \sigma_0) > \delta$, where the segment from x' to z' is upright on σ_0. Since the right triangle $x'y'z'$ is similar to the right triangle $uy'v$ and $d(x', y') \leqslant \operatorname{diam}_d \sigma_0 = \sqrt{2} < 2$, it follows that

$$d(x_0, v) \geqslant d(u, v) = \frac{d(x', z')}{d(x', y')} \cdot d(u, y') > \delta^2/4.$$

This completes the proof. ∎

Now we can prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 Since $K'^{(0)}$ is discrete in $|K'|$, it suffices to show the "if" part. Observe that $\beta_v^K(x) = \sum_{w \in K'^{(0)}} \beta_w^{K'}(x)\beta_v^K(w)$ for each $x \in |K|$. Then it follows that

$$\begin{aligned}\rho_K(x, y) &= \sum_{v \in K^{(0)}} |\beta_v^K(x) - \beta_v^K(y)| \leqslant \sum_{v \in K^{(0)}} \sum_{w \in K'^{(0)}} \beta_v^k(w)|\beta_w^{K'}(x) - \beta_w^{K'}(y)| \\ &= \sum_{w \in K'^{(0)}} |\beta_w^{K'}(x) - \beta_w^{K'}(y)| = \rho_{K'}(x, y),\end{aligned}$$

hence $\mathrm{id}\colon |K'| \to |K|$ is continuous. It remains to show the continuity of $\mathrm{id}\colon |K| \to |K'|$ at each $w \in |K|$. By Lemma 2.3, there is a common subdivision K'' of K_w and K' such that $K''^{(0)}$ is discrete in $|K|$. Then, $\mathrm{id}\colon |K''| \to |K'|$ is continuous. It suffices to show the continuity of $\mathrm{id}\colon |K| = |K_w| \to |K''|$ at w, where $w \in K_w^{(0)}$. Thus, we may assume that $w \in K^{(0)}$.

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For each $x \in |K|$, observe that

$$\begin{aligned}\rho_K(x,w) &= \sum_{v\in K^{(0)}} \left|\beta_v^K(x) - \beta_v^K(w)\right| \\ &= 1 - \beta_w^K(x) + \sum_{v\in K^{(0)}\setminus\{w\}} \beta_v^K(x) = 2(1-\beta_w^K(x)).\end{aligned}$$

By the same reason, we have $\rho_{K'}(x,w) = 2(1-\beta_w^{K'}(x))$.

Let $\delta = \operatorname{dist}_{\rho_K}(w, K'^{(0)} \setminus \{w\}) > 0$. For each $\varepsilon > 0$, we shall show that if $\rho_K(x,w) < \delta\varepsilon/2$, then $\rho_{K'}(x,w) < \varepsilon$. For every $v \in K'^{(0)} \setminus \{w\}$, $\beta_w^K(v) \leqslant 1 - \delta/2$ because $2(1-\beta_w^K(v)) = \rho_K(v,w) \geqslant \delta$. For each $x \in |K|$,

$$\begin{aligned}\beta_w^K(x) = \sum_{v\in K'^{(0)}} \beta_v^{K'}(x)\beta_w^K(v) &\leqslant \beta_w^{K'}(x) + \sum_{v\in K'^{(0)}\setminus\{w\}} \beta_v^{K'}(x)(1-\delta/2) \\ &\leqslant \beta_w^{K'}(x) + (1-\beta_w^{K'}(x))(1-\delta/2) = \delta\beta_w^{K'}(x)/2 + 1 - \delta/2.\end{aligned}$$

Hence, it follows that

$$\rho_{K'}(x,w)/2 = 1 - \beta_w^{K'}(x) \leqslant \frac{2(1-\beta_w^K(x))}{\delta} = \rho_K(x,w)/\delta.$$

Thus, we have $\rho_{K'}(x,w) < \varepsilon$. ∎

Proof of Theorem 1.3 This is a combination of Theorem 1.2 and Lemma 2.3. ∎

3 Remarks

Here we give some remarks and questions. First, we show the existence of PL homeomorphisms which are not admissible PL.

Proposition 3.1 *Let $K = \{v_0, v_i, \langle v_0, v_i\rangle \mid i \in \mathbb{N}\}$ be the countable 1-dimensional simplicial complex, where the metric space $|K|$ is the (countable) hedgehog. Then there exists a homeomorphism $f\colon |K| \to |K|$ with respect to the metric topology of $|K|$ which is PL, but not admissible PL. In fact, f is simplicial with respect to some subdivision K' of K, but K has no admissible subdivisions K'' and K''' such that f is simplicial with respect to K'' and K'''.*

Proof For each $i \in \mathbb{N}$, let $w_i = (1-2^{-i})v_0 + 2^{-i}v_i \in \langle v_0, v_i\rangle$. The following is a non-admissible subdivision of K:

$$K' = \{v_0, w_i, v_i, \langle v_i, w_i\rangle, \langle w_i, v_i\rangle \mid i \in \mathbb{N}\}.$$

Let $f\colon K' \to K'$ be the simplicial isomorphism defined by

$$\begin{gathered} f(0) = 0,\ f(v_{2i-1}) = v_{2i},\ f(w_{2i-1}) = w_{2i}, \\ f(v_{2i}) = v_{2i-1},\ f(w_{2i}) = w_{2i-1} \text{ for each } i \in \mathbb{N}.\end{gathered}$$

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It is easy to see that both $f\colon |K| \to |K|$ and $f^{-1}\colon |K| \to |K|$ are continuous with respect to the metric topology. If f is simplicial with respect to subdivisions K'' and K''' of K, then K'' and K''' must contain w_i, $i \in \mathbb{N}$, as vertices, hence they are not admissible. Thus, f is not admissible. ■

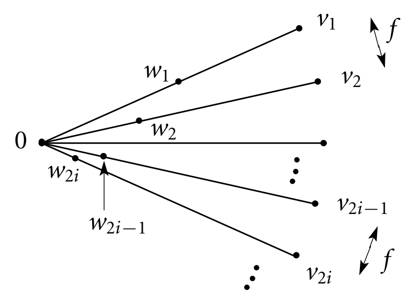

Figure 1: A PL homeomorphism which is not admissible PL

A simplicial complex K is called a *full complex* if every finite set of vertices spans a simplex of K. Recall any derived subdivision K' of K is simplicially isomorphic to the barycentric subdivision $\operatorname{Sd} K$, hence $|K'|$ is homemorphic to $|K|$ $(= |\operatorname{Sd} K|)$ with respect to the metric topology. It should be noticed that the metric topology of $|K'|$ is very different from the one of $|K|$ in general.

Proposition 3.2 *The countable-infinite full complex K has a derived subdivision K' of K such that $K'^{(0)}$ is dense in $|K|$.*

Proof We write $K^{(0)} = \{v_{n,k} \mid n, k \in \mathbb{N}\}$, where $v_{n,k} \neq v_{n',k'}$ if $(n,k) \neq (n',k')$. Since $|K|$ is separable, it has a countable dense subset $D = \{x_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\sigma_n \in K$ be the carrier of x_n and define

$$m(n) = \max\{k \mid v_{n,k} \in \sigma_1^{(0)} \cup \cdots \cup \sigma_n^{(0)}\}.$$

Moreover, for each $k \in \mathbb{N}$, let $\sigma_{n,k} \in \operatorname{St}(\sigma_n, K)$ be the simplex spanned by

$$\sigma_n^{(0)} \cup \{v_{n,m(n)+1}, \dots, v_{n,m(n)+k}\}.$$

Then $x_n \in \sigma_{n,k}$. If $n \neq n'$, then $\sigma_{n,k} \neq \sigma_{n',k'}$ for every $k, k' \in \mathbb{N}$, but it is possible that $\sigma_n = \sigma_{n'}$. We can take points $y_{n,k} \in \sigma_{n,k}^{\circ}$, $n, k \in \mathbb{N}$, so that $y_{n,k} \to x_n$ $(k \to \infty)$ in $|K|$. Then $Y = \{y_{n,k} \mid n, k \in \mathbb{N}\}$ is dense in $|K|$, and it follows that the interior σ° of each $\sigma \in K$ contains at most one point of Y. For each $\sigma \in K$, define $w_\sigma \in \sigma^{\circ}$ as follows:

$$w_\sigma = \begin{cases} y_{n,k} & \text{if } \sigma^{\circ} \cap Y = \{y_{n,k}\} \text{ for some } n, k \in \mathbb{N}, \\ \hat{\sigma} & \text{if } \sigma^{\circ} \cap Y = \varnothing, \end{cases}$$

where $\hat{\sigma}$ is the barycenter of σ. Let K' be the derived subdivision of K defined by these points w_σ, $\sigma \in K$. Then $K'^{(0)}$ is dense in $|K|$ because $Y \subset K'^{(0)}$. ■

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Recall that two simplicial complexes K and L are *combinatorially equivalent* if they have simplicially isomorphic subdivisions. When they have simplicially isomorphic admissible subdivisions (equivalently there exists an admissible PL homeomorphism between K and L), it is said that K and L are *admissible combinatorially equivalent*. By Corollary 1.4, the admissible combinatorial equivalence is an equivalence relation between simplicial complexes. The following is open.

Question 1. When two simplicial complexes K and L are combinatorially equivalent, are they admissible combinatorially equivalent?

Related to the above, we have the following question.

Question 2. Is every simplicial subdivision of K simplicially isomorphic to an admissible subdivision of K?

The following question is also open.

Question 3. Does every simplicial subdivision of K have a simplicial subdivision which is simplicially isomorphic to an admissible subdivision?

Added in Proof.

In the introduction it is mentioned that D. W. Henderson proved the metric version of Whitehead's theorem on small subdivisions [1]. However, his proof is valid only for a locally finite-dimensional simplicial complex. The second author recently gave a complete proof of this result without local finite-dimensionality in the following paper: K. Sakai, *Small subdivisions of simplicial complex with the metric topology*, to appear in J. Math. Soc. Japan.

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