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Exponents, attractors and Hopf decompositions for interval maps

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Abstract. Our main results, specialized to unimodal interval maps T with negative Schwarzian derivative, are the following.

(1) There is a set C_T such that the ω-limit of Lebesgue-a.e. point equals C_T. C_T is a finite union of closed intervals or it coincides with the closure of the critical orbit.

(2) There is a constant λ_T such that $\lambda_T = \overline{\lim}_{n\to\infty} 1/n \log |(T^n)'(x)|$ for Lebesgue-a.e. x.

(3) $\lambda_T > 0$ if and only if T has an absolutely continuous invariant measure of positive entropy.

(4) $\lambda_T \geq \inf\{p^{-1} \log |(T^p)'(z)|: T^p z = z\}$, i.e. uniform hyperbolicity on periodic points implies $\lambda_T > 0$, and $\lambda_T < 0$ implies the existence of a stable periodic orbit.

1. *Introduction and main results*

In Keller (1987) we proved that the canonical Markov extensions of $\mathscr{S}$-unimodal maps (i.e. unimodal maps with negative Schwarzian derivative) are either dissipative or (essentially) conservative and ergodic with respect to Lebesgue measure and that, in the conservative case, they have a finite or σ-finite invariant density. This classification enabled us to show that an $\mathscr{S}$-unimodal map has a finite ergodic invariant density if and only if it has positive upper Lyapunov exponents on a set of positive Lebesgue-measure.

In this paper we attempt to extend this result in two directions: on one side we describe further consequences of the Hopf-decomposition. We prove, for example, that for each $\mathscr{S}$-unimodal map there is a unique compact set C which is the ω-limit of Lebesgue-a.e. trajectory. In the conservative case C is a finite union of intervals, whereas it is the closure of the critical orbit in the dissipative case. This answers a question of Milnor (1985). Our second goal is to clarify which properties of $\mathscr{S}$-unimodal maps are vital to the above results. To this end we prove the existence of a nice Hopf-decomposition for a more abstract class of dynamical systems that we call regular Markov systems. It includes in particular the canonical Markov extensions of multimodal maps with negative Schwarzian derivative (Blokh and

Ljubich (1987) showed that these maps have no homtervals if they have no stable periodic points) In view of recent work of de Melo and van Strien (1986), van Strien (1988), and Nowicki and van Strien (1988) there is some hope that also more general smooth maps (not necessarily with negative Schwarzian derivative) give rise to Markov extensions covered by this result In the following we give an outline of this paper

In § 2 we investigate regular Markov systems let X be a metric space which comes with a finite or countable partition $\mathscr{X}$ We assume that each $D \in \mathscr{X}$ is σ-compact Fix a subset Y of X and a finite or countable partition $\mathscr{Y}$ of Y such that for each $Z \in \mathscr{Y}$ there is $D \in \mathscr{X}$ with $Z \subseteq D$ and assume that $T \; Y \to X$ is such that $T(Z) \in \mathscr{X}$ and $T \; Z \to T(Z)$ is a homeomorphism for all $Z \in \mathscr{Y}$ Writing $Y_1 = Y$, $Y_{n+1} = Y \cap T^{-1} Y_n$ and $Y_\infty = \bigcap_{n \geq 1} Y_n$, one can consider $T^n \; Y_n \to X$ Let

$$\mathscr{Y}_n = \{Z_0 \cap T^{-1} Z_1 \cap \quad \cap T^{-(n-1)} Z_{n-1} \; Z_i \in \mathscr{Y} \; \forall i\}$$

Then $T^n(Z) \in \mathscr{X}$ and $T^n \; Z \to T^n(Z)$ is a homeomorphism for all $Z \in \mathscr{Y}_n$ We call such a system (X, T) a *Markov system*

Next we introduce a Borel-measure m on X In concrete examples this will usually be Lebesgue measure or any other measure naturally associated with the metric structure of X Hence we assume that m gives positive measure to each open set Two minor additional assumptions are that $m(Z) > 0$ for all $Z \in \mathscr{Y}$ and that there is $k > 0$ such that $\mathrm{cl}\,(Z)$ is compact and $m(Z) < \infty$ for all $Z \in \mathscr{Y}_k$ The main assumptions relating T and m are

(1) T is nonsingular with respect to m, i e there is a positive linear contraction $P \; L_m^1 \to L_m^1$ such that $\int_A Pf \, dm = \int_{T^{-1}A} f \, dm$ for all Borel sets $A \subseteq X$ In particular there is a measurable function $g \; X \to \mathbb{R}_+$ such that

$$Pf = \sum_{Z \in \mathscr{Y}} (f \; g) \circ T_Z^{-1}, \quad T_Z = T_{|Z} \tag{1 1}$$

$1/g$ is the 'derivative' of T with respect to m

(2) There is a positive cone $\mathscr{H} \subseteq \mathscr{C}(X)$ containing the functions χ_D, $D \in \mathscr{X}$, such that

$$f \in \mathscr{H}, \; Z \in \mathscr{Y} \Rightarrow P(f\chi_Z) \in \mathscr{H} \tag{1 2}$$

$$\mathscr{H} \text{ is closed in the topology of uniform convergence on compact subsets} \tag{1 3}$$

$$\mathscr{H} - \mathscr{H} \text{ is dense in } L_m^1 \tag{1 4}$$

For all $D \in \mathscr{X}$ and for all compact $K \subseteq D$ the set

$$\{\log f_{|K} \; f \in \mathscr{H}, f \not\equiv 0\} \text{ is equicontinuous} \tag{1 5}$$

As $P^n f = \sum_{Z \in \mathscr{Y}} P(P^{n-1} f \; \chi_Z)$, it follows inductively from (1 2), (1 3) and (1 5) that

$$P^n f \in \mathscr{H} \text{ or } P^n f \equiv \infty \quad \text{for all } n \geq 0 \text{ and } f \in \mathscr{H} \tag{1 6}$$

Observe also that (1 5) implies $f_{|D} \equiv 0$ or $f_{|D} > 0$ for $f \in \mathscr{H}$ and $D \in \mathscr{X}$ A quadruple $(X, T, m, \mathscr{H})$ as above is called a *regular Markov system*

The transformation T induces a combinatorial Markov structure on $\mathscr{Y}$ For $U, V \in \mathscr{Y}$ write

$$U \to V \quad \text{if } V \subseteq TU,$$

$$U \preceq V \quad \text{if there are } U = U_0 \to U_1 \to \quad \to U_n = V,$$

$$U \approx V \quad \text{if } U \preceq V \text{ and } V \preceq U \text{ or if } U = V$$

This yields equivalence classes $[U]$, $[V]$, etc, ordered by $[U] \preceq [V]$ if $U \preceq V$ We call $[U]$ maximal, if $[U] \preceq [V]$ implies $[U] = [V]$

Let $\operatorname{supp}[U] = \{x \in X \quad x \in V \text{ for some } V \approx U\}$ The sets $\operatorname{supp}[U]$ provide a finite or countable partition of X into *irreducible subsets* X_i, $i \in I$ As $TU \in \mathscr{X}$ for $U \in \mathscr{Y}$, each X_i is a union of elements from $\mathscr{X}$ except when $X_i = U$ for some $U \in \mathscr{Y}$ with $U \not\to U$ We call X_i maximal if it is the support of a maximal equivalence class As in the theory of countable state Markov chains (e g Ch 7 of Breiman, 1968), each X_i has a minimal period p_i, and each maximal X_i is a disjoint union of sets $X_{i,1}, \quad, X_{i,p_i}$ which are cyclically permuted by T The $X_{i,j}$ are unions of elements from $\mathscr{X}$ For later use we remark that for each X_i holds

$$f \in \mathscr{H},\ Pf = f \Rightarrow f_{|X_i} \equiv 0 \quad \text{or} \quad f_{|X_i} > 0 \tag{1 7}$$

Finally let $B_i = \bigcup_{n \geq 0} T^{-n} X_i$

In this situation we have

THEOREM 1 *Let $f \in \mathscr{H} \cap L^1_m$ and $S_n f = \sum_{k=0}^{n-1} P^k f$ Fix some $i \in I$ and $x_0 \in X_i$ Let $s_n = S_n f(x_0)$ and suppose that $s_n > 0$ for some n Then $s_n = O(n)$, $\bar{f}(y) = \lim_{n \to \infty} s_n^{-1} S_n f(y)$ exists for all $y \in B_i$, $0 < \bar{f} < \infty$ on X_i, and $\bar{f} \quad \chi_{X_i} \in \mathscr{H}$ Furthermore*

(1) *If (s_n) is bounded, then P is dissipative on X_i*

(2) *If (s_n) is unbounded, then P is conservative on X_i, $m(X_i \setminus Y_\infty) = 0$, and X_i is maximal*

(3) *If $\bigcup_{n \geq 0} T^{-n} x_0$ is dense in X_i and if (s_n) is unbounded, then there is a unique $h_i \in \mathscr{H}$, such that $Ph_i = h_i$, $h_i(x_0) = 1$, $h_i > 0$ on X_i, and $h_i \equiv 0$ on $X \setminus X_i$ h_i has the following properties*

  (a) *$\bar{f} \quad \chi_{X_i}$ is a constant multiple of h_i and $\bar{f} \equiv 0$ on all nonmaximal X_j for each $f \in \mathscr{H} \cap L^1_m$ If $\phi \in L^1_m$, then $\lim_{n \to \infty} s_n^{-1} S_n \phi = h_i \int_{B_i} \phi \, dm / \int_{B_i} f \, dm$ m-a e on X_i As a consequence, the system $(T, h_i\, dm)$ is pointwise dual ergodic*

  (b) *$\int h_i \, dm < \infty$ if and only if $n^{-1} S_n f \to \gamma \quad \bar{f}$ as $n \to \infty$ m-a e on B_i* for some $\gamma > 0$ *In particular the system $(T, h_i\, dm)$ is ergodic If, even more, for each compact $K \subseteq X$ and each $\delta > 0$ there is $k > 0$ such that $K \cap X_{i,j}$ is contained in the δ-neighbourhood of $T^{-kp_i} x_0$ (for that $X_{i,j}$ which contains x_0), then we have also*

  (c) *If $\int h_i \, dm < \infty$ then the measure-preserving system $(T, h_i\, dm)$ is the product of an exact system with a finite rotation*

  (d) *If $\int h_i \, dm = \infty$, then $P^n \psi \to 0$ as $n \to \infty$ uniformly on compact subsets for all $\psi \in \mathscr{H} \cap L^1_m$ with $0 \leq \psi \leq h_i$*

Recall that P is *dissipative* on X_i if $\sum_{n \geq 0} P^n f < \infty$ m-a e for each $f \geq 0$ in L^1_m and that P is *conservative* on X_i if there is some $f \in L^1_m$, positive on X_i, such that $\sum_{n \geq 0} P^n f = \infty$ m-a e on X_i (cf § 3 1 in Krengel, 1985) The measure-preserving system (T, μ) is *exact*, if it has a trivial tail-field, and it follows from Lin (1971) that this is equivalent to $\lim_{n\to\infty} \|P^n f - \int f\,d\mu\| = 0$ for all $f \in L^1_m$. The conservative, ergodic measure-preserving system (T, μ) is *pointwise dual ergodic*, if for the dual operator $T^*: L^1_\mu \to L^1_\mu$ defined by $\int \psi \cdot T^*\phi\,d\mu = \int \psi \circ T \cdot \phi\,d\mu$ $(\psi \in L^\infty_\mu, \phi \in L^1_\mu)$ there is a sequence (a_n) of positive reals such that $\lim_{n\to\infty} a_n^{-1} \sum_{k=0}^{n-1} T^{*k}\phi = \int \phi\,d\mu$ μ-a.e. for all $\phi \in L^1_\mu$ (see Aaronson, 1981).

We pause here for a moment to see how maps with nonpositive Schwarzian derivative fit into the framework of regular Markov systems and to discuss a first application of Theorem 1.

Let U and V be two finite open intervals and suppose that $F: U \to V$ is a $\mathscr{C}^3$-diffeomorphism with nonpositive Schwarzian derivative, i.e.

$$\mathscr{S}F = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 \le 0.$$

We denote by $\mathscr{D}^r(U)$ the set of all positive functions f in $\mathscr{C}^r(U)$ for which $1/\sqrt{f}$ is concave. Misiurewicz (1980) noticed that if $f \in \mathscr{D}^2(U)$, then $(f/|F'|) \circ F^{-1} \in \mathscr{D}^2(V)$, and an approximation argument shows that the same is true with $\mathscr{D}^0(U)$ and $\mathscr{D}^0(V)$ instead of $\mathscr{D}^2(U)$ and $\mathscr{D}^2(V)$. Using another observation of Misiurewicz's, namely that $\mathscr{S}F \le 0$ if and only if $1/\sqrt{|F'|}$ is convex, one can actually prove the same statement assuming only that F is $\mathscr{C}^1$ and $1/\sqrt{|F'|}$ is convex.

Misiurewicz also proved that $\mathscr{D}^0(U)$ is closed in the u.c.s. topology and that $\mathscr{D}^0(U) - \mathscr{D}^0(U)$ is dense in L^1_m. Finally observe that the concavity of $1/\sqrt{f}$ implies

$$\frac{(b-y)^2}{(b-x)^2} \le \frac{f(x)}{f(y)} \le \frac{(y-a)^2}{(x-a)^2}$$

if U has endpoints a and b and if $a < x \le y < b$.

Suppose now we are dealing with a Markov system where X is a finite or countable disjoint union of finite open intervals and where $T: Z \to T(Z)$ has nonpositive Schwarzian derivative for all $Z \in \mathscr{Y}$. Let

$$\mathscr{H} = \{f \in \mathscr{C}(X): f_{|D} \in \mathscr{D}^0(D) \quad \text{for all } D \in \mathscr{X}\} \tag{1.8}$$

In view of the above discussion $(X, T, m, \mathscr{H})$ satisfies (1.1)-(1.5), i.e. it is a regular Markov system. Since it is one-dimensional, we can prove the following remark, which facilitates the application of Theorem 1.

Remark 1. Let $(X, T, m, \mathscr{H})$ be a regular Markov system with nonpositive Schwarzian derivative as described above. If X_i is an irreducible subset on which P is conservative and if $x_0 \in X_{i,j}$, then for all compact $K \subseteq X$ and all $\delta > 0$ there is a $k > 0$ such that $K \cap X_{i,j}$ is contained in the δ-neighbourhood of $T^{-kp_i}\{x_0\}$. In particular T has no homtervals. (Recall that J is a homterval, if it is a nontrivial interval and if $T^n{}_{|J}$ is monotone for all n.)

Proof. Suppose the assertion is wrong. Since P is conservative on X_i, $m(X_i \setminus Y_\infty) = 0$ by Theorem 1. Hence there is a point $x \in K \cap X_{i,j} \cap Y_\infty$ which has an open interval-neighbourhood I such that $x_0 \notin T^{kp_i}(I)$ for infinitely many $k > 0$. Let $U_n(x)$ be that element of $\mathscr{Y}_n$ which contains x, $J = \bigcap_{n>0} U_n(x)$. Since X_i is irreducible, $I \cap J$ is a nontrivial interval, whence J is a homterval. Since it cannot be wandering – this would contradict the conservativity of P on X_i – it must be cyclic, i.e. $T^nJ \subseteq J$ for some $n > 0$ (n minimal with this property). But then J contains a stable (possibly one-sided stable) periodic point of T with period n, which again contradicts the conservativity of P on X_i. (I want to mention here that Blokh and Ljubich (1987) actually proved the non-existence of wandering homtervals for maps with negative Schwarzian derivative.)

Now general interval maps with negative Schwarzian derivative do not come as Markov maps. However, there are several useful ways to derive Markov systems from a given interval map and to study the map using the derived systems.

A rather naive, but yet fruitful approach is the following. Consider $T: [0, 1] \to [0, 1]$ with

$$\mathcal{S}T \leq 0 \text{ and } A_T = \{0, 1\} \cup \{x : T'(x) = 0\} \text{ finite} \tag{1.9}$$

Let

$$K_T = \bigcup_{a \in A_T} \mathrm{cl}\,\{T^n a : n \geq 0\}$$

and $X = [0, 1] \setminus K_T$, $Y = X \cap T^{-1}X$. For $\mathscr{X}$ (resp. $\mathscr{Y}$) we take the partition of X (resp. $\mathscr{Y}$) into maximal open intervals. Obviously $T(Z) \in \mathscr{X}$ and $T: Z \to T(Z)$ is a homeomorphism for all $Z \in \mathscr{Y}$, i.e. (X, T) is a Markov system. The above discussion shows that Theorem 1 applies.

Let X_d be the union of all those X_i on which P is dissipative, X_c the union of those X_i on which P is conservative. Observe that X_d and X_c are open sets and that $[0, 1]$ is the disjoint union of X_d, X_c, and K_T.

If $T_{|X_i}$ is dissipative, then for each compact $L \subseteq X_i$

$$\sum_{n \geq 0} m\{x : T^n x \in L\} = \sum_{n \geq 0} \int_L P^n 1 \, dm = \int_L \lim_{n \to \infty} S_n f \, dm < \infty,$$

the finiteness of the integral being a consequence of (1.5). Hence $\omega(x) \cap \mathrm{int}\,(L) = \varnothing$ for m-a.e. $x \in X$, i.e. $\omega(x) \cap X_d = \varnothing$ for m-a.e. $x \in X$. On the other hand, if $T^n x \in X_d$ for large $n \geq 0$, then $\omega(x) \subseteq \mathrm{cl}\,(X_d) \subseteq X_d \cup K_T$. Hence $\omega(x) \subseteq K_T$ for m-a.e. $x \in \bigcap_{n \geq 0} T^{-n} X_d$.

If $x \in \bigcup_{n \geq 0} T^{-n} K_T$, then $\omega(x) \subseteq K_T$, as $TK_T \subseteq K_T$ and K_T is closed.

Finally we must consider those x, for which $T^n x \in X_c$ for some $n \geq 0$. As X_c is a union of maximal irreducible subsets, for each such x there is a maximal X_i such that $\omega(x) \subseteq \mathrm{cl}\,(X_i)$. We claim that there is even equality for m-a.e. such x. Since T is nonsingular, we may assume $x \in X_i$, and by Remark 1, $T_{|X_i}$ is Lebesgue-ergodic. Hence there is equality, and we have deduced from Theorem 1 the following.

COROLLARY 1. *In the situation just described, for m-a.e. $x \in X$ holds $\omega(x) \subseteq K_T$ or $\omega(x) = \mathrm{cl}\,(X_i)$ for some X_i on which P is conservative.*

More information about the sets $\omega(x)$ can be obtained from another regular Markov system associated with a map $T: [0, 1] \to [0, 1]$ which satisfies (1.9), namely from its *canonical Markov extension* that we introduce in § 3. The dynamics of this extension are so closely related to those of T that Theorem 1, applied to the extension, is the key to the following refinement of Corollary 1 for $\mathcal{S}$-unimodal maps. (These are maps T of $[0,1]$ with $\mathscr{S}T\leq 0$, $T(0)=T(1)=0$, $T'(0)>1$, and just one point $c\in[0,1]$ where $T'(c)=0$)

THEOREM 2 *Suppose T is $\mathscr{S}$-unimodal Then one of the following is true* $\omega(x)=C=$ *(a finite union of compact intervals) for m-*a e $x\in[0,1]$ *or* $\omega(x)=\omega(c)$ *for m-*a e $x\in[0,1]$

This theorem, whose final proof is given in § 4, answers a question of Milnor (1985) A similar result was obtained by Guckenheimer and Johnson (1990) and, as I was told, also by Blokh and/or Lyubich

Note added in proof Blokh and Lyubich published this result in [Ergodic properties of transformations of an interval, *Funct Anal Appl* **23** (1989), 48–49] Full proofs are supplied in preprint 1990/2 of the Institute for Mathematical Sciences at the SUNY Stony Brook Another proof of Theorem 2 is contained in the PhD thesis of M Martens on Interval Dynamics, TU Delft (1990) Both papers contain further interesting results about Cantor attractors

In § 3 we also investigate various growth numbers associated with a map T of $[0,1]$ which satisfies (1 9) Let

$$\bar{\lambda}(x)=\overline{\lim_{n\to\infty}}\frac{1}{n}\log|(T^n)'(x)|, \tag{1 10}$$

$$\bar{I}(x)=\overline{\lim_{n\to\infty}}-\frac{1}{n}\log m(Z_n(x)), \tag{1 11}$$

$$H_{m_i}(\xi_n)=-\sum_{Z\in\xi_n} m_i(Z)\log m_i(Z), \tag{1 12}$$

where $Z_n(x)$ is the maximal monotonicity interval of T^n containing x, ξ_n is the collection of all $Z_n(x)$ for a given n, and m_i is normalized Lebesgue measure on B_i By $\lambda(x)$ and $I(x)$ we denote the corresponding limits of the expressions in (1 10) and (1 11) if they exist

From the classification of the canonical Markov extension given by Theorem 1 we derive

THEOREM 3

(a) *Suppose T satisfies* (1 9) *Then there are a measurable partition* $(B_1, \quad, B_p)$ *of* $[0,1]$ *and constants* $\lambda^+_{T,i}\geq 0$ *such that* $\max\{\bar{\lambda}(x),0\}=\lambda^+_{T,i}$ *and* $I(x)=\lim_{n\to\infty}n^{-1}H_{m_i}(\xi_n)=\lambda^+_{T,i}$ *for m-*a e $x\in B_i$ *For every B_i holds*

  $\lambda^+_{T,i}>0$ *if and only if $T_{|B_i}$ has an absolutely continuous invariant probability measure μ_i of positive entropy In this case μ_i is unique and* $\lambda(x)=I(x)=h_{\mu_i}(T)=\int\log|T'|\,d\mu_i=\lambda^+_{T,i}$ *for m-*a e $x\in B_i$, *and there is $X_i\subseteq B_i$ such that μ_i and m restricted to X_i are equivalent measures and* $B_i=\bigcup_{n\geq 0}T^{-n}X_i$

(b) *If T is unimodal as in Theorem* 2, *then* $p=1$, $B_1=[0,1]$, *and if* $T''(c)\neq 0$ *we can say a bit more There is a constant λ_T such that* $\bar{\lambda}(x)=\lambda_T$ *for m-*a e x *and*

(1) $\lambda_T>0$ *if and only if T has an absolutely continuous invariant probability measure μ of positive entropy μ has all the properties of the measures μ_i in* (a)

(2) $\lambda_T<0$ *if and only if there is a strictly stable periodic orbit* $\{z, Tz, \quad, T^{q-1}z\}$ *In this case* $\lambda(x)=\lambda(z)=\lambda_T$ *for m-*a e x

This theorem has the following

COROLLARY 2 *Each of the following conditions implies the existence of a unique absolutely continuous T-invariant measure of positive entropy on B_i*

(1) $\bar{\lambda}(x) > 0$ *on a subset of B_i of positive Lebesgue measure*
(2) $\bar{I}(x) > 0$ *on a subset of B_i of positive Lebesgue measure*
(3) *There are $C > 0$ and $\alpha < 1$ such that $m(Z) \leq C\alpha^n$ if $Z \in \xi_n$*
(4) *T is $\mathcal{S}$-unimodal with $T''(c) \neq 0$ and there are $C > 0$ and $\beta > 1$ such that $|(T^n)'(z)| \geq C\beta^n$ if $T^n z = z$*

For conditions (1)-(3) this is an immediate consequence of Theorem 3 The fact that condition (4) also implies $\lambda_T > 0$ was observed by Tomasz Nowicki (personal communication) We give its proof in § 3

The reader will have observed that Theorem 3 does not provide any information in the case $\lambda_{T,i} = 0$ It turns out that a great variety of different asymptotic behaviours can occur in this case, and the full range of these possibilities is already displayed by unimodal maps, e g by the quadratic family $T_a(x) = ax(1-x)$ Therefore we restrict our further discussion to $\mathcal{S}$-unimodal maps T of $[0, 1]$, and we write λ_T instead of λ_{T1}

Guckenheimer (1979) classified these maps into three types according to their asymptotic topological behaviour An $\mathcal{S}$-unimodal map T has either

(I) a unique stable periodic orbit $z = T^p z$, or
(II) an invariant zero-dimensional attractor restricted to which T acts like an irreducible rotation on a compact group (generalized adding machine, also called register shift), or
(III) T is sensitive to initial conditions, i e there is $\varepsilon > 0$ such that for every interval $I \subseteq [0, 1]$ there is some $n > 0$ with $m(T^n I) > \varepsilon$

In case I, $\lambda_T \leq 0$, and $\lambda_T = 0$ if and only if $|(T^p)'(z)| = 1$ In case II, $\lambda_T = 0$ by Theorem 3, since T has neither a stable periodic orbit nor an absolutely continuous invariant measure of positive entropy The unique invariant probability measure on the attractor (the Haar-measure of the group rotation) has entropy zero In case III there is no stable periodic orbit, whence $\lambda_T \geq 0$ by Theorem 3 On the other hand this case comprises all maps T with $\lambda_T > 0$ Any hope that all case III maps have $\lambda_T > 0$ was destroyed by a counterexample of Johnson (1986) who constructs a transformation with sensitive dependence to initial conditions but without any absolutely continuous invariant measure and hence with $\lambda_T = 0$

Still one might hope that the following conjecture is true for each $\mathcal{S}$-unimodal T there is a probability measure ν_T on $[0, 1]$ such that $\nu_T = \text{weak-lim}_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} m \circ T^{-k}$ This conjecture is true in the following situations In case I, ν_T is the uniform distribution on the unique stable periodic orbit, in case II, it is the unique invariant probability measure on the attractor, and if $\lambda_T > 0$ in case III, ν_T is the unique absolutely continuous invariant probability measure of positive entropy In all these cases $\int \log |T'| \, d\nu_T = \lambda_T$ For case I this is nearly trivial, for case II this was proved in Keller (1989b), and if $\lambda_T > 0$, this is just the Ergodic Theorem applied to the function $\log |T'|$ In general, however, the above conjecture turns out to be hopelessly wrong if $\lambda_T = 0$ Reinterpreting Johnson's construction for our canonical Markov extensions the following is proved in Hofbauer and Keller (1990)

Remark 2 The family of quadratic maps contains infinitely many examples of maps with sensitive dependence to initial conditions, $\lambda_T = 0$, but without a natural measure There are also examples where $\lambda_T = 0$, a natural measure ν_T exists, but where ν_T has positive entropy

There is also a positive result in this direction intimately related to Theorem 2 For a probability measure ν on [0, 1] let

$$\omega^*(\nu) = \left\{\text{weak accumulation points of } \left(\frac{1}{n}\sum_{k=0}^{n-1} \nu \circ T^{-k}\right)_{n>0}\right\},$$

and for a set M of probability measures on [0, 1] denote by concl (M) the convex closure of M in the weak topology In § 4 we prove

THEOREM 4 *$\omega^*(m) \subseteq$ concl $(\omega^*(\delta_c))$ and $\omega^*(\delta_x) \subseteq$ concl $(\omega^*(\delta_c))$ for Lebesgue-*a e $x \in [0, 1]$, *if T has no finite absolutely continuous invariant measure of positive entropy*

The quantities $\bar{\lambda}(x)$ and $\bar{I}(x)$ measure to some extent the degree of unpredictability of the orbit of x, see Shaw (1981) for a discussion However, $\lambda(x) = I(x) = 0$ for m-a e x does not exclude the possibility that each single trajectory is very complicated It only means that there are so few essentially different trajectories or itineraries that the amount of information (in the sense of Shannon) gained by realizing the first n symbols of a particular itinerary is of the order $o(n)$ Indeed, Remark 2 shows that there are situations where m-a e orbit is generic for the same asymptotic distribution ν_T of positive entropy, and it will turn out in Theorem 5 that at the same time m-a e itinerary is very similar to the itinerary of c

Another measure of the unpredictability of a trajectory – its algorithmic complexity $K(x)$ (with respect to a finite partition of the phase space) – was introduced by Brudno (1982, 1983) It is not based on Shannon's statistical concept of information but on the amount of information needed in order to construct an individual itinerary Roughly speaking, for unimodal maps complexity is defined as follows we fix a universal Turing machine whose alphabet contains the symbols L, R, 0, 1 Following Kolmogorov, the complexity $k(w)$ of a finite word w over the alphabet $\{L, R\}$ is defined to be the length of the shortest word over $\{0, 1\}$ which, given as an input, causes the Turing machine to produce w and nothing else as output For an infinite L, R-sequence $w = w_1 w_2 w_3$ we then define the complexity $K(w) = \overline{\lim}_{n\to\infty} n^{-1} k(w_1 \quad w_n)$ The value of $K(w)$ is actually independent of the particular universal Turing machine chosen for reference, see Brudno (1983) Next, given a unimodal map T on [0, 1] with critical point c, we define the itinerary $w(x) = w_1(x) w_2(x) w_3(x)$ of a point $x \in [0, 1]$ by $w_i(x) = L$ if $T^i x \leq c$ and $w_i(x) = R$ if $T^i x > c$ Finally let $K(x) = K(w(x))$ It is not hard to see that $K(Tx) = K(x)$ For a probability measure ν on [0, 1] define $\bar{I}_\nu(x)$ as in (1 11) but with respect to the measure ν instead of m The two basic relations between $K(x)$ and $\bar{I}_\nu(x)$ are (cf Brudno, 1983)

$$\bar{I}_\nu(x)/\log 2 \le K(x) \quad \text{for } \nu\text{-a e } x, \text{ if } \nu \text{ is a probability measure on } [0,1], \tag{1 13}$$

$$K(x) \le \sup\{h_\nu(T)/\log 2 \quad \nu \in \omega^*(\delta_x)\} \tag{1 14}$$

We sketch the proofs of these inequalities in § 4, where we also prove

THEOREM 5 *Let T be $\mathcal{S}$-unimodal*

(1) *If* $\lambda_T > 0$, *then* $K(x) = \lambda_T/\log 2$ m-a e

(2) *If* $\lambda_T \le 0$, *then* $K(x) \le K(c)$ m-a e *More precisely,* $K(x|w(c)) = 0$ m-a e , *i e if the Turing machine has a second (read only) tape on which* $w(c)$ *is stored and if it can use this information freely, then the length of the shortest* 0, 1-*input which causes the output* $w_1(x)$ $\quad w_n(x)$ *is of the order* $o(n)$

(3) *If* $K(c) > 0$, *then T has sensitive dependence to initial conditions*

As a matter of fact, the construction of Hofbauer and Keller (1990) shows that there are examples of maps in the quadratic family for which $\lambda_T = 0$ but $K(c) > 0$ I have no idea, however, whether $\lambda_T = 0$ implies $K(x) = 0$ for m-a e x

2 *Hopf decomposition and ergodic properties of regular Markov systems*

Let (X, T) be a regular Markov system as described in § 1 with $g \quad X \to \mathbb{R}_+$ as in (1 1) For $n > 0$ set $g_n(x) = g(x)g(Tx) \quad\quad g(T^{n-1}x)$

Proof of Theorem 1 We start by observing the following consequence of (1 5) for each $z \in X$ and each compact neighbourhood N of z there is a constant $c = c(z, N)$ such that

$$c^{-1} \le f(y)/f(z) \le c \quad \text{for all } y \in N \text{ and } 0 \not\equiv f \in \mathcal{H}$$

Let $f \in \mathcal{H} \cap L^1_m$ Then $\int P^k f\, dm \le \int f\, dm < \infty$ and $0 \le P^k f \in \mathcal{H} \cap L^1_m$ by (1 6) for all $k \ge 0$ Hence,

$$P^k f(z) \le c(z, N) \quad m(N)^{-1} \int f\, dm < \infty \quad \text{uniformly in } k \tag{2 1}$$

Observe that $0 \le S_n f = \sum_{k=0}^{n-1} P^k f \in \mathcal{H} \cap L^1_m$ for all $n > 0$ Fix $\iota \in I$ and $x_0 \in Z \in \mathcal{Y}$ for some $Z \subseteq X_\iota$, and consider any $y \in Z$ In view of (1 5), $S_n f(y) = 0$ if and only if $S_n f(x_0) = s_n = 0$ Hence, if $s_n > 0$ for at least one n, then $0 < \overline{\lim}_{n\to\infty} s_n^{-1} S_n f(y) < \infty$ Next consider $z \in T^{-j} y$ for some $j > 0$ and fix a compact neighbourhood N of z By (2 1)

$$\begin{aligned} S_n f(z) &= \sum_{k=0}^{n-j-1} \sum_{u \in T^{-k} z} (f \quad g_k)(u) + \sum_{k=n-j}^{n-1} P^k f(z) \\ &\le \sum_{k=0}^{n-j-1} \sum_{u \in T^{-(k+j)} y} (f \quad g_{k+j})(u)/g_j(z) + \sum_{k=n-j}^{n-1} P^k f(z) \\ &\le S_n f(y)/g_j(z) + j \quad c(z, N) \quad m(N)^{-1} \int f\, dm \end{aligned}$$

Since z can be any point in B_i and since $s_n > 0$ for some n,

$$\bar{f}(z) = \overline{\lim_{n\to\infty}}\, s_n^{-1} S_n f(z) < \infty$$

for all $z \in B_i$. $\bar{f} > 0$ on X_i follows by interchanging the roles of y and z.

As $s_n^{-1} S_n f(x_0) = 1$ for all n, (1.5) implies that the sequence $(s_n^{-1} S_n f) \cdot \chi_{X_i}$ has nontrivial u.c.s. accumulation points, and in view of (1.3) all these accumulation points belong to $\mathscr{H}$.

If (s_n) is bounded, then $\bar{f} = \lim_{n\to\infty} s_n^{-1} S_n f$ pointwise and hence also u.c.s. on X_i, i.e. $\bar{f} \cdot \chi_{X_i} \in \mathscr{H}$. As $\sup_n S_n f \le \sup_n s_n \cdot \bar{f} < \infty$, P is dissipative on X_i in this case.

So assume from now on that (s_n) is unbounded, i.e. P is conservative on X_i. By (2.1), $s_n = O(n)$. Our first remark is that $m(X_i \setminus Y) = m(X_i \setminus T^{-1}X) = 0$, whence X_i is maximal. Let $U = X_i \setminus T^{-1}X$. As T is not defined on U, all sets $T^{-k}U$ $(k \ge 0)$ are pairwise disjoint. Hence $\int_U S_n f\, dm = \sum_{k=0}^{n-1} \int_{T^{-k}U} f\, dm \le \int f\, dm < \infty$ for all n, i.e. $\int_U \sup_n S_n\, dm < \infty$, and since P is conservative on X_i, it follows that $m(U) = 0$. Now $m(X_i \setminus Y_\infty) = 0$, because X_i is maximal and T is nonsingular with respect to m.

Let ϕ be any u.c.s. accumulation point of $(s_n^{-1} S_n f) \cdot \chi_{X_i}$. We saw already that $\phi \in \mathscr{H}$. Now we prove

$$P\phi = \phi \text{ on } X_i \quad \text{if } \phi = \lim_{j\to\infty} (s_{n_j}^{-1} S_{n_j} f) \cdot \chi_{X_i} \tag{2.2}$$

(Observe that if $x \in B_i \setminus X_i$, then x belongs to a nonmaximal X_j, whence $\lim_{j\to\infty} s_{n_j}^{-1} S_{n_j} f(x) = 0$.) We interpret $\rho_x = (g(y) : y \in T^{-1}x)$ for each x as a σ-finite discrete measure on $T^{-1}x$. By Fatou's Lemma and (2.1) we have for $x \in X_i$

$$\begin{aligned} P\phi(x) &= \sum_{y \in T^{-1}x} \left(\lim_{j\to\infty} s_{n_j}^{-1} S_{n_j} f(y) \right) g(y) \\ &\le \underline{\lim}_{j\to\infty}\, s_{n_j}^{-1} \sum_{y \in T^{-1}x} S_{n_j} f(y) \cdot g(y) \\ &= \underline{\lim}_{j\to\infty}\, s_{n_j}^{-1} P S_{n_j} f(x) \\ &= \underline{\lim}_{j\to\infty}\, s_{n_j}^{-1} (S_{n_j} f(x) - f(x) + P^{n_j} f(x)) \\ &= \phi(x) \end{aligned}$$

Hence $P\phi = \phi$ on X_i, since P is conservative on X_i.

Let $\mathscr{F}_i = \{h \in \mathscr{H} \cap L_m^1 : Ph = h, \int h\, dm = 1, \text{ and } h_i \equiv 0 \text{ outside } X_i\}$. We claim

$$\operatorname{card}(\mathscr{F}_i) \le 1 \tag{2.3}$$

In order to prove this, let $\psi \in L_m^1$ be continuous, $\psi \equiv 0$ outside X_i, and suppose that $P\psi = \psi$ and $\int \psi\, dm = 0$. By positivity of P we have $P\psi^{\pm} \ge \psi^{\pm} \ge 0$, and since $\int P\psi^{\pm}\, dm \le \int \psi^{\pm}\, dm$, it follows that $P\psi^+ = \psi^+$ and $P\psi^- = \psi^-$, where ψ^+, ψ^- are continuous. As $\bigcup_{n\ge 0} T^{-n}x_0$ is dense in X_i, this implies $\psi^+ \equiv 0$ on X_i or $\psi^+ > 0$ on

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X_i and the same for ψ^- Hence $\psi \equiv 0$, because $\int \psi \, dm = 0$ and supp $(m) = X$ Now (2 3) follows immediately

We consider the case card $(\mathcal{F}_i) = 1$ first Let h_i be the unique element in $\mathcal{F}_i$ $h_i > 0$ on X_i by (1 7), and a variant of Hopf's ergodic theorem (Theorem 3 3 12 in Krengel, 1985) implies

$$\lim_{n \to \infty} \frac{1}{n} S_n f = \int f \, dm \quad h_i \quad m\text{-a e on } B_i$$

Since the sequence $(n^{-1} S_n f) \quad \chi_{X_i}$ is u c s relatively compact, the convergence is also in the u c s sense on X_i In particular, $s_n / n \to \int f \, dm \quad h_i(x_0)$ and hence

$$\lim_{n \to \infty} \frac{S_n f}{s_n} = \frac{h_i}{h_i(x_0)} = \bar{f} \quad m\text{-a e on } B_i$$

Now the 'only if' part of assertion (b) and the ergodicity of the system $(T, h_i \, dm)$ follow from (1 4)

Suppose next that $\mathcal{F}_i = \emptyset$ Because of (1 5) and (2 2) we still have at least one P-invariant u c s accumulation point ϕ of $s_n^{-1} S_n f) \chi_{X_i}$ (which is not integrable in this case) In view of our assumptions in § 1 we can find $Z \subseteq X_i$, $Z \in \mathcal{Y}_n$ for some $n > 0$ such that cl (Z) is compact and $m(Z) < \infty$ Consider the first return map $T_Z \quad Z \to Z$, $T_Z(x) = T^{n(x)} x$, where $n(x) = \min \{ n > 0 \quad T^n x \in Z \}$ As P is conservative on X_i, $n(x) < \infty$ for m-a e $x \in Z$ and T_Z is m-a e defined It is routine to check that (Z, T_Z) is a Markov system Restricting also m and $\mathcal{H}$ to Z it is not hard to see that one obtains a regular Markov system with associated transfer operator P_Z It is well known that $P\phi = \phi$ implies $P_Z(\phi_{|Z}) = \phi_{|Z}$, and since cl (Z) is compact and $m(Z) < \infty$, $\phi_{|Z}$ is m-integrable Hence the above considerations apply to the system $(T_Z, \phi_{|Z} \, dm)$, and the ergodicity of this system follows Now the system $(T, \phi \quad \chi_{X_i} \, dm)$ must also be ergodic, since $\bigcup_{k \geq 0} T^k Z = X_i$ mod m In particular, $h_i = \phi$ is the unique (up to constant multiples) P-invariant density which does not vanish on X_i It satisfies $Ph_i = h_i$, $h_i > 0$ on X_i and $h_i \equiv 0$ on $X \backslash X_i$ We fix the arbitrary constant factor by requiring $h_i(x_0) = 1$ Then h_i is the only u c s accumulation point of $(s_n^{-1} S_n f) \chi_{X_i}$, i e this sequence converges u c s to $h_i = \bar{f} \quad \chi_{X_i}$ Now the 'if' part of (b) follows from Birkhoff's ergodic theorem, which asserts that $n^{-1} S_n f \to 0$ $h_i \, dm$-a e

For the following considerations let $h_i = Ph_i$, where h_i can be integrable or not If $\phi \in L^1_m$, then

$$s_n^{-1} S_n \phi = s_n^{-1} S_n f \quad (S_n \phi / S_n f) \to h_i \quad \int_{B_i} \phi \, dm \bigg/ \int_{B_i} f \, dm \quad m\text{-a e on } X_i$$

by the Chacon–Ornstein Theorem (see Theorems 3 2 7 and 3 3 4 in Krengel, 1985) Let $d\mu_i = h_i \, dm$ and denote by T_i^* the dual operator of $T \quad L^\infty_{\mu_i} \to L^\infty_{\mu_i}$ Then the pointwise dual ergodicity of (T, μ_i) follows, because $T_i^* \phi = P(\phi h_i)/h_i$, as is easily checked This finishes the proof of (a)

We are left with the proofs of (c) and (d) For $\psi \in L^1_m$, $0 \leq \psi \leq h_i$, let $\tilde{\psi} = \overline{\lim}_{n \to \infty} P^n \psi$ As $P^n \psi \leq P^n h_i \leq h_i$ for all n, we have $\tilde{\psi} \leq h_i$, whence $P^k \tilde{\psi} \leq h_i$ for all k Let $\rho_{x k} = (g_k(y) \quad y \in T^{-k} x)$ As $\sum_{y \in T^{-k} x} h_i(y) g_k(y) = P^k h_i(x) = h_i(x) < \infty$, we have

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$0 \le P^n \psi \le h_i \in L^1_{\rho_i, k}$ for all n and x Hence, by Fatou's Lemma,

$$\begin{aligned} P^k \tilde{\psi}(x) &= \sum_{y \in T^{-k} x} \left(g_k \overline{\lim_{n\to\infty}} P^n \psi \right)(y) \\ &\ge \overline{\lim_{n\to\infty}} \sum_{y \in T^{-k} x} (g_k \, P^n \psi)(y) \\ &= \overline{\lim_{n\to\infty}} P^{n+k} \psi(x) \\ &= \tilde{\psi}(x) \end{aligned} \tag{2 4}$$

Let $f = P\tilde{\psi} - \tilde{\psi}$ Then $f \ge 0$ and $\sum_{k=0}^{\infty} P^k f = \lim_{n\to\infty} P^n \tilde{\psi} - \tilde{\psi} \le h_i < \infty$ As P is conservative on X_i, this implies $f = 0$ m-a e on X_i, i e $P\tilde{\psi} = \tilde{\psi}$ m-a e on X_i In particular, $\tilde{\psi} = c \; h_i$ for some $c \ge 0$

Now suppose additionally that $\psi \in \mathscr{H}$ Fix $\varepsilon > 0$ and $K \subseteq X_i$ compact with $m(K) < \infty$ Because of (1 5) there is $\delta > 0$ such that $|\log f(x) - \log f(y)| < \varepsilon$ whenever $x, y \in K$ with $d(x, y) < \delta$ and $f \in \mathscr{H}, f > 0$ Let $X_{i,1}, \quad , X_{i\,p_i}$ be the cyclic decomposition of X_i There *is* $j \in \{0, \quad , p_i - 1\}$ such that $x_0 \in X_{i\,j}$ By assumption, there is some $N = kp_i > 0$ such that $K \cap X_{i\,j}$ is contained in the δ-neighbourhood of $T^{-N} x_0$ Hence there is some finite subset of $T^{-N} x_0$ the δ-neighbourhood of which contains $K \cap X_{i\,j}$ Choose positive integers r_n such that $\tilde{\psi}(x_0) = \lim_{n\to\infty} P^{r_n + N} \psi(x_0)$ and such that $\lim_{n\to\infty} P^{r_n} \psi(y)$ exists for all $y \in T^{-N} x_0$ Let $\psi^* = \overline{\lim}_{n\to\infty} P^{r_n} \psi$ Then $\psi^* \le \tilde{\psi}$, and, as in (2 4),

$$P^N \tilde{\psi}(x_0) = \tilde{\psi}(x_0) = \overline{\lim_{n\to\infty}} P^N (P^{r_n} \psi)(x_0) \le P^N \psi^*(x_0) \le P^N \tilde{\psi}(x_0),$$

i e $P^N(\tilde{\psi} - \psi^*)(x_0) = 0$, and as $\tilde{\psi} - \psi^* \ge 0$, this implies $\tilde{\psi} = \psi^* = \lim_{n\to\infty} P^{r_n} \psi$ on $T^{-N} x_0$ Hence, for large n,

$$\int_{K \cap X_{i\,j}} \tilde{\psi} \, dm \le e^{3\varepsilon} \int_{K \cap X_{i\,j}} P^{r_n} \psi \, dm \le e^{3\varepsilon} \int \psi \, dm$$

In the limit $\varepsilon \to 0$ and $K \nearrow X$ this yields

$$\int_{X_{i\,j}} \tilde{\psi} \, dm \le \int \psi \, dm \tag{2 5}$$

In particular

$$\frac{c}{p_i} \int h_i \, dm = c \int_{X_{i\,j}} h_i \, dm = \int_{X_{i\,j}} \tilde{\psi} \, dm \le \int \psi \, dm$$

If $\int h_i \, dm = \infty$, this implies $c = 0$, i e $\lim_{n\to\infty} P^n \psi = 0$ pointwise and hence also u c s This proves (d)

If $\int h_i \, dm < \infty$, choose ψ such that $\psi \equiv 0$ outside $X_{i\,j}$ By dominated convergence $\int (P^n \psi - \tilde{\psi})^+ \, dm \to 0$ as $n \to \infty$ Hence

$$\begin{aligned} 0 &\le \int_{X_{i\,j}} |P^{np_i} \psi - \tilde{\psi}| \, dm \\ &= 2 \int_{X_{i\,j}} (P^{np_i} \psi - \tilde{\psi})^+ \, dm - \int_{X_{i\,j}} (P^{np_i} \psi - \tilde{\psi}) \, dm \end{aligned}$$

$$\leq 2\int (P^{np_i}\psi - \tilde{\psi})^+ \, dm + \int_{X_{i,j}} \tilde{\psi}\, dm - \int \psi\, dm$$

$$\leq 2\int (P^{np_i}\psi - \tilde{\psi})^+ \, dm \quad \text{by (2.5)}$$

$$\to 0 \quad \text{as } n\to\infty,$$

which proves in view of (1.4) that $(T^{p_i}, h_i \cdot \chi_{X_{i,j}}\, dm)$ is exact. In order to finish the proof of (c) observe that for each $k=1,\ldots,p_i$ the system $(T^{p_i}, h_i \cdot \chi_{X_{i,(j+k)\bmod p_i}}\, dm)$ is a factor of $(T^{p_i}, h_i \cdot \chi_{X_{i,j}}\, dm)$ via the factor map $T^k: X_{i,j}\to X_{i,(j+k)\bmod p_i}$, whence it is also exact. □

In the remainder of this section we prove some general results relating the existence of an integrable invariant density to certain growth-numbers and an entropy-like quantity of the underlying system. These results are basic for the proof of the more specialized Theorem 3.

For $W\subseteq X$ and $U\in\mathscr{Y}$ let

$$\mathscr{Y}_n[W,U]=\{Z\in\mathscr{Y}_n : Z\subseteq U \text{ and } \exists x\in Z \text{ with } T^jx\in W\ (j=1,\ldots,n-1)\},$$

$$N_n[W]=\sup_{U\in\mathscr{Y}}\operatorname{card}\mathscr{Y}_n[W,U], \quad\text{and} \tag{2.6}$$

$$h^*[T,W]=\overline{\lim_{n\to\infty}}\frac{1}{n}\log N_n[W].$$

Finally, for a T^{-1}-invariant subset Ω of X, let

$$h_\infty(T_{|\Omega})=\inf\{h^*[T,\Omega\backslash K] : K\subseteq X \text{ compact}\}.$$

$h_\infty(T_{|\Omega})$ might be called the topological entropy at infinity of the system $(\Omega, T_{|\Omega})$.

PROPOSITION 1. *Suppose $(X,T,m,\mathscr{H})$ is a regular Markov system with $\sup_{C\in\mathscr{X}} m(C)<\infty$. Let $F_n: X\to(0,\infty)$ be a sequence of measurable functions such that $\Gamma=\sup\{\int_Z F_n\, dm : n>0, Z\in\mathscr{Y}_n\}<\infty$. If $\Omega=T^{-1}\Omega$ is such that*

(i) *P is dissipative on Ω, or*

(ii) *There is no m-integrable P-invariant density on Ω and there is $x_0\in\Omega$ such that $\Omega\subseteq\operatorname{cl}(\bigcup_{n\geq 0}T^{-n}x_0)$,*

then

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log F_n(x)\leq h_\infty(T_{|\Omega}) \quad\text{for } m\text{-a.e. } x\in\Omega.$$

We note the following corollary.

COROLLARY 3. *Suppose $(X,T,M,\mathscr{H})$ is a regular Markov system with $\sup_{C\in\mathscr{X}} m(C)<\infty$ and $\Omega=T^{-1}\Omega$ is a measurable subset of X. If $\limsup_{n\to\infty}-n^{-1}\log m(Z_n(x))>h_\infty(T_{|\Omega})$ or $\limsup_{n\to\infty} n^{-1}\log g_n^{-1}(x)>h_\infty(T_{|\Omega})$ on a subset of Ω of positive m-measure, then T is conservative, and if $\Omega\subseteq\operatorname{cl}(\bigcup_{n\geq 0}T^{-n}x_0)$ for some $x_0\in\Omega$, then there exists a T-invariant probability measure $\mu\ll m$ with $d\mu/dm\in\mathscr{H}$, $\operatorname{supp}(\mu)\subseteq\Omega$, and $h_\mu(T)>0$.*

Proof Let $F_n(x)=1/m(Z_n(x))$ or $F_n(x)=1/g_n(x)$, and observe that $\int_Z g_n^{-1}\,dm = \int P^n(\chi_Z g_n^{-1})\,dm=\int \chi_{T^nZ}\,dm \le \sup_{C\in X} m(C)<\infty$ Now, in view of Proposition 1 and Theorem 1, each of the two conditions implies the assertion of the corollary □

Proof of Proposition 1 We use the notation of Theorem 1 Let $\varepsilon>0$, $\delta>0$ and fix $K\subseteq X$ compact such that $h^*[T,\Omega\backslash K]<h_\infty(T_{|\Omega})+\delta$ Remember that each $f\in\mathcal{H}$ is bounded on K in view of (1 5) Hence, if P is dissipative on X_i, and if $Z\in\mathcal{Y}_k$ has finite m-measure, then $(\sum_{n\ge k} P^n\chi_Z)\ \chi_{X_i}\in\mathcal{H}$ and $\sum_{n\ge k} m(Z\cap T^{-n}(K\cap X_i))=\int_{K\cap X_i}\sum_{n\ge k} P^n\chi_Z\,dm<\infty$, such that $\sum_{n\ge 0}\chi_{K\cap X_i}(T^n x)<\infty$ for m-a e $x\in Z$ Since we assumed that there is some k with $m(Z)<\infty$ for all $Z\in\mathcal{Y}_k$, it follows *a fortiori*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_{K\cap X_i}(T^j x)=0 \quad \text{for } m\text{-a e } x\in X_i \tag{2 7}$$

On the other hand, if P is conservative on X_i, then assumption (ii) of the proposition together with Theorem 1 guarantees the existence of an infinite ergodic absolutely continuous invariant measure μ_i on X_i, and (2 7) follows from Birkhoff's Ergodic Theorem In any case, (2 7) holds for a e $x\in\Omega$ As K is compact, there are only finitely many X_i with $K\cap X_i\ne\emptyset$ Therefore there is $N=N(\varepsilon,\delta)$ such that

$$m(\Omega\backslash A_{N\delta})<\varepsilon, \text{ where } A_{N\delta}=\left\{x\in\Omega\ \ \sum_{j=0}^{n-1}\chi_K(T^j x)<\delta n \text{ for all } n\ge N\right\} \tag{2 8}$$

Let $h_\delta=h^*[T,\Omega\backslash K]+\delta$ Then

$$N_n[\Omega\backslash K]\le e^{h_\delta n} \quad \text{for large } n \tag{2 9}$$

Denote by $S(\delta,n)$ the family of all sets $M\subseteq\{0,\ \ ,n-1\}$ with card $(M)\le\delta n$ Given U and $n\ge N$ we have

$$\{Z\in\mathcal{Y}_n\ \ Z\subseteq U,\ Z\cap A_{N\delta}\ne\emptyset\}\subseteq\bigcup_{M\in S(\delta\ n)} B(M,n,U), \tag{2 10}$$

where $B(M,n,U)=\{Z\in\mathcal{Y}_n\ \ Z\subseteq U$ and $\exists x\in Z$ s th $j\in M\Leftrightarrow T^j x\in K\}$ Fix $M\in S(\delta,n)$ and denote the elements of $M\cup\{0,n\}$ by $0=k_0<k_1<\quad<k_r=n$ Then $r\le\text{card}\,(M)+2\le\delta n+2$ and

$$\begin{aligned}\text{card}\,(B(M,n,U)) &\le \prod_{i=1}^{r} N_{k_i-k_{i-1}}[\Omega\backslash K]\\ &\le \prod_{i=1}^{r} e^{h_\delta(k_i-k_{i-1})} \quad \text{by (2 9)}\\ &= e^{h_\delta n}\end{aligned}$$

Hence, by (2 10),

$$\begin{aligned}\textit{card}\,\{Z\in\mathcal{Y}_n\ \ Z\subseteq U,\ Z\cap A_{N\delta}\ne\emptyset\} &\le \text{card}\,(S(\delta,n))\quad e^{h_\delta n}\\ &\le e^{n(H(\delta)+\delta)}\quad C^2\quad e^{h_\delta n}\end{aligned}$$

for large n, where $H(\delta)=-\delta\log\delta-(1-\delta)\log(1-\delta)$, and we used Stirling's formula to estimate card $(S(\delta,n))$ Now

$$\begin{aligned}\int_{U\cap A_{N\delta}} F_n\,dm &\le \text{card}\,\{Z\in\mathcal{Y}_n\ \ Z\subseteq U,\ Z\cap A_{N\delta}\ne\emptyset\}\quad\Gamma\\ &\le \Gamma(e^{H(\delta)+\delta+h_\delta})^n,\end{aligned}$$

which allows the estimate

$$m(U \cap A_{N,\delta} \cap \{F_n \geq (e^{H(\delta)+2\delta+h_\delta})^n\}) \leq \Gamma e^{-\delta n}$$

Now the Borel-Cantelli Lemma yields

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log F_n(x) \leq H(\delta) + 2\delta + h_\delta$$

$$\leq H(\delta) + 4\delta + h_\infty(T_{|\Omega})$$

for m-a e $x \in U \cap A_{N,\delta}$, and since $U \in \mathscr{U}$ was arbitrary, this holds for m-a e $x \in A_{N,\delta}$ As $m(\Omega \setminus A_{N,\delta}) < \varepsilon$ (see (2 8)), we obtain the assertion of the proposition in the limit $\varepsilon, \delta \to 0$ □

3 *Canonical Markov extensions for interval maps*

Throughout this section let T $[0, 1] \to [0, 1]$ be a *piecewise monotone C^1-map* with a finite number of critical points, i e there are $0 < a_1 < a_2 < \quad < a_{N-1} < 1$ such that $T'(a_i) = 0$ for $i = 1, \quad , N-1$ and $T'(x) \neq 0$ otherwise Let $a_0 = 0$ and $a_N = 1$ Then $T_{|[a_{i-1}, a_i]}$ is a homeomorphism from $[a_{i-1}, a_i]$ to $[Ta_{i-1}, Ta_i]$ ($i = 1, \quad , N$)

In a series of papers, Hofbauer (1979, 1980, 1981a, b, 1986) constructed certain countable state topological Markov chains for such maps (called Markov diagrams), which admit the given system as a topological factor He showed how knowledge about the chains can be turned into knowledge about the asymptotic topological properties of the transformations T Inspired by Hofbauer's construction, we used a variant of the Markov diagrams (called canonical Markov extensions) to study Ruelle-zeta-functions of piecewise analytic interval maps (Keller, 1989a) The main advantage of the extensions over the diagrams is that they are locally smooth with the same degree of smoothness as the underlying transformation T We shall use these canonical Markov extensions (more exactly, a technical variant of them) to construct a regular Markov system very closely related to the given map T

Let ξ be the partition of $[0, 1] \setminus \{a_0, a_1, \quad , a_N\}$ into maximal open intervals, and define $\mathscr{X}$ recursively by

$$(0, 1) \in \mathscr{X} \quad \text{and} \tag{3 1}$$

$$\text{if } D \in \mathscr{X} \text{ and } I \in \xi \text{ with } D \cap I \neq \emptyset, \quad \text{then } T(D \cap I) \in \mathscr{X} \tag{3 2}$$

Let $\hat{X}$ be the disjoint union of intervals from $\mathscr{X}$, formally

$$\hat{X} = \{\hat{x} = (x, D) \quad D \in \mathscr{X} \text{ and } x \in D\}$$

Define π $\hat{X} \to (0, 1)$ and $\pi_{\mathscr{X}}$ $\hat{X} \to \mathscr{X}$ by

$$\pi(x, D) = x, \quad \pi_{\mathscr{X}}(x, D) = D$$

With the discrete metric on $\mathscr{X}$ and the usual distance on $(0, 1)$, $\hat{X}$ becomes in a natural way a metric space, whose subsets $\pi_{\mathscr{X}}^{-1} D$ can be identified with the subsets D of $(0, 1)$ $\pi_{\mathscr{X}}^{-1} D$ must not be confused with $\pi^{-1} D$, however Denote by $\hat{\mathscr{X}}$ the partition of $\hat{X}$ into the sets $\pi_{\mathscr{X}}^{-1} D$

Since the Lebesgue measure m is defined on each $D \in \mathscr{X}$, it carries over immediately to $\hat{X}$, where we denote it by $\hat{m}$ ($\hat{m}(M) = m(\pi M)$ for measurable $M \subseteq \pi_{\mathscr{X}}^{-1} D$) The corresponding σ-algebras of Lebesgue measurable sets are denoted by $\mathscr{B}$ (for m) and $\hat{\mathscr{B}}$ (for $\hat{m}$)

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Let $\hat{Y}=\hat{X}\setminus\pi^{-1}\{a_0, \quad , a_N\}$, and denote by $\hat{\mathcal{Y}}$ the partition of $\hat{Y}$ into maximal open intervals Obviously $\hat{\mathcal{Y}}=\hat{\mathcal{X}}\vee\pi^{-1}\xi$ Indeed, $\hat{\mathcal{Y}}=\bigcup_{k\geq 0}\hat{\mathcal{Y}}^{(k)}$, where $\hat{\mathcal{Y}}^{(0)}=\{\pi_{\mathcal{X}}^{-1}(0,1)\cap\pi^{-1}I \quad I\in\xi\}$ and $\hat{\mathcal{Y}}^{(k+1)}=\hat{\mathcal{Y}}^{(k)}\cup\hat{\mathcal{D}}^{(k+1)}$ with

$$\hat{\mathcal{D}}^{(k+1)}=\{\hat{T}(U)\cap\pi^{-1}I\neq\varnothing \quad U\in\hat{\mathcal{Y}}^{(k)}\setminus\hat{\mathcal{Y}}^{(k-1)}, I\in\xi\}$$

Observe that usually $\hat{\mathcal{D}}^{(k+1)}\cap\hat{\mathcal{Y}}^{(k)}\neq\varnothing$ In fact, it may happen that $\hat{\mathcal{D}}^{(k+1)}\subseteq\hat{\mathcal{Y}}^{(k)}$, in which case $\hat{\mathcal{Y}}$ is finite

The map T lifts to the following transformation $\hat{T} \quad \hat{Y}\to\hat{X}$

$$\hat{T}(x, D)=(Tx, C) \quad \text{where } C=T(D\cap I) \text{ for that } I\in\xi \text{ which contains } x$$

The basic relation between T and $\hat{T}$ is

$$\pi\circ\hat{T}=T\circ\pi \tag{3 3}$$

Let $Z=\pi_{\mathcal{X}}^{-1}D\cap\pi^{-1}I\in\hat{\mathcal{Y}}$, $D\in\mathcal{X}$, $I\in\xi$ Then $T(\pi Z)=T(D\cap I)\in\mathcal{X}$, i e $(\hat{X},\hat{T})$ is a Markov system for the partitions $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$

Suppose $(x, D)\in\hat{Y}_n(=\bigcap_{k=0}^{n-1}\hat{T}^{-k}\hat{Y})$ Then $\hat{T}^k(x, D)\in\hat{Y}$ for $k=0, \quad , n-1$, i e $T^kx\notin\{a_0, \quad , a_N\}$ for $k=0, \quad , n-1$, and it is easily checked that

$$\hat{T}^n(x, D)=(T^nx, T^n(D\cap Z_n(x))), \tag{3 4}$$

where $Z_n(x)$ is the maximal interval in $[0,1]\setminus\bigcup_{k=0}^{n-1}T^{-k}\{a_0, \quad , a_N\}$ that contains x

In particular, if $x\notin\bigcup_{k\geq 0}T^{-k}\{a_0, \quad , a_N\}$ and if $\bigcap_{n\geq 0}Z_n(x)=\{x\}$, then for each pair $\hat{x}_1, \hat{x}_2\in\pi^{-1}x$ there is $n\in\mathbb{N}$ such that $\hat{T}^n\hat{x}_1=\hat{T}^n\hat{x}_2$ Thus, if $\bigcap_{n\geq 0}Z_n(x)=\{x\}$ for all $x\notin\bigcup_{k\geq 0}T^{-k}\{a_0, \quad , a_N\}$ and if $\hat{X}_i$ and $\hat{X}_j$ are irreducible subsets of $\hat{X}$ with $\pi\hat{X}_i\cap\pi\hat{X}_j\neq\varnothing$, then there is an irreducible $\hat{X}_k$ with $\hat{X}_i\leq\hat{X}_k$ and $\hat{X}_j\leq\hat{X}_k$

Another simple consequence of (3 3) and (3 4) is

LEMMA 1 (See Lemma 1 in Keller, 1989b) *Let T and $\hat{T}$ be as above, $A\in\hat{\mathcal{B}}$ All identities are to be read modulo null sets Then*

(a) $\hat{T}^{-1}A=A$ *if and only if* $\pi^{-1}(\pi A)=A$ *and* $T^{-1}(\pi A)=\pi A$

(b) $A\in\bigcap_{n\geq 0}\hat{T}^{-n}\hat{\mathcal{B}}$ *if and only if* $\pi^{-1}(\pi A)=A$ *and* $\pi A\in\bigcap_{n\geq 0}T^{-n}\mathcal{B}$

Quite generally, if $\Phi \quad (X_1, m_1)\to(X_2, m_2)$ is a nonsingular, measurable map between two measure spaces (i e $m_2(A)=0\Rightarrow m_1(\Phi^{-1}A)=0$), we can define the transfer operator $P_\Phi \quad L^1_{m_1}\to L^1_{m_2}$ by

$$\int P_\Phi f \quad g\,dm_2=\int f \quad (g\circ\Phi)\,dm_1 \quad \text{for all } g\in L^\infty_{m_2} \tag{3 5}$$

P_Φ is a positive, linear operator with $\|P_\Phi\|=1$ If X_1 has an at most countable measurable partition such that Φ restricted to each element of the partition is bijective, bimeasurable and nonsingular forwards and backwards, then

$$P_\Phi f(x)=\sum_{y\in\Phi^{-1}x}\frac{f(y)}{\Phi'(y)}, \tag{3 6}$$

where $\Phi'=d(m_2\circ\Phi)/dm_1$ is the Radon–Nikodym derivative of Φ with respect to m_2 and m_1 In particular, P_Φ extends naturally to the space of finite-valued measurable functions if the partition is finite

For $\Phi=T$ or $\Phi=\hat{T}$ we obtain the Perron–Frobenius operators corresponding to T and $\hat{T}$ respectively T' and $\hat{T}'$ are just the absolute values of the usual derivatives,

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and

$$\hat{T}' = T' \circ \pi \quad \text{and} \quad \pi' \equiv 1 \tag{3 7}$$

Obviously

$$P_\pi \circ P_{\hat{T}} = P_{\pi \circ \hat{T}} = P_{T \circ \pi} = P_T \circ P_\pi \tag{3 8}$$

LEMMA 2 *Let T and $\hat{T}$ be as above*

(a) *If P_T is dissipative, then so is $P_{\hat{T}}$*

(b) *If $P_{\hat{T}}$ is conservative on $A \subseteq \hat{X}$, then so is P_T on πA*

(c) *If $P_{\hat{T}}\hat{h} = \hat{h}$ for some $\hat{h} \in L^1_{\hat{m}}$ then $P_T(P_\pi \hat{h}) = P_\pi \hat{h}$, $P_\pi \hat{h} \in L^1_m$, and if Theorem 1 applies to $\hat{T}_1$ then the system $(T, P_\pi \hat{h}\, dm)$ has positive entropy*

Proof (a), (b) and the first two assertions of (c) are immediate consequences of (3 7) and (3 8) Let $d\hat{\mu} = \hat{h}\, d\hat{m}$, $d\mu = P_\pi \hat{h}\, dm$ By Theorem 1(c), $\bigcap_{n\geq 0} \hat{T}^{-n}\hat{\mathscr{B}}$ is finite mod $\hat{\mu}$, and by Lemma 1(b), its cardinality coincides with that of $\bigcap_{n\geq 0} T^{-n}\mathscr{B}$ mod μ Hence (T, μ) has positive entropy □

Suppose now that $\mathscr{S}T \leq 0$ As $\hat{T}' = T' \circ \pi$, this implies $\mathscr{S}\hat{T} \leq 0$, and in view of the discussion after Theorem 1, $(\hat{X}, \hat{T}, \hat{m}, \mathscr{H})$ is a regular Markov system, where $\mathscr{H}$ is defined as in (1 8) In particular, Theorem 1 and Proposition 1 apply to this system

In § 1 we introduced the relation $\rightarrow$ on $\hat{\mathscr{Y}}$, namely $U \rightarrow V$ if $V \subseteq \hat{T}U$ $\hat{\mathscr{Y}}$ together with $\rightarrow$ is a directed graph $\mathscr{G} = (\hat{\mathscr{Y}}, \rightarrow)$, and in order to obtain knowledge about T from information about $\hat{T}$ provided by Theorem 1 and Proposition 1, we must have a closer look at $\mathscr{G}$ We claim

$$\text{If } (c, d) \in \hat{\mathscr{Y}}^{(k)} \quad \text{then } c, d \in (T^j a_i \ \ 0 \leq i \leq N, 0 \leq j \leq k\} \tag{3 9}$$

For $k = 0$ this is true by definition, and if it is true for some $k \geq 0$, then it must be true also for $(c, d) = \hat{T}(U) \cap \pi^{-1}I \in \hat{\mathscr{D}}^{(k+1)}$, where $U \in \hat{\mathscr{Y}}^{(k)}$, $I \in \xi$ This reasoning also shows that there are at most $N+1$ maximal irreducible subsets of $\hat{X}$, one corresponding to each a_i

Proof of Theorem 3(*a*) Let $\hat{X}_1, \ldots, \hat{X}_{p-1}$ $(p \geq 1)$ be those irreducible subsets of $\hat{X}$ on which $P_{\hat{T}}$ is conservative with a unique integrable invariant density $\hat{h}_j$, $\hat{B}_j = \bigcup_{n\geq 0} \hat{T}^{-n}\hat{X}_j$ $(j = 1, \ldots, p-1)$ Let $\hat{B}_p = \hat{X} \setminus \bigcup_{j=1}^{p-1} \hat{B}_j$ Then $P_{\hat{T}}$ has no integrable invariant density on $\hat{B}_p$, and $\hat{T}^{-1}\hat{B}_j = \hat{B}_j$ for $j = 1, \ldots, p$ Set $X_j = \pi(\hat{X}_j)$ and $B_j = \pi(\hat{B}_j)$ $(j = 1, \ldots, p)$, $d\mu_j = P_\pi \hat{h}_j\, dm$ $(j = 1, \ldots, p-1)$ By Lemma 1, the B_j are disjoint, measurable, T-invariant subsets of $(0, 1)$ (modulo m-null sets), and for $j = 1, \ldots, p-1$, $\mu_{j|X_j} \approx m_{|X_j}$ and $B_j = \bigcup_{n\geq 0} T^{-n}\pi\hat{X}_j$ modulo m-null sets

In order to apply Corollary 3 we prove $h_\infty(\hat{T}_{|\hat{B}_j}) \leq h_\infty(\hat{T}) = 0$ for all j The inequality is trivial For the proof of $h_\infty(\hat{T}) = 0$ we need a result of Hofbauer (1986, Corollary 1 to Theorem 9)

Let $N_n[W]$ be as in (2 6), and set $\hat{X}^{(k)} = \bigcup_{U \in \hat{\mathscr{Y}}^{(k)}} U$ Then

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}}\, n^{-1} \log N_n[\hat{X} \setminus \hat{X}^{(k)}] = 0$$

(For an earlier version see Hofbauer, 1979, Lemma 13 A generalization of this result, closer in notation to the present paper, can be found in Keller, 1989b)

Now fix $\varepsilon>0$ Choose $k\in\mathbb{N}$ and $C>0$ such that

$$N_n[\hat{X}\backslash\hat{X}^{(k)}]\le C\quad e^{\varepsilon n}\quad \text{for all } n \tag{3 10}$$

We consider compact subsets K of $\hat{X}$ such that for each of the finitely many $U\in\hat{\mathscr{Y}}^{(k)}$ the set $U\backslash K$ consists of two (small) intervals both having one endpoint with U in common Fix $l\in\mathbb{N}$ Then K can be chosen such that card $\mathscr{Y}_j[\hat{X}\backslash K, U]=2$ for all $U\in\hat{\mathscr{Y}}^{(k)}$ and $j\le l$ Subdividing the integer interval $\{1, \quad , n\}$ into subintervals of length l (the last one may be shorter) and observing (3 10), we obtain the following estimate

$$\begin{aligned}N_n[\hat{X}\backslash K]&\le\left(\max_{j\le l}\left\{N_j[\hat{X}\backslash\hat{X}^{(k)}]\quad\max_{U\in\hat{\mathscr{Y}}^{(k)}}\text{card }\mathscr{Y}_{l-j}[\hat{X}\backslash K, U]\right\}\right)^{1+n/l}\\&\le(2C\,e^{\varepsilon l})^{1+n/l}\end{aligned}$$

Taking logarithms on both sides and dividing by n this yields in the limit $n\to\infty$

$$h^*[\hat{T}, \hat{X}\backslash K]\le\frac{1}{l}\log 2C+\varepsilon$$

In the limit $l\to\infty$ and $\varepsilon\to 0$ we obtain $h_\infty(\hat{T})=0$

Now Corollary 3 applied to $(\hat{X}, \hat{T})$ implies in view of Remark 1

$$\begin{aligned}\bar{\lambda}(x)&=\overline{\lim_{n\to\infty}}\frac{1}{n}\log|(T^n)'(x)|=\overline{\lim_{n\to\infty}}\frac{1}{n}\log|(\hat{T}^n)'(x, (0, 1))|\\&=\overline{\lim_{n\to\infty}}\frac{1}{n}\log\hat{g}_n^{-1}(x, (0, 1))\le 0\quad\text{for } m\text{-a e } x\in B_p,\end{aligned}$$

whereas the ergodic theorem applied to the system $(\hat{T}, \hat{\mu}_j)$, $\hat{\mu}_j=\hat{h}_j\hat{m}$, and to the function $\log|\hat{T}'|$ implies

$$\begin{aligned}\lambda(x)&=\lim_{n\to\infty}\frac{1}{n}\log|(T^n)'(x)|=\lim_{n\to\infty}\frac{1}{n}\log|(\hat{T}^n)'(x, (0, 1))|\\&=\int\log|T'|\circ\pi\quad\hat{h}_j\,d\hat{m}=\int\log|T'|\quad P_\pi\hat{h}_j\,dm\\&=\int\log|T'|\,d\mu_j\quad\text{for } m\text{-a e } x\in B_j\quad(j=0, \quad , p-1)\end{aligned}$$

Hence $\max\{\bar{\lambda}(x), 0\}=0=\lambda^+_{T\,p}$ for m-a e $x\in B_p$, and $\max\{\bar{\lambda}(x), 0\}=\int\log|T'|\,d\mu_j=\lambda^+_{T\,j}$ for m-a e $x\in B_j$, $j=0, \quad , p-1$

$I(x)=0=\lambda^+_{T\,p}$ for m-a e $x\in B_p$ follows from Corollary 3 $I(x)=h_{\mu_j}(T)$ for m-a e $x\in B_j$ $(j=0, \quad , p-1)$ is a consequence of the Shannon-McMillan-Breiman theorem and the martingale theorem, because ξ is a finite generator for T

$$\begin{aligned}h_{\mu_j}(T)=h_{\mu_j}(T, \xi)&=-\lim_{n\to\infty}\frac{1}{n}\log\mu_j(Z_n(x))+\lim_{n\to\infty}\frac{1}{n}\log\frac{d\mu_j}{dm}(x)\\&=\lim_{n\to\infty}\frac{1}{n}\left(-\log\mu_j(Z_n(x))+\log\frac{\mu_j(Z_n(x))}{m(Z_n(x))}\right)\\&=\lim_{n\to\infty}-\frac{1}{n}\log m(Z_n(x))\quad\text{for }\mu\text{-a e } x\in B_j,\end{aligned}$$

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and as $B_j = \bigcup_{n\geq 0} T^{-n}X_j$, the same is true for m-a e $x \in B_j$ The identity $h_{\mu_j}(T) = \int \log |T'|\, d\mu_j$ is the Rohlin formula, and

$$\frac{1}{n} H_{m_j}(\xi_n) = -\frac{1}{n} \sum_{Z \in \xi_n} m_j(Z) \log m_j(Z) = -\int \frac{1}{n} \log m_j(Z_n(x))\, dm_j(x)$$

$$\to \int I(x)\, dm_j(x) = h_{\mu_j}(T, \xi) \quad \text{as } n \to \infty,$$

because the sequence $(n^{-1} \log m_j(Z_n(x)))_{n>0}$ is uniformly integrable and converges m_j-a e to $I(x) \equiv h_{\mu_j}(T, \xi)$, cf Lemma 9 2 6 of Krengel (1985)

To finish the proof of (a) of Theorem 3, we note that $h_{\mu_j}(T, \xi) = h_{\mu_j}(T) > 0$ for $j = 0, \quad , p-1$ The positivity is a consequence of the fact that $\bigcap_{n\geq 0} T^{-n}\mathscr{B}$ is finite mod μ_j (see Lemma 1 and Theorem 1) For the identity $h_{\mu_j}(T, \xi) = h_{\mu_j}(T)$ we must show that ξ is a generator for the system (T, μ_j) Suppose this is not the case Then T has a homterval contained in supp $(\mu_j) = \pi\hat{X}_j$, and *a fortiori* also $\hat{T}$ has a homterval contained in $\hat{X}_j$ But we showed in Remark 1 that this contradicts the conservativity of $P_{\hat{T}}$ on $\hat{X}_j$

Before we can turn to the proof of Theorem 3(b), we need some more information about the graph $\mathscr{G} = (\hat{\mathscr{Y}}, \to)$ for unimodal T The following lemma can be extracted from Hofbauer (1980, § 1) and Hofbauer (1981, end of § 2) Since Hofbauer's notation differs largely from ours, we include its proof

As a notational convenience let (a, b) denote the interval of points between a and b, no matter whether $a < b$ or $b < a$, and define $\acute{x}$ for $x \in (0, 1)$ by $T\acute{x} = Tx$ and $\acute{x} \neq x$

LEMMA 3 *Suppose T is unimodal, and the orbit of c is not eventually periodic Let*

$$c_{-k} = \inf \{x < c \quad (T^k)'(y) \neq 0\ \forall y \in (x, c)\} \quad (k = 1, 2, \quad)$$

(a) *There is a sequence $(\iota_k)_{k\geq 2}$ of integers, $1 \leq \iota_k < k$, such that $\mathscr{X} = \{V_k \quad k \geq 0\}$ where $V_0 = (0, 1)$, $V_1 = (0, Tc)$, and $V_k = (T^k c, T^{\iota_k} c)$ $(k \geq 2)$ T^k maps (c_{-k}, c) diffeomorphically onto V_k $(k \geq 1)$ $\iota_{k+1} = \iota_k + 1$ if $c \notin V_k$ and $\iota_{k+1} = 1$ if $c \in V_k$ If $c \in V_k$, then $T^k(c_{-(k+1)}) = c$ Observe that $\hat{\mathscr{X}}$ is in a natural way isomorphic to $\{\hat{V}_k = V_k \times \{k\} \quad k \geq 0\}$*

(b) *If $c \notin V_k$, let $\hat{D}_k = \hat{V}_k$ $(k \geq 1)$ If $c \in V_k$, let $\hat{D}_k = (T^k c, c) \times \{k\}$ $(k \geq 1)$, $\hat{E}_k = (c, T^{\iota_k} c) \times \{k\}$ $(k \geq 2)$, and $\hat{E}_1 = (0, c) \times \{1\}$ Then $\hat{\mathscr{Y}}^{(0)} = \{(0, c) \times \{0\}, (c, 1) \times \{0\}\}$, and for $k \geq 1$, $\hat{\mathscr{D}}^{(k)} = \{\hat{D}_k\}$ if $c \notin V_k$ and $\hat{\mathscr{D}}^{(k)} = \{\hat{D}_k, \hat{E}_k\}$ if $c \in V_k$*

(c) *Let $0 = R_0 < R_1 < R_2 <$ be the finite or infinite sequence of those nonnegative integers k, for which $c \in V_{k+1}$ There is a map Q $\mathbb{N} \to \mathbb{N}_0$, $Q(j) < j$, such that $R_j - R_{j-1} = 1 + R_{Q(j)}$, and $\mathscr{G}$ has the following four kinds of edges*
   (1) *$\hat{D} \to \hat{E}_1$ and $\hat{D} \to \hat{D}_1$ if $\hat{D} \in \hat{\mathscr{Y}}^{(0)}$ or if $\hat{D} = \hat{E}_1$,*
   (2) *$\hat{D}_k \to \hat{D}_{k+1}$ $(k \geq 1)$,*
   (3) *$\hat{D}_{R_j} \to \hat{E}_{R_j+1}$ for all $j \geq 1$,*
   (4) *$\hat{E}_{R_j+1} \to \hat{D}$ if $\hat{D}_{R_{Q(j)}+1} \to \hat{D}$, $\hat{D} \in \hat{\mathscr{Y}}$ and $j \geq 1$*

(d) *If* $n = R_j + 1$ *and* $\hat{T}^n\hat{x} \in \hat{V}_{R_j+1}$, *then* $\hat{x} \in \hat{V}_0$ *or* $\hat{x} \in \hat{V}_{R_i - R_{Q(i)}}$ *for some* i *with* $Q(i) < j$ *and* $\pi\hat{x} \in (c_{-n}, \acute{c}_{-n})$

Proof (a) $(0, 1) \in \mathscr{X}$ by (3 1) and $(0, Tc) \in \mathscr{X}$ by (3 2) As $c_{-1} = 0$, T maps (c_{-1}, c) diffeomorphically onto $(0, Tc)$ Next, $(T^2c, Tc) \in \mathscr{X}$ by (3 2), and T^2 maps (c_{-2}, c) diffeomorphically onto $V_2 = (T^2c, Tc)$ If $c \in V_2$, then clearly $T^2(c_{-3}) = c$ So let $i_2 = 1$ and suppose there are $i_2, \quad , i_k$ with properties as in (a)

If $c \notin V_k = (T^kc, T^{i_k}c)$, then $V_{k+1} = (T^{k+1}c, T^{i_k+1}c) \in \mathscr{X}$ by (3 2), i e $i_{k+1} = i_k + 1$ Also $(T^{k+1})'(y) \neq 0$ for all $y \in (c_{-k}, c)$, whence $c_{-(k+1)} = c_{-k}$ and T^{k+1} maps $(c_{-(k+1)}, c)$ diffeomorphically onto V_{k+1}

If $c \in V_k$, then $V_{k+1} = (T^{k+1}c, Tc)$ and $(T^{i_k+1}c, Tc)$ are in $\mathscr{X}$ by (3 2), i e $i_{k+1} = 1$ Obviously T^{k+1} maps $(c_{-(k+1)}, c)$ diffeomorphically onto V_{k+1} Observe that $k = R_j + 1$ for some R_j from (c), $j \geq 1$ We must show that $(T^{i_k+1}c, Tc) \in \mathscr{X}$, and in fact we will show a bit more, namely that

$$i_k = R_j - R_{j-1} = R_{Q(j)} + 1 \tag{3 11}$$

for some integer $Q(j)$, $0 \leq Q(j) < j$ Let $m = R_{j-1} + 1$ Then $c \notin V_i$ for $i = m+1, \quad , k-1$, whence $i_k = k - m = R_j - R_{j-1}$ (3 11) follows once we have shown that $c \in V_{i_k} = T^{i_k}(c_{-i_k}, c)$ But suppose this is not the case Then $c_{-(i_k+1)} = c_{-i_k}$ As $T^{m+i_k}(c_{-k}, c) = V_k = (T^{i_k}c, T^kc)$ and as $T^{m+i_k}(c_{-(k+1)}) = c$ by inductive hypothesis, it follows that $T^{m+i_k}(c_{-k}, c_{-(k+1)}) = (T^{i_k}c, c)$, whence $T^m(c_{-k}, c_{-(k+1)}) \subseteq (c_{-(i_k+1)}, c) = (c_{-i_k}, c)$ Hence $V_{i_k} = T^{i_k}(c_{-i_k}, c) \supseteq T^{m+i_k}(c_{-k}, c_{-(k+1)}) = (T^{i_k}c, c)$, a contradiction to $c \notin V_{i_k}$ because the orbit of c is not eventually periodic

In both cases, if $c \in V_{k+1}$, then $T^{k+1}c_{-(k+2)} = c$ This finishes the inductive proof of (a)

(b) is an easy consequence of (a) and of the definition of $\hat{\mathscr{Y}}$, and (c) follows from the proof of (a) and (b) and from (3 11)

For the proof of (d) we use the structure of $\mathscr{G}$ as described in (c) We simply list all possible backwards-paths of length n in $\mathscr{G}$ starting at $\hat{D}_{R_j+1}$ or $\hat{E}_{R_j+1}$ $n = R_j + 1$ steps may either lead straight down to $\hat{V}_0$, or there is some minimal $m \leq n$ such that at the mth step back we arrive at some $\hat{E}_{R_i+1}$ where i is such that $Q(i) < j$ As

$$m = R_j + 1 - (R_{Q(i)} + 1) = R_j - R_{Q(i)},$$

we have in view of $n = R_j + 1$

$$n - m = R_{Q(i)} + 1 = R_i - R_{i-1},$$

such that $\hat{x} \in \hat{V}_{R_i - R_{Q(i)}}$ So we end up with $\hat{x}$ in $\hat{V}_0$ or in $\hat{D}_{R_{i-1}+1}$, cf (3 11) In the first case, $\pi\hat{x} \in (c_{-n}, \acute{c}_{-n})$ by (a) In the second case, $\pi\hat{x} \in (c_{-(R_{Q(i)}+2)}, \acute{c}_{-(R_{Q(i)}+2)})$ as $\hat{T}^{R_{Q(i)}+1}\hat{x} \in \hat{E}_{R_i+1} = (c, T^{R_{Q(i)}+1}c) \times \{R_i + 1\}$, and similarly $\pi\hat{T}^{R_{Q(i)}+1}\hat{x} \in (c_{-(n-R_{Q(i)}-1)}, \acute{c}_{-(n-R_{Q(i)}-1)})$, such that $\pi\hat{x} \in (c_{-n}, \acute{c}_{-n})$ □

Remark 2 If T is unimodal, and if c is eventually periodic, then $\hat{\mathscr{X}}$ and $\hat{\mathscr{Y}}$ are finite The numbers $0 = R_0 < R_1 < R_2 < \quad$ can be defined as before, however

Proof of Theorem 3(*b*) Suppose the orbit of c is not eventually periodic By Lemma 3(c), $\hat{\mathscr{Y}}$ either has an infinite chain of finite irreducible subsets

$$\hat{X}_1 \leq \hat{X}_2 \leq \hat{X}_3 \leq \tag{3 12}$$

or a finite chain of irreducible subsets

$$\hat{X}_1 \le \quad \le \hat{X}_s, \tag{3 13}$$

where all sets in the chain but the last one are finite (By a finite irreducible set we mean an irreducible set which is the union of a finite number of equivalence classes Note also that the numbering $\hat{X}_1, \hat{X}_2,$ has nothing to do with the numbering used at the beginning of the proof of Theorem 3(a))

If the orbit of c is eventually periodic, then $\hat{\mathscr{Y}}$ is finite, and it is easily seen that there is a finite chain of finite irreducible sets as in (3 13)

In case (3 12), $P_{\hat{T}}$ is dissipative on all $\hat{X}_i$, whereas in case (3 13), $P_{\hat{T}}$ is dissipative on $\hat{X}_1, \quad, \hat{X}_{s-1}$, but may be conservative on $\hat{X}_s$, see Theorem 1 Let $\hat{W}$ be a finite union of sets from $\hat{\mathscr{Y}}$ We claim that in any case

$$\hat{m}\left(\bigcap_{n\ge 0} \hat{T}^{-n}\hat{W}\right) = 0 \text{ if } \hat{W} \text{ does not contain a maximal irreducible set} \tag{3 14}$$

In order to prove this claim suppose first that $\hat{W} = \hat{X}_i$ for some nonmaximal $\hat{X}_i$ By construction of $\hat{\mathscr{Y}}$ and by Lemma 3, there are $l \in \mathbb{N}$ and a compact $K \subseteq \hat{W}$ such that $\hat{T}^l(\hat{W}\backslash K) \subseteq \bigcup_{j>i} \hat{X}_j$ As $P_{\hat{T}}$ is dissipative on nonmaximal $\hat{X}_i$, (3 14) follows in this case For general $\hat{W}$ we may now assume w l o g that $\hat{W}$ is contained in a maximal component $\hat{X}_s$ but $\hat{W} \ne \hat{X}_s$ If $P_{\hat{T}}$ is dissipative on $\hat{X}_s$, then a similar reasoning as above applies If $P_{\hat{T}}$ is conservative on $\hat{X}_s$, then $\hat{T}$ is Lebesgue-ergodic on $\hat{X}_s$ by Theorem 1, and (3 14) follows as $\hat{m}(\hat{X}_s \backslash \hat{W}) > 0$

Hence, either $P_{\hat{T}}$ is dissipative on all of $\hat{X}$, or $\hat{m}(\hat{X}\backslash\bigcup_{n\ge 0} \hat{T}^{-n}\hat{X}_s) = 0$ In any case, $p = 1$ in part (a) of the Theorem (cf the definition of p at the beginning of the proof of that part) The rest of (b) is a consequence of the following Lemma, which is a slight variation of Lemma 3 6 of Nowicki (1985)

LEMMA 4 *Suppose T is $\mathscr{S}$-unimodal and $T''(c) \ne 0$ Let*

$$\lambda_\pi = \inf\left\{\frac{1}{n}\log|(T^n)'(y)| \quad y = T^n y,\ n \ge 1\right\}$$

*Then $\bar{\lambda}(x) \ge \lambda_\pi$ for m-*a e *x*

Before we prove the lemma, let us see how it finishes the proof of Theorem 3(b) and how it implies Corollary 2(4)

Observe that

$$\lambda_\pi \le \bar{\lambda}(x) \le \lambda_{T1}^+ \quad \text{for } m\text{-a e } x \tag{3 15}$$

If $\lambda_{T1}^+ > 0$, then $\lambda_T = \lambda_{T1}^+$, and everything was proved in (a) If $\lambda_\pi < 0$, then T has a unique strictly stable periodic orbit $\{z, \quad, T^{q-1}z\}$ which attracts m-a e x, and $\bar{\lambda}(x) = \lambda(x) = \lambda(z) = \lambda_\pi = \lambda_T$ for m-a e x, see Proposition II 5 7 of Collet and Eckmann (1980) The remaining case is where $0 \le \lambda_\pi \le \bar{\lambda}(x) \le \lambda_{T1}^+ \le 0$ But then $\bar{\lambda}(x) = \lambda_\pi = \lambda_{T1}^+ = 0 = \lambda_T$ for m-a e x □

Proof of Corollary 2(4) Under the assumptions of this Corollary, $\bar{\lambda}(x) \ge \lambda_\pi \ge \log\beta > 0$ for m-a e x, whence $\lambda_T > 0$

Proof of Lemma 4 If T has a (possibly one-sided) stable periodic orbit, then $\lambda(x)=\lambda_\pi$ for m-a e x as noted above So suppose that T has no stable periodic orbit For $n\geq 1$ let

$$\mathcal{K}^n=\{x\in(0,1)\quad T^ix\notin(x,\acute{x})\ \forall i=1,\quad ,n-1,\ T^nx\in(x,\acute{x})\}$$

$\mathcal{K}^n$ is an open set, and each connected component of it is of the form $(u,\acute{v})$ with $T^nu=u$ and $T^nv=v$ Moreover, T^n is monotone on every component of $\mathcal{K}^n$, in particular dist $(\mathcal{K}^n,c)>0$ This is Lemma II 5 6 of Collet and Eckmann (1980)

Fix a component $(u,\acute{v})$ of $\mathcal{K}^n$ As $\mathcal{S}T\leq 0$, $|(T^n)'|$ has no positive strict local minimum on $(u,\acute{v})$, i e

$$\inf\{|(T^n)'(x)|\quad x\in(u,\acute{v})\}\geq\min\{|(T^n)'(u)|,|(T^n)'(\acute{v})|\}\tag{3 16}$$

Let $M=\sup\{|T'(x)/T'(\acute{x})|\quad x\in(0,1)\backslash\{c\}\}$ As $T''(c)\neq 0$, we have $1\leq M<\infty$ (cf Lemma 3 4 of Nowicki, 1985) Hence $|(T^n)'(\acute{v})|=|(T^{n-1})'(T\acute{v})|\ \ |T'(\acute{v})|\leq M\ \ |(T^{n-1})'(Tv)|\ \ |T'(v)|=M\ \ |(T^n)'(v)|$, and (3 16) implies

$$\log|(T^n)'(x)|\geq-\log M+n\lambda_\pi\quad\text{for all } x\in\mathcal{K}^n\tag{3 17}$$

Suppose now that $x\in(0,1)$ is such that there are integers $0=n_0<n_1<n_2<\quad$ with

$$(n_{i+1}-n_i)\to\infty\text{ as } i\to\infty\quad\text{and}\quad T^{n_i}x\in\mathcal{K}^{n_{i+1}-n_i}\text{ for all } i\geq 0\tag{3 18}$$

Then

$$\begin{aligned}\log|(T^{n_i})'(x)|&=\sum_{j=1}^{i}\log|(T^{n_j-n_{j-1}})'(T^{n_{j-1}}x)|\\&\geq\sum_{j=1}^{i}(-\log M+(n_j-n_{j-1})\lambda_\pi)\\&=-i\ \ \log M+n_i\lambda_\pi,\end{aligned}$$

whence $\bar{\lambda}(x)\geq\overline{\lim}_{i\to\infty}n_i^{-1}\log|(T^{n_i})'(x)|\geq\lambda_\pi$

So we have to show that (3 18) holds for m-a e x By definition of $\mathcal{K}^n$ and by the fact that dist $(\mathcal{K}^n,c)>0$ for all n, it suffices to show that

$$c\in\omega(x)\quad\text{for } m\text{-a e } x\tag{3 19}$$

One way to realize this is to note that $c\notin\omega(x)$ implies $\bar{\lambda}(x)>0$ (see Theorem II 5 2 of Collet and Eckmann (1980) or Theorem 1 3 of Misiurewicz (1981), which is the original source) Hence, if $m\{x\quad c\notin\omega(x)\}>0$, then $\lambda^+_{T1}>0$ by Theorem 3(a), and $\hat{T}$ is Lebesgue-ergodic on $\hat{X}_\varsigma$ (for $\hat{X}_\varsigma$ see (3 13)) In particular, $\pi\hat{T}^n\hat{x}$ comes arbitrarily close to c for $\hat{m}$-a e $\hat{x}$, i e $c\in\omega(x)$ for m-a e x, a contradiction

Another proof of (3 19), which does not rely on Misiurewicz's theorem, uses Lemma 3(d) $\hat{m}$-a e trajectory is unbounded in the sense that it leaves any finite union $\hat{W}$ of elements of $\hat{\mathcal{X}}$ at some time (This is (3 14)) In particular, for any $n=R_j+1$ and $\hat{m}$-a e $\hat{x}\in(0,1)\times\{0\}$ there is $k\geq n$ such that $\hat{T}^k\hat{x}\in\hat{V}_n$ Thus, by Lemma 3(d), $T^{k-n}x=\pi\hat{T}^{k-n}(x,(0,1))\in(c_{-n},\acute{c}_{-n})$ As $n=R_j+1$ can be arbitrarily large, $c\in\omega(x)$ for m-a e $x\in(0,1)$, i e (3 19) □

4 *Shadowing by the critical orbit*

For the proofs of Theorems 2, 4, and 5 we need some finer information about how typical trajectories of $\mathcal{S}$-unimodal maps (typical in the sense of Lebesgue measure)

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are shadowed by initial pieces of the critical orbit During this whole section T is an $\mathcal{S}$-unimodal map and $\hat{T}$ its canonical Markov extension In order to avoid the distinction between finite and infinite Markov extensions, we also assume that c is not eventually periodic If it is, Theorems 2, 4, and 5 follow easily from the work of Misiurewicz (1981) or can be proved in a straightforward way along the lines of this section

Let $\hat{E} = \bigcup_{j \geq 0} \hat{E}_{R_j+1}$

LEMMA 5 *For $M \in \mathbb{N}$ and $\varepsilon > 0$ there are $\delta > 0$ and a compact set $\hat{K} \subseteq \hat{X}$ such that*

(i) $x, y \in Z \in \xi_i,\ i \leq M,\ |x-y| > \varepsilon \Rightarrow |T^i x - T^i y| > \delta$

(ii) $\hat{x} \in \hat{V}_k \backslash \hat{K},\ k \leq M \Rightarrow \text{dist}\,(\pi\hat{x}, \text{endpoints of } V_k) < \delta$

(iii) $\hat{x} \in \hat{E}_{R_j+1} \backslash \hat{K},\ R_j + 1 \leq M \Rightarrow \hat{T}^i \hat{x} \notin \hat{E}\ (i = 1, \quad , M)$

Proof Given M and ε, there is $\delta > 0$ satisfying (i), because the monotone branches of T are strictly monotone Now the existence of a compact $\hat{K}$ satisfying (ii) and (iii) is obvious, since $c \in V_{R_j+1}$ for all j by definition, and since $\hat{x} \in \hat{E}_{R_j+1} \backslash \hat{K}$ implies that $\hat{T}^i \hat{x}$ is close to the endpoint $T^{R_{Q(j)}+1+i} c$ of $\hat{V}_{R_{Q(j)}+1+i}$ for $i = 1, \quad , M$ □

Next we introduce the following first entrance stopping time For $\hat{x} \in \hat{X}$ let

$$\begin{aligned} \tau(\hat{x}) &= \min\{n \geq 1 \quad \hat{T}^n \hat{x} \in \hat{E}\} \text{ if such an } n \text{ exists,} \\ \tau(\hat{x}) &= \infty \text{ otherwise} \end{aligned} \tag{4 1}$$

Observe that $\tau(\hat{x}) < \infty$ unless $T^n \pi \hat{x} \in \bigcap_{k \geq 1} (c_{-k}, \acute{c}_{-k})$ for some n In particular, if T has no stable periodic orbit, then $\bigcap_{k \geq 1} (c_{-k}, \acute{c}_{-k}) = \{c\}$, and $\tau(\hat{x}) < \infty$ except on the countably many preimages of c Define recursively

$$\tau_1(\hat{x}) = \tau(\hat{x}) \quad \text{and} \quad \tau_{n+1}(\hat{x}) = \tau_n(\hat{x}) + \tau(\hat{T}^{\tau_n(\hat{x})}(\hat{x})) \tag{4 2}$$

Define also numbers $\rho_n(\hat{x})$ by

$$\rho_n(\hat{x}) = j \quad \text{if } \hat{T}^{\tau_n(\hat{x})}(\hat{x}) \in \hat{E}_{R_j+1} \quad (n \geq 1) \tag{4 3}$$

Then

$$R_{\rho_{n+1}(\hat{x})} = R_{Q(\rho_n(\hat{x}))} + \tau(\hat{T}^{\tau_n(\hat{x})}(\hat{x})) \quad \text{and} \quad \rho_{n+1}(\hat{x}) \geq Q(\rho_n(\hat{x})) + 1 \tag{4 4}$$

Finally let

$$\sigma_n(\hat{x}) = \tau_n(\hat{x}) - (R_{\rho_n(\hat{x})} - R_{\rho_n(\hat{x})-1}) = \tau_n(\hat{x}) - R_{Q(\rho_n(\hat{x}))} - 1 \geq \tau_{n-1}(\hat{x}) \tag{4 5}$$

Then, skipping the argument $\hat{x}$, we have

$$d_n = \tau_{n+1} - \sigma_n = \tau(\hat{T}^{\tau_n}\hat{x}) + R_{Q(\rho_n)} + 1 = R_{\rho_{n+1}} + 1 \quad \text{by (4 4),} \tag{4 6}$$

and

$$\hat{T}^{\sigma_n} \hat{x} \in \hat{V}_j \quad \text{where } j = R_i + 1 \text{ and } i = \rho_n - 1 \text{ or } Q(i) = \rho_n - 1 \tag{4 7}$$

Now Lemma 3(d) implies

$$T^{\sigma_n} \pi \hat{x} = \pi \hat{T}^{\sigma_n} \hat{x} \in (c_{-d_n}, \acute{c}_{-d_n}) \tag{4 8}$$

This describes exactly in which sense the critical orbit is shadowing the trajectory of $\hat{x}$ from time σ_n to time τ_{n+1}

The following theorem relates the quality of shadowing of a typical trajectory by initial pieces of the critical orbit to the classification of the Perron–Frobenius operator $P_{\hat{T}}$

THEOREM 6

(a) *$P_{\hat{T}}$ is dissipative if and only if* $\sigma_{n+1}-\sigma_{n-1}\to\infty$ *as* $n\to\infty$ $\hat{m}$-a e

(b) *$P_{\hat{T}}$ is conservative on the maximal irreducible subset of $\hat{X}$ with a nonintegrable invariant density if and only if* $\underline{\lim}_{n\to\infty}(\sigma_{n+1}-\sigma_{n-1})<\infty$ *but* $\sigma_n/n\to\infty$ *as* $n\to\infty$ $\hat{m}$-a e

(c) *$P_{\hat{T}}$ is conservative on the maximal irreducible subset of $\hat{X}$ with an invariant probability density $\hat{h}$ if and only if* $\lim_{n\to\infty}\sigma_n/n$ *is finite* $\hat{m}$-a e *In this case the limit is* $1/\hat{\mu}(\hat{E})$ *where* $\hat{\mu}=\hat{h}\ \hat{m}$

Proof In view of Theorem 1 it is enough to prove the 'only if' implications

(a) For $M\in\mathbb{N}$ choose $\hat{K}\subseteq\hat{X}$ as in Lemma 5 Fix $\hat{x}$ and suppose that $\max\{R_{\rho_n},R_{\rho_{n+1}}\}<M$ for some n By Lemma 5(iii), $\hat{T}^{\tau_n}(\hat{x})\in\hat{K}$ Hence, if $P_{\hat{T}}$ is dissipative, then $\underline{\lim}_{n\to\infty}\max\{R_{\rho_n},R_{\rho_{n+1}}\}\geq M$ $\hat{m}$-a e , and since $M\in\mathbb{N}$ was arbitrary,

$$\max\{R_{\rho_n},R_{\rho_{n+1}}\}\to\infty \text{ as } n\to\infty \quad \text{for } \hat{m}\text{-a e } \hat{x} \tag{4 9}$$

By (4 5) and (4 6),

$$\begin{aligned}\sigma_{n+1}-\sigma_n&=\tau_{n+1}-R_{\rho_{n+1}}+R_{\rho_{n+1}-1}-\tau_{n+1}+R_{\rho_{n+1}}+1\\&=R_{\rho_{n+1}-1}+1\end{aligned} \tag{4 10}$$

Hence (4 9) implies

$$\begin{aligned}\sigma_{n+1}-\sigma_{n-1}&=R_{\rho_{n+1}-1}+R_{\rho_n}-1+2\\&\geq\tfrac{1}{2}(R_{\rho_{n+1}}+R_{\rho_n})\to\infty \text{ as } n\to\infty \quad \text{for } \hat{m}\text{-a e } \hat{x}\end{aligned}$$

(b) If $P_{\hat{T}}$ is conservative on the maximal irreducible subset of $\hat{X}$ then $\hat{T}$ is Lebesgue-ergodic on this set (see Theorem 1(3)), and there is some $j>0$ such that for $\hat{m}$-a e $\hat{x}$ holds $\rho_n(\hat{x})=j$ and $\rho_{n+1}(\hat{x})=Q(j)+1$ for infinitely many n, i e $\underline{\lim}_{n\to\infty}\max\{R_{\rho_n},R_{\rho_{n+1}}\}<\infty$

Next observe that $\sigma_n/n\to\infty$ will follow from $\tau_n/n\to\infty$ So fix $M\in\mathbb{N}$ and choose $\hat{K}\subseteq\hat{X}$ as in Lemma 5 Let $\hat{K}_M=\bigcup_{j=0}^{M}\hat{T}^{-j}\hat{K}$ Birkhoff's ergodic theorem implies

$$\lim_{n\to\infty}\frac{1}{\tau_n}\sum_{i=1}^{\tau_n}\chi_{\hat{K}_M}(\hat{T}^i\hat{x})=0 \quad \text{for } \hat{m}\text{-a e } \hat{x},$$

such that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{n}{\tau_n}&=\overline{\lim_{n\to\infty}}\frac{1}{\tau_n}\sum_{i=1}^{\tau_n}\chi_{\hat{E}}(\hat{T}^i\hat{x})\\&=\overline{\lim_{n\to\infty}}\frac{1}{\tau_n}\sum_{i=1}^{\tau_n}\chi_{\hat{E}\setminus\hat{K}_M}(\hat{T}^i\hat{x}) \quad \text{for } \hat{m}\text{-a e } \hat{x}\end{aligned} \tag{4 11}$$

By definition, $\hat{T}^i\hat{x}\in\hat{E}$ if and only if $i=\tau_k$ for some k So we consider triplets $\hat{u}=\hat{T}^{\tau_{k-1}}\hat{x}$, $\hat{v}=\hat{T}^{\tau_k}\hat{x}$, and $\hat{w}=\hat{T}^{\tau_{k+1}}\hat{x}$, and we assume that $\tau(\hat{u})+\tau(\hat{v})<M$ By (4 4) and Lemma 3(c), $R_{\rho_{k+1}}=R_{Q(\rho_k)}+\tau(\hat{v})=R_{\rho_k}-R_{\rho_k-1}-1+\tau(\hat{v})<R_{\rho_k}-R_{Q(\rho_{k-1})}+\tau(\hat{v})=\tau(\hat{u})+\tau(\hat{v})<M$ Hence, by Lemma 5(iii),

$$\hat{w}\in\hat{K} \quad \text{or} \quad \tau(\hat{w})>M$$

But if $\hat{w} \in \hat{K}$ and $\tau(\hat{u}) + \tau(\hat{v}) < M$, then $\hat{u}, \hat{v}, \hat{w} \in \hat{K}_M = \bigcup_{j=0}^{M} \hat{T}^{-j}\hat{K}$ Therefore, if a triplet $\hat{u}, \hat{v}, \hat{w}$ contributes to the sum $\sum_{i=1}^{\tau_n} \chi_{\hat{E} \setminus \hat{K}_M}(\hat{T}^i\hat{x})$, then $\tau(\hat{u}) + \tau(\hat{v}) + \tau(\hat{w}) \geq M$, whence

$$\overline{\lim_{n\to\infty}} \frac{n}{\tau_n} \leq \frac{3}{M} \quad \hat{m}\text{-a e}$$

by (4 11) As $M \in \mathbb{N}$ was arbitrary, this finishes the proof of (b)

(c) If $P_{\hat{T}}$ has an invariant probability density $\hat{h}$ on the maximal irreducible component of $\hat{X}$, then $(\hat{T}, \hat{\mu})$ is ergodic ($\hat{\mu} = \hat{h}\hat{m}$), and

$$\lim_{n\to\infty} \frac{n}{\tau_n} = \lim_{n\to\infty} \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} \chi_{\hat{E}}(\hat{T}^i\hat{x}) = \hat{\mu}(\hat{E}) > 0 \quad \text{for } \hat{m}\text{-a e } \hat{x}$$

by Birkhoff's ergodic theorem The observation that $\tau_{n-1} \leq \sigma_n \leq \tau_n$ (see (4 5)) finishes the proof □

Proof of Theorem 2 In view of Theorem 3(b) and its proof we must consider the two cases that $P_{\hat{T}}$ is conservative on some maximal $\hat{X}_s$ and that $P_{\hat{T}}$ is dissipative on all of $\hat{X}$ In the first case, the trajectory of $\hat{m}$-a e $\hat{x} \in \hat{X}$ finally enters $\hat{X}_s$ and follows in the sequel the regime of the Lebesgue-ergodic $\hat{T}_{|\hat{X}_s}$ In particular, $\hat{m}$-a e trajectory is dense in $\hat{X}_s$, whence $\omega(x) = \mathrm{cl}\,(\pi\hat{X}_s)$ for m-a e $x \in (0, 1)$ (observe (3 3)) As $\hat{X}_s$ is maximal, there is $n > 0$ such that $\hat{X}_s = \bigcup_{k\geq n} \hat{V}_k = \bigcup_{k\geq 0} \hat{T}^k\hat{V}_n$ By ergodicity of $\hat{T}$ on $\hat{X}_s$, $\hat{V}_n \cap \hat{T}^k\hat{V}_n \neq \varnothing$ for some $k > 0$, whence $V_n \cap T^kV_n \neq \varnothing$ This shows that $\pi\hat{X}_s = \bigcup_{k\geq 0} T^kV_n$ is a finite union of intervals

Now consider the case where $P_{\hat{T}}$ is dissipative If T has a stable periodic orbit, then $\omega(x)$ coincides with this orbit for m-a.e x, and nothing remains to show (Proposition II 5 7 of Collet and Eckmann, 1980) Hence, we may assume that the preimages of c are dense in (0, 1), see Corollary II 5 5 of Collet and Eckmann, 1980) In particular,

$$\gamma_n = \max\{\mathrm{diam}\,(Z_n(x)) \;\; x \in (0, 1)\} \to 0 \quad \text{as } n \to \infty \tag{4 12}$$

In view of (3 14), $\hat{m}$-a e trajectory in $\hat{X}$ is unbounded, i e $\sup_{n>0} R_{\rho_n(\hat{x})} = \infty$ for $\hat{m}$-a e $\hat{x}$ Fix such an $\hat{x}$ As $d_n = R_{\rho_{n+1}} + 1$ is unbounded, it follows from (4 8) that $\{T^nc \;\; n \geq 0\} \subseteq \omega(\pi\hat{x})$, whence

$$K_T = \mathrm{cl}\,\{T^nc \;\; n \geq 0\} \subseteq \omega(x) \quad \text{for } m\text{-a e } x \tag{4 13}$$

For the converse inclusion consider the sets

$$\begin{aligned} I_k = I_k(\hat{x}) &= \{j \geq 0 \;\; \exists n \geq 0 \text{ s th } \pi\hat{T}^j\hat{x} \in Z_k(T^nc)\} \\ &\subseteq \{j \geq 0 \;\; \mathrm{dist}\,(T^j(\pi\hat{x}), K_T) \leq \gamma_k\} \end{aligned}$$

and

$$\begin{aligned} I_k^n = I_k^n(\hat{x}) &= \{\sigma_n, \sigma_n + 1, \quad , \sigma_{n+1} + R_{Q(\rho_{n+1})} - k\} \\ &= \{\sigma_n, \sigma_n + 1, \quad , \tau_{n+1} - 1 - k\} \end{aligned}$$

In view of (4 8), $I_k^n \subseteq I_k$ for all n

For all $k \in \mathbb{N}$ and $\hat{m}$-a e $\hat{x}$, $\bigcup_{n\geq 1} I_k^n$ covers all of $\mathbb{N}$ except of some finite initial segment and of the sets

$$J_k^n = \{\sigma_{n+1} + R_{Q(\rho_{n+1})} + 1 - k, \quad , \sigma_{n+1} - 1\} \quad (n \geq 1).$$

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Fix $M\in\mathbb{N}$, $\varepsilon>0$, and choose $\hat{K}\subseteq\hat{X}$ and $\delta>0$ as in Lemma 5 If $j\in\bigcup_{n\geq 1} I_M^n\subseteq I_M$, then $\operatorname{dist}(\pi\hat{T}^j\hat{x}, K_T)\leq\gamma_M$ As $P_{\hat{T}}$ is dissipative, there is $l_0=l_0(\hat{x})\in\mathbb{N}$ such that $\hat{T}^l\hat{x}\in\hat{X}\backslash\hat{K}$ for $l\geq l_0$ If $J_M^n\neq\emptyset$ for some n so large that $\tau_{n+1}\geq l_0$, then $q=R_{Q(\rho_{n+1})}+2\leq M$ and $\hat{y}=\hat{T}^{\tau_{n+1}+1}\hat{x}\in\hat{V}_q$, see Lemma 3(c)(4) Now Lemma 5(ii) implies

$$\operatorname{dist}(\pi\hat{y}, \text{endpoints of } V_q)<\delta,$$

i e , in view of Lemma 3(a),

$$\text{either (a) } |\pi\hat{y}-Tc|<\delta, \quad \text{or (b) } |\pi\hat{y}-T^q c|<\delta$$

Fix $j\in J_M^n$ In both cases, Lemma 5(i) implies that there is $z\in\bigcup_{r=1}^M T^{-r}\{c\}$ such that $|T^j(\pi\hat{x})-z|<\varepsilon$ This is obvious in case (a), and it follows in case (b) upon observing that $\tau_{n+1}+1-q=\tau_{n+1}-1-R_{Q(\rho_{n+1})}=\sigma_{n+1}$ by (4 5)

Putting everything together, we see that

$$\omega(\pi\hat{x})\subseteq\{y \ \operatorname{dist}(y, K_T)\leq\gamma_M\}\cup U(\varepsilon, M)$$

where $U(\varepsilon, M)$ denotes the ε-neighbourhood of $\bigcup_{r=1}^M T^{-r}\{c\}$ In the limit $\varepsilon\to 0$ (for fixed M) this yields

$$\omega(\pi\hat{x})\subseteq\{y \ \operatorname{dist}(y, K_T)\leq\gamma_M\}\cup\bigcup_{r=1}^{M} T^{-r}\{c\},$$

and in view of (4 13) and (4 12) we have in the limit $M\to\infty$

$$K_T\subseteq\omega(x)\subseteq K_T\cup\bigcup_{r=1}^{\infty} T^{-r}\{c\} \quad \text{for } m\text{-a e } x$$

But by Corollary 1 from § 1, $\omega(x)\subseteq K_T$ or $\omega(x)$ is the closure of an open set for m-a e x, whence $\omega(x)=K_T$ m-a e □

Proof of Theorem 4 If T has no absolutely continuous invariant measure of positive entropy, then $P_{\hat{T}}$ has no invariant probability density by Lemma 2(c) Hence, if for $\hat{x}\in\hat{X}$ and $N\in\mathbb{N}$ we let $n(N)\in\mathbb{N}$ be such that $\sigma_{n(N)}\leq N<\sigma_{n(N)+1}$, then

$$\frac{n(N)}{N}\leq\frac{n(N)}{\sigma_{n(N)}}\to 0 \quad \text{as } N\to\infty \quad \hat{m}\text{-a e}$$

by Theorem 6 Denote $\mu_n=n^{-1}\sum_{i=0}^{n-1}\delta_{T^ic}$ If we set $\sigma_0=0$, then

$$\left(\frac{1}{N}\sum_{i=0}^{N-1}\delta_{T^ix}-\left(\sum_{k=1}^{n(N)}\frac{\sigma_k-\sigma_{k-1}}{N}\mu_{\sigma_k-\sigma_{k-1}}+\frac{N-\sigma_{n(N)}}{N}\mu_{N-\sigma_{n(N)}}\right)\right)(\psi)\to 0$$

as $N\to\infty$ for each $\psi\in C([0,1])$ by (4 8), and some routine arguments involving the weak compactness of $\omega^*(\delta_c)$ show that $\omega^*(\delta_x)\subseteq\operatorname{concl}(\omega^*(\delta_c))$ for m-a e x

As

$$\frac{1}{n}\sum_{i=0}^{n-1} m\circ T^{-i}=\int\left(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^ix}\right)dm(x),$$

similar routine arguments show now that $\omega^*(m)\subseteq\operatorname{concl}(\omega^*(\delta_c))$

Sketch of proof of (1 13) For $\alpha>0$, $\varepsilon>0$ and large $n\in\mathbb{N}$ there are not more than $2^{n(\alpha+\varepsilon)}$ different L, R-strings of finite length with complexity $\leq n(\alpha+\varepsilon)$ Hence the total measure of the points x with $\log\nu(Z_n(x))/\log 2>\alpha+2\varepsilon$ and

$k(w_1(x), \ldots, w_n(x)) \leq n(\alpha+\varepsilon)$ is, for large n, bounded by $2^{-n\varepsilon}$, and the Borel-Cantelli lemma yields $\bar{I}_\nu(x)/\log 2 \leq \alpha + 2\varepsilon$ for ν-a e x with $K(x) \leq \alpha$ Let $\varepsilon \to 0$ and observe that $\alpha > 0$ was arbitrary □

Sketch of proof of (1 14) Let $K(x) = \lim_{j\to\infty} n_j^{-1} k(w_1(x), \ldots, w_{n_j}(x))$, and assume w l o g that $n_j^{-1} \sum_{i=0}^{n_j-1} \delta_{T^i x} \to \nu$ weakly as $j \to \infty$ The distribution of blocks of length l in $w_1(x), \ldots, w_{n_j}(x)$ is, for large l, close to the distribution of these blocks under ν Fix a prefix-code from the blocks of length l to $\{0, 1\}^*$ with average length close to $l \cdot h_\nu(T)/\log 2$ Making l larger, the average length per block size l can be made arbitrarily close to $h_\nu(T)/\log 2$ using some standard coding techniques This yields a coding of $w_1(x), \ldots, w_{n_j}(x)$, which leads to $K(x) \leq h_\nu(T)/\log 2$ □

Proof of Theorem 5

(1) If $\lambda_T > 0$, then T has the unique absolutely continuous invariant probability measure μ, and $K(x) = h_\mu(T)/\log 2$ for m-a e x follows from (1 13), (1 14) and from Theorem 3

(2) If $\lambda_T \leq 0$, then T has no absolutely continuous invariant probability measure of positive entropy, whence $P_{\hat{T}}$ has no invariant probability density (see Lemma 2(c)), and $n/\sigma_n \to 0$ m-a e by Theorem 6 Let $n(N)$ be as in the proof of Theorem 4, i e $\sigma_{n(N)} \leq N < \sigma_{n(N)+1}$ In view of the shadowing property (4 8), the first N digits $w_1(x), \ldots, w_N(x)$ of the itinerary of x can be recovered from the numbers $\sigma_1, \sigma_2 - \sigma_1, \ldots, \sigma_{n(N)} - \sigma_{n(N)-1}$ and N provided the itinerary of c is given as additional information There is a prefix-code over the alphabet $\{0, 1\}$ associating to each positive integer n a codeword of length at most $2(1+\log_2 n)$ (actually $(1+\varepsilon) \times (1+\log_2 n)$ is possible) Hence, for fixed $M \in \mathbb{N}$ and with $\sigma_0 = 0$,

$$\begin{aligned}
&\frac{1}{N} k(w_1, \ldots, w_N \,|\, \text{itinerary of } c) \\
&\leq \frac{2}{N}\left(\sum_{i=1}^{n(N)} (1+\log_2(\sigma_i - \sigma_{i-1})) + 1 + \log_2 N\right) \\
&\leq \frac{2}{N}\left(\sum_{i=1}^{n(N)} \left(1+\log_2 M + \frac{\sigma_i - \sigma_{i-1}}{M}\log_2 e\right) + 1 + \log_2 N\right) \\
&\leq 2\left((1+\log_2 M)\frac{n(N)}{N} + \log_2 e \cdot \frac{\sigma_{n(N)}}{MN} + \frac{1+\log_2 N}{N}\right) \\
&\leq 2\left((1+\log_2 M)\frac{n(N)}{\sigma_{n(N)}} + \frac{\log_2 e}{M} + \frac{1+\log_2 N}{N}\right) \\
&\to \frac{2\log_2 e}{M} \quad \text{as } N \to \infty \text{ for } m\text{-a e } x
\end{aligned}$$

As $M \in \mathrm{N}$ was arbitrary, this proves the claim

(3) If T does not have sensitive dependence, we are in case I or II of the Guckenheimer classification In case I, $T^n c$ tends to a stable periodic orbit, whence the itinerary of c is eventually periodic, and in particular $K(c) = 0$ In case II, $K(c) = 0$ can be deduced e g from the infinite *-product structure of the kneading sequence (the itinerary of c) Another possibility is to apply (1 14), which says that

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$K(c) \leq h_{\nu_T}(T) = 0$, where $\{\nu_T\} = \omega^*(\delta_c)$ is the unique invariant measure on the attractor □

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