BULL. AUSTRAL. MATH. SOC. 47B99
VOL. 21 (1980), 125-130. (46J15)

NON-COMPACT COMPOSITION OPERATORS

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In this note sufficient conditions for non-compactness of composition operators on two different functional Hilbert spaces have been obtained.

1. Introduction

Let $H(X)$ denote a functional Hilbert space on a set X and let T be a mapping from X into itself. Then define the composition transformation C_T on $H(X)$ into the vector space of all complex valued functions on X as

$$C_T f = f \circ T \quad \text{for every} \quad f \in H(X) .$$

If the range of C_T is a subspace of $H(X)$, then by the closed graph theorem C_T is a bounded operator on $H(X)$ and we call it a composition operator induced by T. In [5] we have studied these operators on $H^2(\pi^+)$, the Hilbert space of functions f holomorphic in π^+ (the upper half plane) for which

$$\|f\| = \sup_{y>0} \left\{ \left(\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \right)^{\frac{1}{2}} \right\} < \infty .$$

In this note we are interested in studying non-compact composition operators on $H^2(\pi^+)$ and $H^2(D)$, the classical Hardy space of functions f holomorphic on the unit disc D for which

Received 13 August 1979.

https://doi.org/10.1017/S0004972700011345 Published online by Cambridge University Press

$$\sup_{0<r<1} \left\{(2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right\} < \infty .$$

The Banach algebra of all bounded linear operators on $H(X)$ into itself is denoted by $B(H(X))$.

2. Non-compact composition operators on $H^2(\pi^+)$

If T is a holomorphic function from π^+ into itself such that the only singularity that T can have is a pole at infinity, then it has been shown in [5] that C_T is a bounded operator on $H^2(\pi^+)$ if and only if the point at infinity is a pole of T . In this section we shall give sufficient conditions for the non-compactness of C_T on $H^2(\pi^+)$. A study of compact composition operators on another interesting functional Hilbert space has been made by Swanton [6]. Our results and techniques are entirely different from those of Swanton [6]. At this stage we need the following result of Nordgren [2].

LEMMA 2.1. *A sequence in a functional Hilbert space is a weak null sequence if and only if it is norm bounded and pointwise null.*

We now proceed towards the main results of this section.

THEOREM 2.2. *Let* $T : \pi^+ \to \pi^+$ *be a holomorphic function such that* $C_T \in B(H^2(\pi^+))$. *If* $T_*(x) = \lim_{y\to 0} T(x+iy)$ *exists almost everywhere on* R *(the real line) and* $T_*(x) \in R$ *for* $x \in R$ *, then* C_T *is not compact.*

Proof. Consider the functions s_n defined by

$$s_n(w) = (1/\sqrt{\pi})\left((w-i)^n/(w+i)^{n+1}\right) , \quad n = 0, 1, 2, \ldots .$$

Clearly $s_n \to 0$ pointwise and the sequence $\{s_n\}$ is norm bounded. Since $H^2(\pi^+)$ is a functional Hilbert space, by Lemma 2.1, $\{s_n\}$ is a weak null sequence in $H^2(\pi^+)$. Since for every n ,

$$\|C_T s_n\|^2 = \int_{-\infty}^{\infty} |(s_n \circ T)_*(x)|^2 dx$$

$$= (\pi)^{-1} \int_{-\infty}^{\infty} |(T_*(x)-i)^n/(T_*(x)+i)^{n+1}|^2 dx$$

$$= (\pi)^{-1} \int_{-\infty}^{\infty} [1+(T_*(x))^2]^{-1} dx ,$$

it follows that $\{\|C_T s_n\|\}$ is bounded away from zero. Hence $\{C_T s_n\}$ does not converge to the zero function strongly. This shows that C_T is not compact. Thus the proof is complete.

THEOREM 2.3. *Let T be a holomorphic function from π^+ into itself such that $C_T \in B(H^2(\pi^+))$. If there exists an $M > 0$ such that $|(i+nT(w))/(i+nw)| \le M$ for every $w \in \pi^+$ and $n \in N$, then T induces a non-compact composition operator on $H^2(\pi^+)$.*

Proof. For each $n \in N$ define f_n by

$$f_n(w) = n^{-\frac{1}{2}}(in^{-1}+w)^{-1} .$$

A simple computation shows that the sequence $\{f_n\}$ is a norm bounded sequence in $H^2(\pi^+)$ with

$$\|f_n\|^2 = \pi \quad \text{for every } n \in N .$$

Also $\{f_n\}$ is a pointwise null sequence in $H^2(\pi^+)$ and again since $H^2(\pi^+)$ is a functional Hilbert space, by an application of Lemma 2.1 we can conclude that $\{f_n\}$ is a weak null sequence in $H^2(\pi^+)$. To complete the proof it is enough to show that $\{\|C_T f_n\|\}$ is bounded away from zero. In this regard we have

$$|(C_T f_n)(x+iy)|^2 = |n^{-\frac{1}{2}}(in^{-1}+T(w))^{-1}|^2 \quad \text{where } w = x + iy ,$$

$$= n^{-1}|(in^{-1}+T(w))^{-1}(in^{-1}+w)(in^{-1}+w)^{-1}|^2$$

$$\ge M^{-2}|f_n(x+iy)|^2 .$$

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Therefore,

$$\int_{-\infty}^{\infty} |(C_T f_n)(x+iy)|^2 dx \geq M^{-2} \int_{-\infty}^{\infty} |f_n(x+iy)|^2 dx .$$

Hence

$$\|C_T f_n\|^2 \geq M^{-2}\pi ,$$

which completes the proof of the theorem.

3. Non-compact composition operators on $H^2(D)$

In this section we restrict ourselves to those holomorphic functions T from π^+ into itself whose only singularity is the pole at the point at infinity. If T is a holomorphic function from π^+ into π^+, then we define a holomorphic function t of D into itself by $t(z) = (L^{-1} \circ T \circ L)(z)$, where L is the linear fractional transformation from D onto π^+ defined by $L(z) = i(1+z)/(1-z)$ with L^{-1} defined by $L^{-1}(w) = (w-i)/(w+i)$. Next we need a result of [5] which we put in the form of a lemma.

LEMMA 3.1. *If C_T is a bounded operator on $H^2(\pi^+)$, then the multiplication operator M_β induced by the function $\beta(z) = (1-t(z))/(1-z)$ is a bounded operator on $H^2(D)$, where $t = L^{-1} \circ T \circ L$.*

THEOREM 3.2. *If $C_T \in B(H^2(\pi^+))$, then t induces a non-compact composition operator on $H^2(D)$, where $t = L^{-1} \circ T \circ L$.*

Proof. Since $C_T \in B(H^2(\pi^+))$, by Lemma 3.1, $M_\beta \in B(H^2(D))$. Hence

$$\begin{aligned}\sup\{|(1-t(z))/(1-z)| : z \in D\} &= \|\beta\|_\infty \\ &= \|M_\beta\| < \infty ,\end{aligned}$$

which by the theorem of Julia-Carathéodory [1, Section 299, p. 32] shows that t has an angular derivative at 1. So by Theorem 2.1 of [3], C_t is not compact.

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But the converse of this theorem is false. We cite the following example.

EXAMPLE. Let $T(w) = \{(i-3)w-3i+1\}/\{(i-1)w+i-1\}$. Then it is easy to see that T maps π^+ into π^+ . Also $T(w) \to (i-3)/(i-1)$ as $w \to \infty$ which shows that C_T is not a bounded operator on $H^2(\pi^+)$. But we shall show that the function t defined by $t(z) = \left(L^{-1} \circ T \circ L\right)(z) = (i+z)/2i$ induces a non-compact composition operator on $H^2(D)$. Here we have

$$\lim_{\substack{z \to i \\ z \in \Delta}} \left(t(z)-1\right)/(z-i) = 1/2i ,$$

where Δ is any triangle contained in D with a vertex at i . This shows that t has an angular derivative at i . So again, by Theorem 2.1 of [3], C_t is not compact.

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