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A NOTE ON NORMAL COMPLEMENTS FOR FINITE GROUPS

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Abstract

Assume that G is a finite group and H is a 2-nilpotent Sylow tower Hall subgroup of G such that if x and y are G-conjugate elements of $H \cap G'$ of prime order or order 4, then x and y are H-conjugate. We prove that there exists a normal subgroup N of G such that $G = HN$ and $H \cap N = 1$.

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1. Introduction

All groups considered in this note will be finite.

We say that a subgroup H of a group G has a *normal complement* in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N = 1$. If H is a Sylow p-subgroup of G, p a prime, then we say that G has a *normal p-complement* or G is *p-nilpotent*.

If a subgroup H of a group G has a normal complement in G, then every pair of G-conjugate elements of H are H-conjugate. If H is a nilpotent Hall subgroup of G, then the converse is also true. This is a consequence of a well-known result of Wielandt (see [6, Corollary 10.41]).

On the other hand, a well-known theorem due to Brauer and Suzuki, whose proof makes use of character theory, established sufficient conditions for a (not necessarily nilpotent) Hall subgroup of a group G to have a normal complement in G.

Theorem 1.1 [5, Theorem 8.22]. *Let H be a Hall π-subgroup of G and suppose that whenever two elements of H are conjugate in G, then they are already conjugate in H. Assume for every elementary subgroup $E \subseteq G$, that if E is a π-subgroup, then E is conjugate to a subgroup of H. Then H has a normal complement in G.*

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Also, as González-Sánchez showed in [3], if H is a Sylow p-subgroup of G, we need only consider elements of prime order or order 4.

Theorem 1.2 [3, Main Theorem]. *Let H be a Sylow subgroup of a group G. Suppose that every pair of G-conjugate elements of prime order or order 4 of H are H-conjugate. Then H has a normal complement in G.*

Theorem 1.2 is a consequence of the following result, which can be proved using the same arguments as those used in the proof of [1, Remark].

Theorem 1.3. *Let p be a prime and H be a Sylow p-subgroup of a group G. Suppose that every pair of G-conjugate elements of prime order or order 4 of $H \cap G'$ are H-conjugate. Then H has a normal complement in G.*

The main goal of this note is to prove that Theorem 1.3 holds not only for Sylow subgroups but also for 2-nilpotent Sylow tower Hall subgroups.

We say that a group G is a *Sylow tower group* if, for some ordering of the distinct primes $p_1, p_2, \ldots, p_r$, there exists a series of normal subgroups of G,

$$1 = G_0 \leq G_1 \leq \cdots \leq G_r = G,$$

such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1} for $i = 1, \ldots, r$ (see [2, Example IV.3.4(c)].

We can now state the main theorem of this note.

Theorem 1.4. *Let H be a 2-nilpotent Sylow tower Hall subgroup of a group G. Suppose that every pair of G-conjugate elements of prime order or order 4 of $H \cap G'$ are H-conjugate. Then H has a normal complement in G.*

As an immediate deduction we have the following result.

Corollary 1.5. *If H is a Sylow tower group and an odd order Hall subgroup of a group G, and every pair of G-conjugate elements of prime order or order 4 of $H \cap G'$ are H-conjugate, then H has a normal complement in G.*

Theorem 1.4 does not hold for a soluble Hall subgroup H of a group G in which every pair of G-conjugate elements of prime order or order 4 of $H \cap G'$ are H-conjugate.

Example 1.6. Let $G = S_5$ be the symmetric group of degree 5 and let H be the stabiliser of the letter 5. Then $H = S_4$ is a $\{2, 3\}$-Hall subgroup of G.

Observe that $G' = A_5$ and $H \cap G' = \{(1)\} \cup a^H \cup b^H = A_4$, where $a = (12)(34)$ and $b = (123)$.

Let $x \in H \cap G'$ be an element of prime order or order 4. Then $x \in a^H$ or $x \in b^H$. It is not difficult to see that $x^G \cap H \cap G' = x^H$. Therefore, every pair of G-conjugate elements of prime order or order 4 of $H \cap G'$ are H-conjugate. However, H has no normal complement in G.

2. Proof of Theorem 1.4

Assume, arguing by contradiction, that (G, H) satisfies the hypotheses of the theorem but H fails to have a normal complement in G. Choose such a pair (G, H) with $|G| + |H|$ as small as possible.

Let $H_{2'}$ be a normal 2-complement of H. Then $H_{2'}$ is a Sylow tower group. Let S be a Sylow p-subgroup of $H_{2'}$ such that $S \trianglelefteq H_{2'}$. Then $p > 2$. Note that S is a Sylow p-subgroup of G which is normal in H. If $H = S$, then H has a normal complement in G by Theorem 1.3, which is not the case. Hence, $S < H$.

According to [6, Theorem 10.30], H has a Hall p'-subgroup, $H_{p'}$ say. We will show that $H_{p'}$ satisfies the hypothesis of the theorem. First of all, note that $H_{p'}$ is a 2-nilpotent Sylow tower Hall subgroup of G. Let $x, y \in H_{p'} \cap G'$ be two elements of prime order or order 4 such that x, y are G-conjugate. By hypothesis, x, y are H-conjugate. Since $H_{p'}$ has a normal complement S in H, it follows that x, y are $H_{p'}$-conjugate in $H_{p'}$. The choice of the pair (G, H) implies that $H_{p'}$ has a normal complement M in G. Then $S = H \cap M$.

Assume that M is p-nilpotent and let C be the normal complement of S in M. Then $G = H_{p'}M = H_{p'}(SC) = HC$ and $H \cap C = 1$. Clearly C is normal in G since C is a characteristic subgroup of M. Therefore, C is the normal complement of H in G and, since this contradicts the choice of the pair (G, H), we conclude that M is not p-nilpotent. We will reach a contradiction through the following steps.

Step 1. Let X be an $H_{p'}$-invariant subgroup of M such that $S \le X < M$. Then X is p-nilpotent.

Let $Y = H_{p'}X$. Then Y is a proper subgroup of G containing H. It is clear that the pair (Y, H) satisfies the hypotheses of the theorem. The choice of (G, H) implies that H has a normal complement, A say, in Y. Also, A is a Hall π-subgroup of $Y = HA$ for some set of primes π. Let $B = H_{p'}A \cap X$. Then B is a π-subgroup of X and so B is contained in A. Moreover, $X = BS$, $H_{p'}A = H_{p'}B$ and $H_{p'} \cap A = H_{p'} \cap B = 1$. Consequently, $B = A$ is a normal subgroup of X and so X is p-nilpotent.

Step 2. Let $N = \mathrm{O}_p(M)$. Then M/N is p-nilpotent.

Note that N is a subgroup of S. If $N = S$, then M/N is a p'-group, so that M/N is p-nilpotent. Suppose that $N < S$. Let $Z/N = Z(J(S/N)) \neq 1$ be the Thompson subgroup of S/N. Since N is a proper subgroup of Z and Z is normal in S, we see that $S \le \mathrm{N}_M(Z) < M$. Also, since S and M are both $H_{p'}$-invariant, it follows that $\mathrm{N}_M(Z)$ is $H_{p'}$-invariant. By Step 1, $\mathrm{N}_M(Z)$ is p-nilpotent, so that $\mathrm{N}_{M/N}(Z/N) = \mathrm{N}_M(Z)/N$ is p-nilpotent. Since p is odd, we can apply [4, Theorem 8.3.1] to conclude that M/N is p-nilpotent.

Step 3. We reach a contradiction.

First, recall that M/N has a normal p-complement T/N by Step 2. Then N is a Sylow p-subgroup of T and M/T is a p-group. Assume that $N \cap T' = 1$. Then T' is a p'-subgroup of T, so that T is p-nilpotent. Since a normal p-complement of T is

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also a normal p-complement of M, it would follow that M is p-nilpotent, which is not the case. Consequently, $N \cap T' \neq 1$. Let $x \in N \cap T'$ have order p. Note that $N \cap T'$ is a subgroup of $H \cap G'$ and so by hypothesis the G-conjugacy class x^G of x and the H-conjugacy class x^H of x coincide. Since N and T' are normal subgroups of G, we have $x^G \subseteq N \cap T'$.

Since $T \trianglelefteq G$, it follows that $|x^T|$ divides $|x^G| = |x^H|$, which implies that $|T : \mathrm{C}_T(x)|$ divides $(|T|, |H|)$, which is a power of p. Thus, $T = N\,\mathrm{C}_T(x)$, since N is a Sylow p-subgroup of T, and so $x^T = x^N$. We have proved that every pair of T-conjugate elements of prime order of $N \cap T'$ are N-conjugate. Applying Theorem 1.3, T is p-nilpotent. Consequently, M is p-nilpotent and this is our final contradiction.

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