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A CLASS OF PROJECTIVE STONE ALGEBRAS

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We prove that a regular double Stone algebra is projective in the category of Stone algebras if and only if its centre is a projective Boolean algebra and its dense set is countably generated as a filter. It follows that every countable regular double Stone algebra is projective as a Stone algebra.

0. Introduction

In his book [5] Grätzer lists as problem 53: Describe the projective Stone algebras. The finite case has been solved first by Balbes and Grätzer [1]; more characterizations have been given by the author [4]. Already for the countable case no worthwhile description is known. Any projective Stone algebra has to be a double Stone algebra (*cf.* [4]), so it seems quite natural to ask which classes of double Stone algebras are projective in the category of all Stone algebras, and in this paper we shall answer this question for the regular double Stone algebras.

1. Preliminaries

A Stone algebra $(L, +, \cdot, {}^*, 0, 1)$ is a pseudocomplemented distributive lattice with 0 and 1, such that for all $x \in L$, $x^* + x^{**} = 1$. We shall identify in the sequel any structure with its underlying set. $B_L = \{x^* \mid x \in L\}$ is called the centre of L; it is the set of complemented elements of L and, as a subalgebra of L, it is a Boolean algebra. $D_L = \{x \in L \mid x^* = 0\}$ is called the set of dense

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elements of L ; it is a filter in L . L is called a double Stone algebra, if it is also dually pseudocomplemented, that is, for every element x of L there is an x^+ in L , such that for all $y \in L$, $x + y = 1$ if and only if $x^+ \leq y$, and if furthermore $x^+ \cdot x^{++} = 0$. We refer the reader to [5] for the basic facts on Stone algebras.

A double Stone algebra is called regular, if for all $x, y \in L$, $x^* = y^*$ and $x^+ = y^+$ imply $x = y$.

LEMMA 1.1 (Varlet [8]). *Let L be a double Stone algebra. Then the following statements are equivalent:*

(1) *L is regular;*

(2) *$x \cdot x^+ \leq y + y^*$ for all $x, y \in L$;*

(3) *every chain of prime ideals of L has at most two elements.*

For a regular L , set $F_L = \{x^{++} \mid x \in D_L\}$; then F_L is a filter in B_L , and the map $d : F_L \to D_L$ which assigns to each element a of F_L the unique element x of D_L satisfying $x^{++} = a$, is a lattice isomorphism. In [7], Katriňák has shown that every regular double Stone algebra L can be constructed from a Boolean algebra B and a (not necessarily proper) filter in B .

A Stone algebra homomorphism is a lattice homomorphism which also preserves *, 0 , and 1 .

LEMMA 1.2 (Chen and Grätzer [3]). *Let L and M be Stone algebras, and suppose $f_1 : B_L \to B_M$ is a Boolean homomorphism, and that $f_2 : D_L \to D_M$ is a lattice homomorphism preserving 1 . Then there is a Stone algebra homomorphism $f : L \to M$, such that $f|B_L = f_1$ and $f|D_L = f_2$ if and only if for all $a \in B_L$, $x \in D_L$ with $a \leq x$, we have $f_1(a) \leq f_2(x)$. In this case, the extension f of f_1 and f_2 over L is unique. Furthermore, f is onto (one-to-one) if and only if f_1 and f_2 are onto (one-to-one).*

If L is a double Stone algebra, Lemma 1.2 obviously implies

LEMMA 1.3. *Let L, M, f_1, f_2 be defined as in Lemma 1.2, and let L*

be a double Stone algebra. Then there is an extension of f_1 *and* f_2 *over* L *if and only if for all* $x \in D_L$, $f_1(x^{++}) \leq f_2(x)$.

(Note that the extension does not necessarily preserve $^+$.)

2. Representations in free Stone algebras

This section consists of a series of - mostly - technical lemmas which are crucially needed in the proof of the main theorem. Let α be a non-zero cardinal; for every $i < \alpha$ let F_i be the free Stone algebra generated by some element a_{i0} .

THEOREM 2.1 (Balbes and Horn [2]). *Let* F *be the free product of* $\{F_i \mid i < \alpha\}$ *in the category of distributive lattices with* 0 *and* 1 . *Then* F *is the free Stone algebra on the free generators* $\{a_{i0} \mid i < \alpha\}$.

For the remainder of the paper let F_α be the free Stone algebra on the free generators $\{a_{i0} \mid i < \alpha\}$. For $i < \alpha$, let $a_{i1} = a_{i0}^*$, and $a_{i2} = a_{i0}^{**}$. For $r \in \{0, 1, 2\}$, set $A^r = \{a_{ir} \mid i < \alpha\}$, and let $A = A^0 \cup A^1 \cup A^2$.

Theorem 2.1 implies that for every $x \in F_\alpha - \{0, 1\}$ there are non-empty finite subsets $T_1, \ldots, T_m$, $S_1, \ldots, S_r$ of A , such that

(1) for all $i \leq m$, $\sup T_i \neq 1$ and $x = \sup T_1 \cdot \ldots \cdot \sup T_m$;

(2) for all $j \leq r$, $\inf S_j \neq 0$ and $x = \inf S_1 + \ldots + \inf S_r$.

The sets used in the representation of an $x \in F_\alpha$ are always assumed to be finite nonempty subsets of A .

For $T \subseteq A$, let $\overline{T} = T \cup \{a_{i0} \mid a_{i2} \in T\}$. Let B_α be the centre of F_α and D_α be the set of its dense elements. The proof of the following two lemmas can be found in Katriňák [6].

LEMMA 2.2. *Let* T *be a nonempty finite subset of* A .

(1) $\sup T = 1$ *if and only if there is an* $i < \alpha$, *such that*

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$\{a_{i1}, a_{i2}\} \subseteq T$.

(2) $\sup T \in D_\alpha$ *if and only if there is an* $i < \alpha$ *, such that* $\{a_{i0}, a_{i1}\} \subseteq \overline{T}$.

(3) $\inf T = 0$ *if and only if there is an* $i < \alpha$ *, such that* $\{a_{i0}, a_{i1}\} \subseteq \overline{T}$.

LEMMA 2.3. *Let* $x, y \in F_\alpha$, $x = \sup T_1 \cdot \ldots \cdot \sup T_m$, $y = \sup W_1 \cdot \ldots \cdot \sup W_p$ *, such that* $\sup W_j \neq 1$ *for all* $j \leq p$ *. Then* $x \leq y$ *if and only if for every* $j \leq p$ *one of the following conditions hold:*

(1) *there is an* $i \leq m$ *such that* $T_i \subseteq \overline{W}_j$;

(2) *for all* $i \leq m$, $T_i - \overline{W}_j \neq \emptyset$ *, and*

$$\sup\left(T_1-\overline{W}_j\right) \cdot \ldots \cdot \sup\left(T_m-\overline{W}_j\right) = 0 .$$

In the same paper, Katriňák has also shown that B_α is the free Boolean algebra freely generated by $\{a_{i2} \mid i < \alpha\}$.

The following construction of a normal representation of an $x \in F_\alpha - \{0, 1\}$ is an application of Grätzer ([5], Section 12), to our situation.

Suppose $x \in F_\alpha - \{0, 1\}$; then there are nonempty finite subsets $S_1, \ldots, S_r$ of A , such that

(1) for all $i < \alpha$, $j \leq r$, $\left|S_j \cap \{a_{i0}, a_{i1}, a_{i2}\}\right| \leq 1$,

(2) $x = \inf S_1 + \ldots + \inf S_r$.

Note that $\inf S_j \neq 0$ for all $j \leq r$ by Lemma 2.2 (3).

Let $J = \{S_1, \ldots, S_r\}$ and

$$C(J) = \left\{f \mid f : J \to \bigcup J,\ f\left(S_i\right) \in S_i \text{ for every } i \leq r\right\} .$$

$C(J)$ is the set of choice functions on J . For every $f \in C(J)$ let

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$f^+(J)$ be the following subset of A :

(1) $a_{i1} \in f^+(J)$ if and only if $a_{i1} \in f(J)$;

(2) $a_{i2} \in f^+(J)$ if and only if $a_{i2} \in f(J)$;

(3) $a_{i0} \in f^+(J)$ if and only if $a_{i0} \in f(J)$ and $a_{i2} \notin f(J)$.

Since $a_{i0} \le a_{i2}$, we have $\sup f(J) = \sup f^+(J)$.

Let $C_r(J)$ be the following subset of $C(J)$:

$f \in C_r(J)$ if and only if $f \in C(J)$ and $\sup f^+(J)$ is a minimal element of the set $\sup\{g^+(J) \mid g \in C(J)\}$.

Then $x = \inf\{\sup f^+(J) \mid f \in C_r(J)\}$. This representation of x is called normal; the normal representation of x is unique up to commutativity; furthermore, if $f, g \in C_r(J)$ and $\sup f^+(J) \le \sup g^+(J)$, then $f^+(J) = g^+(J)$.

LEMMA 2.4. *Let* $x, y \in F_\alpha - \{0, 1\}$; *suppose* $x = \sup T_1 \cdot \ldots \cdot \sup T_m$ *is the normal representation, and* $y = \sup W_1 \cdot \ldots \cdot \sup W_p$ *is a representation, such that* $\sup W_j \neq 1$ *for all* $j \le p$. *Then* $x \le y$ *if and only if for every* $j \le p$ *there is an* $i \le m$, *such that* T_i *is a subset of* $\overline{W}_j$.

Proof. First note that for nonempty finite subsets W, T of A satisfying $\sup W \neq 1$, $\sup T \le \sup W$ if and only if $T \subseteq \overline{W}$. This obviously implies sufficiency.

Now let $S_1, \ldots, S_r$ be nonempty finite subsets of A , such that $x = \inf S_1 + \ldots + \inf S_r$, and for all $i < \alpha$, $j \le r$, we have $|S_j \cap \{a_{i0}, a_{i1}, a_{i2}\}| \le 1$. Let $J = \{S_1, \ldots, S_r\}$, and $C_r(J)$ and $f^+(J)$ be defined as above. Then $\{T_1, \ldots, T_m\} = \{f^+(J) \mid f \in C_r(J)\}$.

Let $j \le p$; since $x \le \sup W_j$, we have $\inf S_i \le \sup W_j$ for every $i \le r$, thus, $S_i \cap \overline{W}_j \neq \emptyset$ by Lemma 2.3. Hence, there is an $f \in C(J)$, such that $f(J)$ is a subset of $\overline{W}_j$, and therefore $\sup f^+(J) \le \sup W_j$.

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By definition of $C_r(J)$, there is a $g \in C_r(J)$ satisfying $\sup g^+(J) \leq \sup f^+(J)$. Let $g^+(J) = T_i$; then T_i is a subset of $\overline{W}_j$.

In order to prove the main theorem, we have to take a closer look at the normal representation of an $x \in F_\alpha - \{0, 1\}$. For a nonempty finite subset T of A define

$$T^* = \{a_{i1} \mid a_{i0} \in T \text{ or } a_{i2} \in T\} \cup \{a_{i2} \mid a_{i1} \in T\} ,$$

and

$$K^0(T) = \{i < \alpha \mid a_{i0} \in T\} .$$

If $x \in F_\alpha - \{0, 1\}$, $x = \sup T_1 \cdot \ldots \cdot \sup T_m$ is the normal representation, set $K^0(x) = K^0(T_1) \cup \ldots \cup K^0(T_m)$.

Clearly, $(\sup T)^* = \inf T^*$, and $K^0(x) = \emptyset$, if $x \in B_\alpha$.

The proof of the following lemma is a straightforward application of the construction of the normal representation from an arbitrary representation and is left to the reader.

LEMMA 2.5. *Let* $x = \sup S_1 \cdot \ldots \cdot \sup S_r$ *be an arbitrary representation. Then* $K^0(x)$ *is a subset of* $K^0(S_1) \cup \ldots \cup K^0(S_r)$.

LEMMA 2.6. *Let* x *be such that* $x + x^* \neq 1$. *Then* $K^0(x+x^*) = K^0(x)$.

Proof. Let $x = \sup T_1 \cdot \ldots \cdot \sup T_m$, $x + x^* = \sup W_1 \cdot \ldots \cdot \sup W_p$ be the normal representations. Suppose $a_{k0} \notin T_1 \cup \ldots \cup T_m$. We have

$$x + x^* = \sup T_1 \cdot \ldots \cdot \sup T_m + \inf T_1^* + \ldots + \inf T_m^* ,$$

so, using distributivity, we can compute a representation $x + x^* = \sup S_1 \cdot \ldots \cdot \sup S_r$, where each S_j is a subset of $T_1 \cup \ldots \cup T_m \cup T_1^* \cup \ldots \cup T_m^*$. Since $T_j^* \cap A^0$ is empty for any $j \leq m$,

a_{k0} is not an element of $S_1 \cup \ldots \cup S_r$, hence, k is not an element of $K^0(x+x^*)$ by the preceding lemma.

The reverse inclusion is proved similarly, using $x = (x+x^*) \cdot x^{**}$.

The proof of the following lemma is an easy application of the two preceding lemmas and is omitted.

LEMMA 2.7. *Let* $x = \sup T_1 \cdot \ldots \cdot \sup T_m$ *be the normal representation, and suppose there is a* $k \leq m$ *, such that* $x + x^* = \sup T_1 \cdot \ldots \cdot \sup T_k + x^*$. *Then* $K^0(x)$ *is a subset of* $K^0(T_1) \cup \ldots \cup K^0(T_k)$.

3. Projectivity of regular double Stone algebras

Recall that a Stone algebra is projective if and only if it is a retract of a free Stone algebra. For the basic facts on projective Stone algebras see Balbes and Grätzer [*1*].

Let L be a Stone algebra and F a filter in L ; a subset S of F is called a set of generators for F , if for every $x \in F$ there are $y_1, \ldots, y_k \in S$ satisfying $y_1 \cdot \ldots \cdot y_k \leq x$.

Now we can prove the main theorem.

THEOREM 3.1. *Let* L *be a regular double Stone algebra. Then* L *is projective in the category of all Stone algebras if and only if* B_L *is a projective Boolean algebra and* D_L *has at most a countable set of generators.*

Proof. Let L be projective; then L is a retract of some F_α , so suppose without loss of generality that L is a subalgebra of F_α , and that $f : F_\alpha \to L$ is a retraction.

Since B_L is a retract of L , and since a Boolean algebra is projective if and only if it is projective in the category of Stone algebras (*cf*. Balbes and Grätzer [*1*]), B_L has to be a projective Boolean algebra.

Also, B_L is a subalgebra of B_α, the centre of F_α. For $x \in L$ let x^+ be the dual pseudocomplement of x in L; this does not have to be the dual pseudocomplement of x in F_α, but since x^+ is an element of B_L, and L is a subalgebra of F_α, we have $x^{++} = x^{+*}$ and $x^{*+} = x^{**}$.

Now we shall show that D_L has a set of generators which is at most countable.

Let D_α be the filter of dense elements of F_α; by Lemma 2.2 (2), $\{a_{i0}+a_{i1} \mid i < \alpha\}$ is a set of generators for D_α.

Let $I = \{i < \alpha \mid f(a_{i0}+a_{i1}) \neq 1\}$; then $\{f(a_{i0}+a_{i1}) \mid i \in I\}$ generates D_L. For every $i \in I$ define $x_i = f(a_{i0}+a_{i1}) \cdot f(a_{i0}+a_{i1})^+$; then $x_i + x_i^* = f(a_{i0}+a_{i1})$, and, since L is regular, $x_i \leq x_j + x_j^*$ for all $i, j \in I$ (*cf.* Lemma 1.1 (2)).

If $x = \sup T_1 \cdot \ldots \cdot \sup T_m$ is the normal representation, then m is called the rank of x, and we write $\operatorname{rk}(x) = m$.

Let M be a nonempty subset of I, and let m, k be natural numbers, such that $1 \leq k \leq m$. M is called an (m, k)-set, if for every $i \in M$, $\operatorname{rk}(x_i) = m$ and there are finite nonempty subsets $T_1, \ldots, T_k$ of A, such that, up to commutativity,

$$x_i = \sup T_1 \cdot \ldots \cdot \sup T_k \cdot \sup T^i_{k+1} \cdot \ldots \cdot \sup T^i_m$$

is the normal representation of x_i.

M is called an $(m, 0)$-set if $\operatorname{rk}(x_i) = m$ for all $i \in M$.

Note that an (m, k)-set is also an (m, r)-set for every $r \leq k$, and that for an (m, m)-set M we have $|\{x_i \mid i \in M\}| \leq 1$.

Next, as an intermediate step, we want to prove the following

PROPOSITION. *Let M be an (m, k)-set. Then $\bigcup\{K^0(x_i) \mid i \in M\}$ is finite or M is the union of finitely many $(m, k+1)$-sets.*

Proof. For every $i \in M$ let

$$x_i = \sup T_1 \cdot \ldots \cdot \sup T_k \cdot \sup T^i_{k+1} \cdot \ldots \cdot \sup T^i_m$$

be the normal representation; if $0 < k < m$, set $p = \sup T_1 \cdot \ldots \cdot \sup T_k$; if $k = 0$, set $p = 1$; if $k = m$, set $p = x_i$.

If, for every $i \in M$, $K^0(x_i)$ is a subset of $K^0(T_1) \cup \ldots \cup K^0(T_k)$, then $\cup\left\{K^0(x_i) \mid i \in M\right\}$ is a finite set. Note that this is the case, if $k = m$.

So suppose there is an $s \in M$ such that $K^0(x_s)$ is not a subset of $K^0(T_1) \cup \ldots \cup K^0(T_k)$. Then $p \nleq x_s + x_s^*$.

Assume the contrary; then $p \neq 1$ and

$$p + x_s^* = p + x_s + x_s^* = x_s + x_s^* ,$$

so, by Lemma 2.7, $K^0(x_s)$ is a subset of $K^0(T_1) \cup \ldots \cup K^0(T_k)$, a contradiction.

Let $x_s + x_s^* = \sup S_1 \cdot \ldots \cdot \sup S_r$ be the normal representation, and suppose without loss of generality that $p \nleq \sup S_1$; then $T_i \nsubseteq \overline{S}_1$ for every $i \leq k$.

Let $j \in M$; then, by regularity of L and by definition of M , $x_j = p \cdot \sup T^j_{k+1} \cdot \ldots \cdot T^j_m \leq x_s + x_s^* \leq \sup S_1$. Since the representation of x_j is normal, there is a $q(j) \in \{k+1, \ldots, m\}$, such that $T^j_{q(j)}$ is a subset of $\overline{S}_1$. For every $q \in \{k+1, \ldots, m\}$ and every subset P of $\overline{S}_1$, let $M(q, P) = \left\{i \in M \mid q(i) = q \text{ and } T^i_{q(i)} = P\right\}$.

Since $\overline{S}_1$ has only finitely many subsets, the set $\{M(q, P) \mid q \in \{k+1, \ldots, m\}, P \subseteq \overline{S}_1\}$ is also finite. If $i \in M$, then

$i \in M\left[q(i),\, T^{i}_{q(i)}\right]$, so we have

$$M = \cup\{M(q,\, P) \mid q \in \{k+1,\, \ldots,\, m\},\, P \subseteq \overline{S}_1\} \;.$$

Every nonempty $M(q,\, P)$ is an $(m,\, k+1)$-set, since, for $i \in M(q,\, P)$, $T^{i}_{q(i)} = P$. This proves the proposition.

For every $m < \omega$, set $I_m = \{i \in I \mid \mathrm{rk}(x_i) = m\}$. We want to show that $\cup\left\{K^0(x_i) \mid i \in I_m\right\}$ is a finite set.

Assume on the contrary that there is an $m < \omega$, such that $\cup\left\{K^0(x_i) \mid i \in I_m\right\}$ is infinite.

Set $I^0_m = I_m$; I^0_m is an $(m,\, 0)$-set, which, by our assumption and the proposition, can be partitioned into finitely many $(m,\, 1)$-sets. Then there is an $(m,\, 1)$-set $I^1_m \subseteq I^0_m$, such that $\cup\left\{K^0(x_i) \mid i \in I^1_m\right\}$ is infinite. We repeat the argument m times, and we arrive at an $(m,\, m)$-set $I^m_m \subseteq I^{m-1}_m$ which satisfies that $\cup\left\{K^0(x_i) \mid i \in I^m_m\right\}$ is infinite. This is a contradiction, since I^m_m is an $(m,\, m)$-set, and therefore $\left|\left\{x_i \mid i \in I^m_m\right\}\right| \le 1$.

Thus, for every $m < \omega$, $\cup\left\{K^0(x_i) \mid i \in I_m\right\}$ is finite, and therefore $\cup\left\{K^0(x_i) \mid i \in I\right\}$ is at most countable.

By Lemma 2.6, $\cup\left\{K^0(x_i + x_i^*) \mid i \in I\right\}$ is also at most countable.

For every $i \in I$ an element y_i of D_α is defined as follows. Let $x_i + x_i^* = \sup S_1 \cdot \ldots \cdot \sup S_r$ be the normal representation; for every $j \le r$, set

$$S^d_j = \{a_{k0} \mid a_{k0} \in S_j \text{ and } a_{k1} \in S_j\} \cup \{a_{k1} \mid a_{k0} \in S_j \text{ and } a_{k1} \in S_j\} \;.$$

Then S^d_j is a subset of S_j , and $\sup S^d_j$ is an element of D_α . Now let

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$y_i = \sup S_1^d \cdot \ldots \cdot \sup S_r^d$. Then

(1) $y_i \le x_i + x_i^*$, hence $f(y_i) \le f(x_i+x_i^*) = x_i + x_i^*$,

(2) $K^0(y_i)$ is a subset of $K^0(x_i+x_i^*)$.

Recall that $\{x_i+x_i^* \mid i \in I\}$ is a set of generators of D_L , hence $\{f(y_i) \mid i \in I\}$ is also a set of generators for D_L . (2) implies that $\{y_i \mid i \in I\}$ is countable, hence $\{f(y_i) \mid i \in I\}$ is, at most, a countable generating set for D_L .

For the converse, let L be a regular double Stone algebra such that B_L is a projective Boolean algebra and that D_L has, at most, a countable set of generators. Let $G = \{x^{++} \mid x \in D_L\}$; then G is a filter in B_L . Since L is regular, the map $d : G \to D_L$ which assigns to every element a of G the unique element x of D_L which satisfies $x^{++} = a$, is a lattice isomorphism.

Let $\{m_i \mid i < \omega\}$ be a set of generators for G , and suppose without loss of generality that $i < j$ implies $m_i \ge m_j$. So, for every $a \in G$, there is a natural number i and an element b of B_L , such that $b \le m_i^*$ and $a = m_i + b$.

Let F be a free Stone algebra, $f : F \to L$ an onto Stone algebra homomorphism, and set $f_1 = f|B_F$, $f_2 = f|D_F$. Since f is onto, so are f_1 and f_2 .

B_L is a projective Boolean algebra, thus there is a Boolean homomorphism $g_1 : B_L \to B_F$ satisfying $f_1 \circ g_1 = \text{id } B_L$.

For every $i < \omega$, let $u_i = g_1(m_i)$. If z is an element of D_F which satisfies $f(z) = d(m_i)$, then

$$f(z+u_i) = f(z) + f(u_i) = d(m_i) + m_i = d(m_i) .$$

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Thus, for every $i < \omega$, we can choose an $x_i \in D_F$, such that $f(x_i) = d(m_i)$, and $u_i \leq x_i$.

We now define inductively

$$y_0 = x_0 \ , \quad y_{i+1} = y_i \cdot (x_{i+1}+u_i^*) \ .$$

Then, for all $i < \omega$,

(1) $y_i \in D_F$: $y_0 = x_0 \in D_F$ by definition; if $y_k \in D_F$, then, since $x_{k+1} \in D_F$ and D_F is a filter, $y_k \cdot (x_{k+1}+u_k^*) \in D_F$;

(2) $f(y_i) = d(m_i)$: $f(y_0) = f(x_0) = d(m_0)$; suppose $f(y_k) = d(m_k)$; then

$$\begin{aligned} f(y_{k+1}) &= f(y_k) \cdot (f(x_{k+1})+f(u_k^*)) \\ &= d(m_k) \cdot (d(m_{k+1})+m_k^*) \\ &= d(m_k) \cdot d(m_{k+1}) + d(m_k) \cdot m_k^* \\ &= d(m_{k+1}) + d(m_k) \cdot m_k^* \text{, since } d(m_{k+1}) \leq d(m_k) \ , \\ &= d(m_{k+1}) + d(m_k) \cdot m_k^+ \text{, since } m_k \in B \ , \\ &= d(m_{k+1}) + d(m_k) \cdot d(m_k)^+ \text{, since } d(m_k)^+ = m_k^+ \ , \\ &= d(m_{k+1}) \text{, since } L \text{ is regular (} cf. \text{ Lemma 1.1 (2));} \end{aligned}$$

(3) $u_i \leq y_i$: $y_0 = x_0 \geq u_0$ by definition; suppose $u_k \leq y_k$; then

$$u_{k+1} = u_k \cdot u_{k+1} = u_k \cdot (u_{k+1}+u_k^*) \leq y_k(x_{k+1}+u_k^*) = y_{k+1} \ ;$$

(4) $y_{i+1} + u_i = y_i$:

$$y_{i+1} + u_i = y_i(x_{i+1}+u_i^*) + u_i = y_i \cdot x_{i+1} + y_i \cdot u_i^* + u_i = y_i \ ,$$

since $u_i \leq y_i$, and $u_i \in B_F$.

Note that (4) implies that for $i, j \in \omega$, $i \leq j$, we have $y_j + u_i = y_i$.

Now we define $g_2 : D_L \to D_F$ as follows: let $x \in D_L$; there is a

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$j < \omega$ and an element b of B, such that $x = d(m_j+b)$; since L is regular, $x = d(m_j) + b$. Define $g_2(x) = y_j + g_1(b)$.

<u>g_2 is well defined</u>

Let $d(m_j+b) = d(m_i+c)$; then, since L is regular, $m_j + b = m_i + c$; hence $u_j + g_1(b) = u_i + g_1(c)$. Suppose without loss of generality $i \leq j$; then

$$\begin{aligned} y_j + g_1(b) &= y_j + u_j + g_1(b) \quad \text{by (3)} \\ &= y_j + u_i + g_1(c) \\ &= y_i + g_1(c) \quad \text{by (4).} \end{aligned}$$

Thus g_2 is well defined.

<u>g_2 is a lattice homomorphism</u>

Suppose $x, y \in D_L$, $x = d(m_j+b)$, $y = d(m_i+c)$, and $i \leq j$. Then

$$\begin{aligned} g_2(x+y) &= g_2\big(d(m_j+b)+d(m_i+c)\big) \\ &= g_2\big(d(m_j+b+m_i+c)\big) \\ &= g_2\big(d(m_i+b+c)\big) \\ &= y_i + g_1(b+c) \\ &= y_i + y_j + g_1(b) + g_1(c) \\ &= g_2(x) + g_2(y) , \end{aligned}$$

$$\begin{aligned} g_2(x \cdot y) &= g_2\big(d(m_j+b) \cdot d(m_i+c)\big) \\ &= g_2\big(d\big((m_j+b) \cdot (m_i+c)\big)\big) \\ &= g_2\big(d(m_j+m_i \cdot b+b \cdot c)\big) , \text{ since } m_j \leq m_i , \\ &= y_j + u_i \cdot g_1(b) + g_1(b) \cdot g_1(c) \\ &= y_j + (y_j+u_i) \cdot g_1(b) + g_1(b) \cdot g_1(c) \\ &= y_j + y_i \cdot g_1(b) + g_1(b) \cdot g_1(c) \quad \text{by (4)} \\ &= (y_j+g_1(b)) \cdot (y_i+g_1(c)) \\ &= g_2(x) \cdot g_2(y) . \end{aligned}$$

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Clearly $g_2(1) = 1$, so g_2 is a lattice homomorphism preserving unit.

g_1, g_2 <u>can be extended to a Stone algebra homomorphism</u> $g : L \to F$

Let $x = d(m_i+b) \in D_L$; then

$$g_1(x^{++}) = g_1(m_i+b) = u_i + g_1(b) \leq y_i + g_1(b) = g_2(x)$$

so, by Lemma 2.4 there is a (unique) extension $g : L \to F$ of g_1, g_2.

<u>$f \circ g = \mathrm{id}\ L$</u>

Suppose without loss of generality $x = d(m_i+b) \in D_L$; then

$$f(g(x)) = f(y_i+g_1(b)) = f(y_i) + f(g_1(b)) = f(y_i) + f_1(g_1(b))$$
$$= d(m_i) + b = d(m_i+b) = x .$$

Thus L is a retract of F and therefore projective.

Any countable Boolean algebra is projective, and we get

COROLLARY 3.2. *If L is a countable regular double Stone algebra, then L is projective in the category of Stone algebras.*

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