J. Aust. Math. Soc. **84** (2008), 145–154
doi:10.1017/S1446788708000062

STRONGLY IRREDUCIBLE IDEALS

A. AZIZI

(Received 24 April 2006; accepted 22 August 2007)

Communicated by J. Du

Abstract

A proper ideal I of a ring R is said to be strongly irreducible if for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$. In this paper we study strongly irreducible ideals in different rings. The relations between strongly irreducible ideals of a ring and strongly irreducible ideals of localizations of the ring are also studied. Furthermore, a topology similar to the Zariski topology related to strongly irreducible ideals is introduced. This topology has the Zariski topology defined by prime ideals as one of its subspace topologies.

2000 *Mathematics subject classification*: 13A15, 13C05, 13E05, 13F99.

Keywords and phrases: absolutely flat rings, arithmetical rings, generalizations of Dedekind domains, Laskerian rings, strongly irreducible ideals, Zariski topology.

1. Introduction

Throughout this paper all rings are commutative with identity. A proper ideal I of a ring R is said to be strongly irreducible if for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ (see [2]). Obviously a proper ideal I is strongly irreducible if and only if for each $x, y \in R$, $Rx \cap Ry \subseteq I$ implies that $x \in I$ or $y \in I$. It is easy to see that every prime ideal is a strongly irreducible ideal. Also every strongly irreducible ideal is an irreducible ideal. In this paper, we find the relations between strongly irreducible ideals, primary ideals and prime ideals in some rings. We also find some properties of strongly irreducible ideals. The relations between strongly irreducible ideals of a ring and strongly irreducible ideals of localizations of the ring are studied. We introduce a topology on the set of strongly irreducible ideals in such a way that the Zariski topology is a subspace of this topology and some properties of this topology are given.

2. Strongly irreducible ideals in different rings

Recall that a ring R is said to be a Laskerian ring, if every proper ideal of R has a primary decomposition. We know that every Noetherian ring is a Laskerian ring.

THEOREM 2.1. *Let R be a ring.*

(i) *If I is a strongly irreducible ideal of R, then I is a prime ideal if and only if $I = \sqrt{I}$ (I is a radical ideal).*

(ii) *For each proper ideal J of R, there is a minimal strongly irreducible ideal over J.*

(iii) *If R is a Laskerian ring, then every strongly irreducible ideal is a primary ideal.*

(iv) *If R is an absolutely flat ring, then an ideal is strongly irreducible if and only if it is a primary ideal.*

PROOF. (i) If I is a prime ideal, then obviously $I = \sqrt{I}$. Now let $I = \sqrt{I}$ and $AB \subseteq I$, where A and B are ideals of R. Then $A \cap B \subseteq \sqrt{A \cap B} = \sqrt{AB} \subseteq \sqrt{I} = I$, and since I is a strongly irreducible ideal of R, $A \subseteq I$ or $B \subseteq I$.

(ii) Let

$$T = \{I \mid I \text{ is a strongly irreducible ideal of } R \text{ containing } J\}.$$

Since every maximal ideal is strongly irreducible, $T \neq \emptyset$. By Zorn's lemma T has a minimal element with respect to $\supseteq$.

(iii) Let I be a strongly irreducible ideal and $\bigcap_{i=1}^{n} q_i$ be a primary decomposition for I. Then, $\bigcap_{i=1}^{n} q_i \subseteq I$, and since I is a strongly irreducible ideal, for some j, $1 \leq j \leq n$, $q_j \subseteq I = \bigcap_{i=1}^{n} q_i \subseteq q_j$.

(iv) We know that in an absolutely flat ring, maximal ideals, prime ideals and primary ideals are the same (see [1, p. 55, Exercise 3 and p. 35, Exercise 27]), so it is enough to show that every strongly irreducible ideal I is a prime ideal. Let $ab \in I$. Since R is an absolutely flat ring, there are elements $r_1, r_2 \in R$ such that $a = a^2 r_1$ and $b = b^2 r_2$. Let $t \in Ra \cap Rb$, then $t = as_1 = bs_2$, for some $s_1, s_2 \in R$. Now $t = as_1 = a^2 r_1 s_1 = as_1 a r_1 = bs_2 a r_1 \in Rab$, then $Ra \cap Rb \subseteq Rab \subseteq I$. Since I is a strongly irreducible ideal, $Ra \subseteq I$ or $Rb \subseteq I$. □

It is easy to see that in a unique factorization domain (UFD) R, every two elements have a least common multiple. For each $x, y \in R$, we denote the least common multiple of x and y by $[x, y]$.

THEOREM 2.2. *Let R be a UFD and let I be a proper ideal of R.*

(i) *I is strongly irreducible if and only if for each $x, y \in R$, $[x, y] \in I$ implies that $x \in I$ or $y \in I$.*

(ii) *I is strongly irreducible if and only if $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \in I$, where p_i are distinct prime elements of R and n_i are natural numbers, implies that $p_j^{n_j} \in I$, for some j, $1 \leq j \leq k$.*

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

(iii) *If I is a nonzero principal ideal, then I is strongly irreducible if and only if the generator of I is a power of a prime element of R.*

(iv) *Every strongly irreducible ideal is a primary ideal.*

PROOF. (i) Let I be a strongly irreducible ideal and for $x, y \in R$, $[x, y] \in I$. If $[x, y] = c$, then obviously $Rx \cap Ry = Rc \subseteq I$. So $Rx \subseteq I$ or $Ry \subseteq I$.

Conversely, for each $x, y \in R$, let $[x, y] \in I$ imply that $x \in I$ or $y \in I$. If $Rx \cap Ry \subseteq I$, we have $[x, y] \in Rx \cap Ry \subseteq I$, so $x \in I$ or $y \in I$.

(ii) If I is strongly irreducible, then clearly by part (i), we have the result.

For the converse, for $x, y \in R \setminus \{0\}$, let $[x, y] \in I$, and

$$x = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k} q_1^{m_1} q_2^{m_2} q_3^{m_3} \ldots q_s^{m_s}, \quad y = p_1^{t_1} p_2^{t_2} p_3^{t_3} \ldots p_k^{t_k} r_1^{l_1} r_2^{l_2} r_3^{l_3} \ldots r_u^{l_u},$$

be prime decompositions for x and y, respectively. Therefore,

$$[x, y] = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k} q_1^{m_1} q_2^{m_2} q_3^{m_3} \ldots q_s^{m_s} r_1^{l_1} r_2^{l_2} r_3^{l_3} \ldots r_u^{l_u},$$

where $\alpha_i = \max\{n_i, t_i\}$ for each i. Since $[x, y] \in I$, by the assumption, we have one of the following:

(a) for some i, $p_i^{\alpha_i} \in I$;

(b) for some i, $q_i^{m_i} \in I$;

(c) for some i, $r_i^{l_i} \in I$.

If (a) holds and $\alpha_i = n_i$, then obviously, $x \in I$. If (a) is satisfied and $\alpha_i = t_i$, then $y \in I$. For the case (b), $x \in I$. If (c) is satisfied, then $y \in I$.

Hence by part (i), I is a strongly irreducible ideal of R.

(iii) Let $I = Ra$ be a nonzero strongly irreducible ideal of R, and $a = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$ be a prime decomposition for a. By part (ii), for some i, $p_i^{n_i} \in I$. Hence, $Rp_i^{n_i} = I$.

Conversely, let for a prime element p of R, $I = Rp^n$. If for distinct prime elements $p_1, p_2, \ldots, p_k$ of R and natural numbers $n_1, n_2, \ldots, n_k$, $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \in I = Rp^n$, then $p^n \mid p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$. So, for some j, $1 \leq j \leq k$, we have $p = p_j$ and $n \leq n_j$. Therefore, $p_j^{n_j} \in I$. Thus, by part (ii), I is a strongly irreducible ideal.

(iv) Let I be a strongly irreducible ideal and $xy \in I$, where $x, y \in R \setminus \{0\}$, and let

$$x = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k} q_1^{m_1} q_2^{m_2} q_3^{m_3} \ldots q_s^{m_s}, \quad y = p_1^{t_1} p_2^{t_2} p_3^{t_3} \ldots p_k^{t_k} r_1^{l_1} r_2^{l_2} r_3^{l_3} \ldots r_u^{l_u},$$

be prime decompositions for x and y, respectively. Since $xy \in I$, by part (ii), we have one of the following:

(a) for some i, $p_i^{n_i + t_i} \in I$;

(b) for some i, $q_i^{m_i} \in I$;

(c) for some i, $r_i^{l_i} \in I$.

If (a) holds, let n be a natural number where $n \geq n_i/t_i$, then $(n+1)t_i \geq n_i + t_i$, so $p_i^{n_i+t_i} \mid p_i^{(n+1)t_i}$, and clearly, $p_i^{(n+1)t_i} \mid y^{n+1}$, therefore, $p_i^{n_i+t_i} \mid y^{n+1}$. Now since $p_i^{n_i+t_i} \in I$, $y^{n+1} \in I$.

For the case (b), since $q_i^{m_i} \mid x$, $x \in I$. For the case (c), evidently $y \in I$. $\square$

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

COROLLARY 2.3. *Let R be a UFD.*

(i) *Every principal ideal of R is a strongly irreducible ideal if and only if it is a primary ideal.*

(ii) *Every strongly irreducible ideal of R can be generated by a set of prime powers.*

PROOF. (i) If I is a strongly irreducible ideal, then by Theorem 2.2(iv), I is a primary ideal. Conversely, let I be a nonzero primary ideal. Then let $I = Ra$, where $0 \neq a \in R$. Let $a = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$ be a prime decomposition for a. If $k > 1$, then since $p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k} \in I$ and I is a primary ideal, there exists a natural number n, and a number j, $1 \leq j \leq k$ such that $p_j^{nn_j} \in I = Rp_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$, which is impossible. Thus, $k = 1$, that is, $I = Rp_1^{n_1}$. Now by Theorem 2.2(iii), I is a strongly irreducible ideal.

(ii) Let X be a generator set for a strongly irreducible ideal I of R. If $0 \neq x \in X$, then let $x = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$ be a prime decomposition for x. By Theorem 2.2(ii), for some i, $p_i^{n_i} \in I$, and obviously $Rx \subseteq Rp_i^{n_i}$. Thus, I can be generated by a set of prime powers. $\square$

EXAMPLE 1. Let K be a field and $R = K[x, y]$, where x and y are independent indeterminates. Then R is a Noetherian UFD. If $I = \langle x, y^2 \rangle$, then I is a primary ideal, since $\sqrt{\langle x, y^2 \rangle} = \langle x, y \rangle$, which is a maximal ideal of $K[x, y]$. Furthermore, I is generated by a set of prime powers. Note that $(x + y)y \in I$, $x + y \notin I$ and $y \notin I$. So, by Theorem 2.2(ii), I is not a strongly irreducible ideal.

3. Localization and strongly irreducible ideals

Let R be a ring and let S be a multiplicatively closed subset of R. For each ideal I of the ring $S^{-1}R$, we consider

$$I^c = \{x \in R \mid x/1 \in I\} = I \cap R, \quad \text{and} \quad C = \{I^c \mid I \text{ is an ideal of } S^{-1}R\}.$$

THEOREM 3.1. *Let R be a ring and S be a multiplicatively closed subset of R. Then there is a one-to-one correspondence between the strongly irreducible ideals of $S^{-1}R$ and strongly irreducible ideals of R contained in C which do not meet S.*

PROOF. Let I be a strongly irreducible ideal of $S^{-1}R$. Obviously, $I^c \neq R$, $I^c \in C$ and $I^c \cap S = \emptyset$. Let $A \cap B \subseteq I^c$, where A and B are ideals of R. Then we have $(S^{-1}A) \cap (S^{-1}B) = S^{-1}(A \cap B) \subseteq S^{-1}(I^c) = I$. Hence, $S^{-1}A \subseteq I$ or $S^{-1}B \subseteq I$, and so $A \subseteq (S^{-1}A)^c \subseteq I^c$ or $B \subseteq (S^{-1}B)^c \subseteq I^c$. Thus, I^c is a strongly irreducible ideal of R.

Conversely, let I be a strongly irreducible ideal of R, $I \cap S = \emptyset$ and $I \in C$. Since $I \cap S = \emptyset$, $S^{-1}I \neq S^{-1}R$. Let $A \cap B \subseteq S^{-1}I$, where A and B are ideals of $S^{-1}R$. Then $A^c \cap B^c = (A \cap B)^c \subseteq (S^{-1}I)^c$. Now since $I \in C$, $(S^{-1}I)^c = I$. So $A^c \cap B^c \subseteq I$. Consequently, $A^c \subseteq I$ or $B^c \subseteq I$. Thus, $A = S^{-1}(A^c) \subseteq S^{-1}I$ or $B = S^{-1}(B^c) \subseteq S^{-1}I$. Therefore, $S^{-1}I$ is a strongly irreducible ideal of $S^{-1}R$. $\square$

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

COROLLARY 3.2. *Let R be a ring and let S be a multiplicatively closed subset of R.*

(i) *If I is a strongly irreducible and a primary ideal of R which does not meet S, then $S^{-1}I$ is a strongly irreducible (and a primary) ideal of $S^{-1}R$.*

(ii) *If R is a UFD, a Noetherian ring or an absolutely flat ring, then there is a one-to-one correspondence between the strongly irreducible ideals of $S^{-1}R$ and the strongly irreducible ideals of R which do not meet S.*

PROOF. (i) Since I is a primary ideal of R and $I \cap S = \emptyset$, $(S^{-1}I)^c = I$, then $I \in C$. Now, by Theorem 3.1, $S^{-1}I$ is a strongly irreducible ideal of $S^{-1}R$.

(ii) By Theorem 2.1(iii) and (iv), and by Theorem 2.2(iv), in UFDs, Noetherian rings and absolutely flat rings, every strongly irreducible ideal I is a primary ideal. Now, by part (i), $S^{-1}I$ is a strongly irreducible ideal of $S^{-1}R$. Also, by Theorem 3.1, for each strongly irreducible ideal I of $S^{-1}R$, I^c is a strongly irreducible ideal of R which does not meet S. □

The next proposition shows the importance of the rings for which every primary ideal is a strongly irreducible ideal.

PROPOSITION 3.3. *If R is a ring for which every primary ideal is a strongly irreducible ideal, then every minimal primary decomposition for each ideal of R is unique.*

PROOF. Let $A = \bigcap_{i=1}^{n} q_i = \bigcap_{i=1}^{m} q'_i$, be two minimal primary decompositions for an ideal A of R. Let $n \le m$. We have $\bigcap_{i=1}^{n} q_i \subseteq q'_1$ and, since q'_1 is a strongly irreducible ideal, for some j, $1 \le j \le n$, $q_j \subseteq q'_1$. On the other hand, $\bigcap_{i=1}^{m} q'_i \subseteq q_j$. Since q_j is a strongly irreducible ideal, for some k, $1 \le k \le m$, we have $q'_k \subseteq q_j \subseteq q'_1$. Since $\bigcap_{i=1}^{m} q'_i$ is a minimal primary decomposition, $q'_k = q'_1$ and so $k = 1$. Hence, $q'_1 = q_j$. Without loss of generality, let $q'_1 = q_1$. Similarly we can show that $q'_2 = q_t$ for some t, $1 \le t \le n$, and since $q'_2 \ne q'_1$, $q_t \ne q_1$. That is, $t \ne 1$. Therefore, without loss of generality, we can assume that $q'_2 = q_2$. The same argument will show that for each t, $1 \le t \le m$, $q'_i = q_i$ and $n = m$. □

The following result will help us to find the rings for which every primary ideal is a strongly irreducible ideal.

LEMMA 3.4. *If R is a ring, then the following are equivalent.*

(i) *Every primary ideal of R is a strongly irreducible ideal.*

(ii) *For any prime ideal P of R, every primary ideal of R_P is a strongly irreducible ideal.*

(iii) *For any maximal ideal P of R, every primary ideal of R_P is a strongly irreducible ideal.*

PROOF. (i) $\Rightarrow$ (ii). Let I be a primary ideal of R_P. We know that I^c is a primary ideal of R, $I^c \cap (R \setminus P) = \emptyset$, $I^c \in C$ and, by the assumption, I^c is a strongly irreducible ideal of R. Now, by Theorem 3.1, $I = (I^c)_P$ is a strongly irreducible ideal of R_P.

(ii) $\Rightarrow$ (iii). The proof is clear.

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

(iii) $\Rightarrow$ (i). Let I be a primary ideal of R and let m be a maximal ideal of R containing I. Then, I_m is a primary ideal of R_m and so, by our assumption, I_m is a strongly irreducible ideal of R_m. Now by Theorem 3.1, $(I_m)^c$ is a strongly irreducible ideal of R, and since I is a primary ideal of R, $(I_m)^c = I$, that is, I is a strongly irreducible ideal of R. $\square$

LEMMA 3.5. *If R is a ring, then the following are equivalent.*
(i) *Every proper ideal of R is a strongly irreducible ideal.*
(ii) *Every two ideals of R are comparable.*

PROOF. (i) $\Rightarrow$ (ii). Let I and J be two proper ideals of R. Note that $I \cap J$ is a strongly irreducible ideal, and $I \cap J \subseteq I \cap J$. So $I \subseteq I \cap J \subseteq J$ or $J \subseteq I \cap J \subseteq I$.

(ii) $\Rightarrow$ (i). The proof is obvious. $\square$

Recall that a ring R is said to be an arithmetical ring, if for all ideals I, J and K of R, $I + (J \cap K) = (I + J) \cap (I + K)$. (See [2] or [3].)

LEMMA 3.6. *A ring R is arithmetical if and only if for each maximal ideal P of R, every two ideals of the ring R_P are comparable.*

PROOF. See [3, Theorem 1]. $\square$

Recall that a ring R is said to be a Zerlegung Primideale ring (ZPI-ring) if every proper ideal of R can be written as a product of prime ideals of R. Note that ZPI-rings, almost multiplication rings, Prüfer domains and almost Dedekind domains are generalizations of Dedekind domains (see [4, Chapters VI and IX]).

THEOREM 3.7. *Let R be a ring.*
(i) *Every localization of a ZPI-ring is a ZPI-ring.*
(ii) *In a local ZPI-ring, every two ideals are comparable.*
(iii) *In an arithmetical ring, every primary ideal is a strongly irreducible ideal.*
(iv) *If R is a ZPI-ring, then an ideal is a strongly irreducible ideal if and only if it is a primary ideal.*
(v) *If R is an almost multiplication ring, a Prüfer domain or an almost Dedekind domain, then every primary ideal is a strongly irreducible ideal.*

PROOF. (i) Let S be a multiplicatively closed subset of R and let J be a proper ideal of $S^{-1}R$. Since R is a ZPI-ring, $J^c = P_1 P_2 \ldots P_n$, where P_i are prime ideals of R. Then, for each i, $S^{-1}(P_i) = S^{-1}R$ or $S^{-1}(P_i)$ is a prime ideal of $S^{-1}R$. So, $J = S^{-1}(J^c)$ can be written as a product of prime ideals of $S^{-1}R$, that is, $S^{-1}R$ is a ZPI-ring.

(ii) Let R be a ZPI-ring and let m be a maximal ideal of R. By [4, Theorem 9.10], R is a Noetherian ring. If $m^2 = m$, then by Nakayama's lemma $m = 0$, so R is a field, and the proof is completed.

If $m^2 \neq m$, let $x \in m \setminus m^2$. Then $m^2 \subset m^2 + Rx \subseteq m$. By [4, Theorem 9.10], there are no ideals of R strictly between m^2 and m. So $m^2 + Rx = m$ and by Nakayama's lemma, $m = Rx$.

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

Now let P be a nonzero prime ideal of R and $0 \neq y \in P$. By the Krull intersection theorem, we have $\bigcap_{n=1}^{+\infty} m^n = 0$. Thus, there is a natural number n such that $y \in m^n$ and $y \notin m^{n+1}$. Since $y \in m^n = Rx^n$, there exists an element $u \in R$ such that $y = ux^n$, and since $y \notin m^{n+1}$, $u \notin m$. Then u is a unit element of R. Hence, $x^n = u^{-1}y \in P$. We know that P is a prime ideal of R, so $x \in P$, that is $m = P$. So m is the only nonzero prime ideal of R and since every ideal is a product of prime ideals, every proper ideal of R is a power of m. Therefore, every two ideals of R are comparable.

(iii) Let R be an arithmetical ring. By Lemma 3.6, for every maximal ideal P of R, every two ideals of R_P are comparable. So, by Lemma 3.5, every proper ideal of R_P and obviously every primary ideal of R_P is a strongly irreducible ideal of R_P. Then, by Lemma 3.4, every primary ideal of R is strongly irreducible ideal.

(iv) According to [4, Theorem 9.10], every ZPI-ring is a Noetherian ring. Then R is a Noetherian ring and so, by Theorem 2.1(iii), every strongly irreducible ideal is a primary ideal.

By part (i), for every prime ideal P of R, R_P is a local ZPI-ring and, by part (ii), every two ideals of R_P are comparable. Hence, by Lemma 3.6, R is an arithmetical ring and, by part (iii), every primary ideal of R is a strongly irreducible ideal.

(v) Let R be an almost multiplication ring. By [4, Theorem 9.23], for every prime ideal P of R, R_P is a ZPI-ring. Thus, by part (iv), every primary ideal of R_P is a strongly irreducible ideal of R_P. Now, by Lemma 3.4, every primary ideal of R is a strongly irreducible ideal of R.

If R is a Prüfer domain, then, by [4, Corollary 6.7], for every maximal ideal P of R, R_P is a valuation ring. Hence, by Lemma 3.6, R is an arithmetical ring. Now, by part (iii), every primary ideal of R is a strongly irreducible ideal.

If R is an almost Dedekind domain, then for every maximal ideal P of R, R_P is a valuation ring (see [4, p. 201]). Hence, by Lemma 3.6, R is an arithmetical ring and, by part (iii), every primary ideal of R is a strongly irreducible ideal. □

4. Zariski topology

If R is a ring, we consider $\mathrm{SSpec}(R)$ to be the set of all strongly irreducible ideals of R. We call $\mathrm{SSpec}(R)$, the strongly irreducible spectrum of R. In this section, a topology on $\mathrm{SSpec}(R)$ is introduced. This topology is defined exactly similar to the Zariski topology defined by prime ideals, and the set of prime ideals ($\mathrm{Spec}(R)$) is a subspace topology of $\mathrm{SSpec}(R)$.

If I is an ideal of R, we define the variety of I, denoted by $\mathcal{SV}(I)$, as

$$\mathcal{SV}(I) = \{P \in \mathrm{SSpec}(R) \mid I \subseteq P\}.$$

Also for each $a \in R$ and for each ideal I of R, let

$$\mathcal{OP}(I) = \{P \in \mathrm{SSpec}(R) \mid I \nsubseteq P\}, \quad \mathcal{OP}(a) = \mathcal{OP}(Ra).$$
$$\mathcal{ST}(R) = \{\mathcal{OP}(I) \mid I \text{ is an ideal of } R\}.$$

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

THEOREM 4.1. *Let R be a ring and $a, b \in R$.*
(i) $\mathcal{ST}(R)$ *is a topology on* $\mathrm{SSpec}(R)$.
(ii) *If* $\mathcal{OP}(a) = \emptyset$, *then a is a nilpotent element of R.*
(iii) $\mathcal{OP}(a) = \mathrm{SSpec}(R)$ *if and only if a is a unit element of R.*
(iv) *For each pair of ideals I and J of R, if* $\mathcal{OP}(I) = \mathcal{OP}(J)$, *then* $\sqrt{I} = \sqrt{J}$.
(v) $\mathrm{SSpec}(R)$ *is a quasi-compact topological space.*
(vi) $\mathrm{SSpec}(R)$ *is a T_0 topological space.*

PROOF. (i) Note that $\mathcal{OP}(0) = \emptyset$ and $\mathcal{OP}(R) = \mathrm{SSpec}(R)$.

Let $\{I_\alpha\}_{\alpha\in J}$ and $\{I_k\}_{k=1}^n$ be two families of ideals of R. We show that

$$\text{(a)} \quad \bigcup_{\alpha\in J} \mathcal{OP}(I_\alpha) = \mathcal{OP}\Big(\sum_{\alpha\in J} I_\alpha\Big).$$

$$\text{(b)} \quad \bigcap_{k=1}^{n} \mathcal{OP}(I_k) = \mathcal{OP}\Big(\bigcap_{k=1}^{n} I_k\Big).$$

(a) $P \in \bigcup_{\alpha\in J} \mathcal{OP}(I_\alpha) \Longleftrightarrow I_{\alpha_0} \nsubseteq P$, for some $\alpha_0 \in J \Longleftrightarrow \sum_{\alpha\in J} I_\alpha \nsubseteq P \Longleftrightarrow P \in \mathcal{OP}(\sum_{\alpha\in J} I_\alpha)$.

(b) $P \in \bigcap_{k=1}^{n} \mathcal{OP}(I_k) \Longleftrightarrow I_k \nsubseteq P$, for each k, $1 \le k \le n$.

Now since P is a strongly irreducible ideal,

$$I_k \nsubseteq P \quad \text{for each } k,\ 1 \le k \le n \Longleftrightarrow \bigcap_{k=1}^{n} I_k \nsubseteq P \Longleftrightarrow P \in \mathcal{OP}\Big(\bigcap_{k=1}^{n} I_k\Big).$$

(ii) If $\mathcal{OP}(a) = \emptyset$, then a is in every strongly irreducible ideal. So a is in every prime ideal, hence, a is a nilpotent element.

(iii) $\mathcal{OP}(a) = \mathrm{SSpec}(R) \Longleftrightarrow a \notin P$, for all $P \in \mathrm{SSpec}(R) \Longrightarrow a \notin P$, for all maximal ideal $P \Longrightarrow a$ is a unit.

Conversely if a is a unit, then obviously a is not in any strongly irreducible ideal. That is, $\mathcal{OP}(a) = \mathrm{SSpec}(R)$.

(iv) Suppose that $\mathcal{OP}(I) = \mathcal{OP}(J)$. Since every prime ideal is a strongly irreducible ideal, every prime ideal of R containing I is also a prime ideal of R containing J, and vice versa. Therefore, $\sqrt{I} = \sqrt{J}$.

(v) Let $\mathrm{SSpec}(R) = \bigcup_{\alpha\in J} \mathcal{OP}(I_\alpha)$, where $\{I_\alpha\}_{\alpha\in J}$ is a family of ideals of R. By equation (a) in the proof of part (i), we have $\mathcal{OP}(R) = \mathrm{SSpec}(R) = \mathcal{OP}(\sum_{\alpha\in J} I_\alpha)$. Thus, by part (iv), we have $R = \sqrt{R} = \sqrt{\sum_{\alpha\in J} I_\alpha}$ and, hence, $1 \in \sum_{\alpha\in J} I_\alpha$. So there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in J$ such that $1 \in \sum_{i=1}^{n} I_{\alpha_i}$, that is, $R = \sum_{i=1}^{n} I_{\alpha_i}$. Consequently $\mathrm{SSpec}(R) = \mathcal{OP}(R) = \mathcal{OP}(\sum_{i=1}^{n} I_{\alpha_i}) = \bigcup_{i=1}^{n} \mathcal{OP}(I_{\alpha_i})$.

(vi) Let P_1, P_2 be two distinct points of $\mathrm{SSpec}(R)$. If $P_1 \nsubseteq P_2$, then obviously $P_2 \in \mathcal{OP}(P_1)$ and $P_1 \notin \mathcal{OP}(P_1)$. □

Recall that a topological space is said to be Noetherian if its closed sets satisfy the descending chain condition. Also a nonempty closed set V is said to be irreducible

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press

(reducible) if V cannot (can) be written as the union of two distinct closed sets (see [5, p. 29, Exercises 4.9 and 4.10]).

PROPOSITION 4.2. *Let R be a ring.*

(i) *If R is a Noetherian ring, then* $\mathrm{SSpec}(R)$ *is a Noetherian topological space.*
(ii) *If I is an ideal of R such that $\mathcal{SV}(I)$ is an irreducible closed set, then there exists an irreducible ideal J of R such that $\mathcal{SV}(I) = \mathcal{SV}(J)$.*
(iii) *If I is an ideal of R and* $\mathrm{SSpec}(R)$ *is a Noetherian topological space, then $\mathcal{SV}(I)$ can be written as a finite union of irreducible closed sets $\mathcal{SV}(I_k)$, $k = 1, 2, \dots, n$ such that for each k, I_k is an irreducible ideal of R.*
(iv) *If I is an ideal of a Noetherian ring R, then $\mathcal{SV}(I)$ can be written as a finite union of irreducible closed sets $\mathcal{SV}(I_k)$, $k = 1, 2, \dots, n$ such that for each k, I_k is an irreducible ideal of R.*

PROOF. (i) For each ideal I of R, we consider

$$\sqrt[3]{I} = \cap\{P : P \text{ is a strongly irreducible ideal of } R \text{ containing } I\}.$$

According to the definition of $\mathcal{SV}(I)$, we have $\mathcal{SV}(I) = \mathcal{SV}(\sqrt[3]{I})$. Also for each ideal J of R, $\mathcal{SV}(I) \subseteq \mathcal{SV}(J)$ if and only if $\sqrt[3]{J} \subseteq \sqrt[3]{I}$.

Now let $\dots \subseteq \mathcal{SV}(I_3) \subseteq \mathcal{SV}(I_2) \subseteq \mathcal{SV}(I_1)$ be a chain of closed sets of $\mathrm{SSpec}(R)$, where $\{I_k\}_{k=1}^{+\infty}$ is a family of ideals of R. These inclusions imply $\sqrt[3]{I_1} \subseteq \sqrt[3]{I_2} \subseteq \sqrt[3]{I_3} \dots \subseteq$, and since R is a Noetherian ring, there exists a positive number n such that for each positive number $m \geq n$, $\sqrt[3]{I_n} = \sqrt[3]{I_m}$. Consequently

$$\mathcal{SV}(I_n) = \mathcal{SV}(\sqrt[3]{I_n}) = \mathcal{SV}(\sqrt[3]{I_m}) = \mathcal{SV}(I_m),$$

which completes the proof.

(ii) Let $S = \{L \mid L \text{ is an ideal of } R \text{ such that } \mathcal{SV}(L) = \mathcal{SV}(I)\}$. By Zorn's lemma S has a maximal element J. If J is reducible, let $J = J_1 \cap J_2$, where J_1 and J_2 are ideals of R. Then $\mathcal{SV}(I) = \mathcal{SV}(J) = \mathcal{SV}(J_1 \cap J_2) = \mathcal{SV}(J_1) \cup \mathcal{SV}(J_2)$. Since $\mathcal{SV}(I)$ is irreducible, let $\mathcal{SV}(I) = \mathcal{SV}(J_1)$. This is impossible, since J is a maximal element of S and $J \subset J_1$.

(iii) According to [5, Exercise 4.11], in a Noetherian topological space every closed subset can be written as a union of finitely many irreducible closed sets. Now by part (ii), the proof is completed.

(iv) Part (i) applies to show that $\mathrm{SSpec}(R)$ is a Noetherian topological space. Now the assertion follows from part (iii). □

Acknowledgement

I would like to thank the referee for many helpful comments on this paper.

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* (Addison-Wesley, Reading, MA, 1969).

[2] W. J. Heinzer, L. J. Ratliff Jr and D. E. Rush, 'Strongly irreducible ideals of a commutative ring', *J. Pure Appl. Algebra* **166** (2002), 267–275.

[3] C. Jensen, 'Arithmetical rings', *Acta Math. Sci. Acad. Sci. Hungar.* **17** (1966), 115–123.

[4] M. D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals* (Academic Press, New York, 1971).

[5] H. Matsumura, *Commutative Ring Theory* (Cambridge University Press, Cambridge, 1992).

A. AZIZI, Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran
e-mail: aazizi@shirazu.ac.ir

https://doi.org/10.1017/S1446788708000062 Published online by Cambridge University Press