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Slopes of overconvergent 2-adic modular forms

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Abstract

We explicitly compute all the slopes of the Hecke operator U_2 acting on overconvergent 2-adic level 1 cusp forms of weight 0: the nth slope is $1 + 2v((3n)!/n!)$, where v denotes the 2-adic valuation. We formulate an explicit conjecture about what these slopes should be for weight k forms.

1. Introduction

Let p be a prime, and let N be a positive integer coprime to p. Let $M_k(\Gamma_1(N); \mathbb{Q}_p)$ denote the weight k modular forms of level $\Gamma_1(N)$ defined over $\mathbb{Q}_p$. In recent years, work of Coleman and others (for example [Col97a, Col96, Col97b, CST98, CM98]) has shown that a very profitable way of studying this finite-dimensional $\mathbb{Q}_p$-vector space is to choose a small positive rational number r and then to embed $M_k(\Gamma_1(N); \mathbb{Q}_p)$ into a (typically infinite-dimensional) p-adic Banach space $\mathbf{M}_k(\Gamma_1(N); \mathbb{Q}_p; p^{-r})$ of p^{-r}-overconvergent p-adic modular forms, that is, sections of $\omega^{\otimes k}$ on the affinoid subdomain of $X_1(N)$ obtained by removing certain open discs of radius p^{-r} above each supersingular point in characteristic p (at least if $N \geqslant 5$; see the Appendix for how to deal with the cases $N \leqslant 4$). The space $\mathbf{M}_k(\Gamma_1(N); \mathbb{Q}_p; p^{-r})$, for $0 < r < p/(p+1)$, comes equipped with canonical continuous Hecke operators, and one of them, namely the operator $U := U_p$, has the property of being compact; in particular U has a spectral theory. Coleman exploited this theory in [Col97a] to prove weak versions of conjectures of Gouvêa and Mazur on families of modular forms.

One of us (KB) has made, in many cases, considerably more precise conjectures [Buz04] than those of Gouvêa and Mazur, predicting the slopes of U, that is, the valuations of all the non-zero eigenvalues of U. These conjectures are very explicit, and display a hitherto unexpected regularity. However, they have the disadvantage of being rather inelegant. See also the forthcoming PhD thesis of Graham Herrick (Northwestern University), who has, perhaps, more conceptual conjectures about these slopes.

We present here a very concrete conjecture in the case $N = 1$ and $p = 2$, which presumably agrees with the conjectures in [Buz04] but which has the advantage of being much easier to understand and compute. Let $S_k := S_k(\Gamma_0(1), \mathbb{Q})$ denote the level 1 cusp forms of weight k. If $F(X)$ is a polynomial with rational coefficients, then by its 2-adic Newton polygon we mean its Newton polygon when considered as a polynomial with 2-adic coefficients.

Conjecture 1. *Let $k \geqslant 12$ be even, and let $m = \dim S_k$. Then the 2-adic Newton polygon of $\det(1 - XT_2)$ on S_k equals the 2-adic Newton polygon of*

$$1 + \sum_{n=1}^{m} X^n \prod_{j=1}^{n} \frac{2^{2j}(k-8j)!(k-8j-3)!(k-12j-2)}{(k-12j)!(k-6j-1)!}.$$

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This conjecture can be verified numerically, and we have verified it for all $k \leqslant 2048$. Control theorems of Coleman [Col96] imply that complete knowledge of all slopes of classical cusp forms at level 2 for all weights is equivalent to complete knowledge of all slopes of finite overconvergent tame level 1 cusp forms for all integer weights. In fact, it is a little tedious but completely elementary to show that we may reformulate Conjecture 1 as follows. Let $\mathbf{S}_k := \mathbf{S}_k(\Gamma_0(1); \mathbb{Q}_2; 2^{-1/2})$ denote the $2^{-1/2}$-overconvergent forms of weight k and tame level 1.

Conjecture 2. *Let $k \leqslant 0$ be an integer. Then the Newton polygon of* $\det(1 - XU)$ *on* $\mathbf{S}_k$ *is the Newton polygon of*

$$1 + \sum_{n=1}^{\infty} X^n \prod_{j=1}^{n} \frac{2^{2j}(-k+2+12j)!(-k+6j)!}{(-k+2+8j)!(-k-2+8j)!(-k-12j)}.$$

One can also find a form of this conjecture that makes sense if $k > 0$, for example if one reformulates the factorials as Gamma-functions and then is careful to make precise what is happening at poles. We leave this reformulation to the reader.

As evidence for this conjecture, we have the following theorem.

Theorem 1. *Conjecture 2 is true when $k = 0$.*

If x is a non-zero rational number, then by its *slope* we mean its 2-adic valuation $v_2(x)$. It is easy to check that for $n \geqslant 0$ an integer we have $v_2\left((2n)!\right) = n + v_2(n!)$, and it follows from this that

$$v_2\left(\frac{2^{2j}(12j+2)!(6j)!}{(8j+2)!(8j-2)!(12j)}\right) = 1 + 2v_2\left(\frac{(3j)!}{j!}\right).$$

Similarly one can check that if D denotes the infinite diagonal matrix whose (j,j)th entry, $j \geqslant 1$, is given by

$$d_{j,j} = \frac{2^{4j+1}(3j)!^2 j!^2}{3(2j)!^4},$$

then $v_2(d_{j,j}) = 1 + 2v_2((3j)!/j!)$. Hence Theorem 1 above is equivalent to the following theorem.

Theorem 2. *The Newton polygons of* $\det(1 - XU)$ *on* $\mathbf{S}_0$ *and* $\det(1 - XD)$ *coincide.*

This is the form of the theorem that we shall actually prove.

Note that the sequence $v_2((3j)!/j!)$ is strictly increasing; this follows from the fact that

$$\frac{(3j+3)!/(j+1)!}{(3j)!/j!} = 3(3j+2)(3j+1)$$

is even for all j. We deduce the following corollary.

Corollary 1. *Let $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots$ be the non-zero eigenvalues (with multiplicities) of U on $\mathbf{S}_0$. Then the slope of λ_n is given by the following formula:*

$$v_2(\lambda_n) = 1 + 2v_2\left(\frac{(3n)!}{(n)!}\right).$$

In particular, the slopes are all distinct, and are all positive odd integers. We also have another corollary.

Corollary 2. *Let $f = q + \cdots \in \mathbf{S}_0 \widehat{\otimes} \mathbb{C}_2$ be a normalised finite slope overconvergent eigenform. Then the coefficients of f are all in $\mathbb{Q}_2$.*

Proof. We use only that the slopes of the non-zero eigenvalues of U are distinct. If λ denotes the eigenvalue of U on f then $1-\lambda^{-1}U$ is not invertible on $\mathbf{S}_0 \widehat{\otimes} \mathbb{C}_2$ and hence by Proposition 11 of [Ser62] we see that λ^{-1} is a zero of the characteristic power series $P(T)$ of U acting on $\mathbf{S}_0$. Note that $P(T) \in \mathbb{Q}_2[[T]]$. Choosing some big affinoid disc containing λ^{-1} and applying the Weierstrass preparation theorem to $P(T)$ shows that λ^{-1} is a root of a polynomial with coefficients in $\mathbb{Q}_2$. Hence $\lambda \in \overline{\mathbb{Q}}_2$. Now all the Galois conjugates of λ have the same valuation and are also roots of the characteristic power series of U; hence, by Corollary 1, $\lambda \in \mathbb{Q}_2$. Finally by Proposition 12 of [Ser62] the subspace of $\mathbf{S}_0$ where U acts as multiplication by λ is one-dimensional over $\mathbb{Q}_2$ and so all the eigenvalues of all the other Hecke operators are also in $\mathbb{Q}_2$. □

Remark. This corollary provides some evidence towards Question 4.3 of [Buz04]. See also Corollary 1.2 of [Kil02].

Note that for $p = 2$ and $N = 1$, the map θ induces a bijection between overconvergent eigenforms of weight 0 and weight 2. Thus, the slopes in weight 2 are precisely each of the slopes in weight 0, plus one.

We have also proved Conjecture 2 for $k = -12$ using similar methods, although the combinatorics are too painful to write here, and the arguments do not seem to generalise to all k.

Lawren Smithline was perhaps the first person to observe that there was some structure in the slopes of overconvergent modular forms of small level; his results [Smi00] were primarily for the prime $p = 3$ but some of the techniques used in this paper for studying the explicit matrix representing U were inspired by his ideas. As far as we know, the first people to get explicit results pinning down all overconvergent slopes at a given weight were L. Kilford [Kil02] and D. Jacobs [Jac03], but their results differ in two respects from ours: Firstly, they consider points nearer the boundary of weight space. Secondly, the slopes at the weights they consider have a much simpler pattern; they form an arithmetic progression.

2. Weight zero

The curve $X_0(2)$ has genus 0. A natural choice of uniformiser is given by the following function (Hauptmodul):

$$f(\tau) = \Delta(2\tau)/\Delta(\tau).$$

In the sequel, we shall write this function simply as f. There is a product formula for f:

$$f = q\prod_{n=1}^{\infty}(1+q^n)^{24} = q\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})^{24}} = q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + 105504q^6 + \cdots,$$

which follows immediately from the usual product formula for Δ. For $p = 2$, f is overconvergent of weight 0 and level 1. Furthermore, if $g = 2^6 f$ then the set $\{1, g, g^2, g^3, \dots\}$ is a Banach basis for the space $\mathbf{M}_0 := \mathbf{M}_0(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q}_2; 2^{-1/2})$. It seems a little difficult to extract these concrete statements from the literature and so we sketch a proof of this in the Appendix, and note that these ideas should easily be adaptable to cover other cases where one might want to do explicit computations. The reader who is happy to accept that the formal Banach space with basis $\{1, g, g^2, g^3, \dots\}$ is some kind of p-adic space of modular forms might well want to avoid these technical details.

To determine the spectral theory of U on $\mathbf{M}_0$, we shall explicitly compute a matrix for U acting on $\mathbf{M}_0$. These calculations are much in the spirit of classical congruences for coefficients of modular functions such as j; see for example Watson [Wat38], or Atkin and O'Brien [AO67]. In this optic, the *a fortiori* presence of a spectral theory greatly simplifies matters. Concretely then, our task is to compute $U(f^k)$ as a power series in f, for all $k \geqslant 0$.

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Lemma 1. *The following identities are satisfied:*

$$U(f) = 24f + 2^{11}f^2,$$
$$f\left(\frac{\tau}{2}\right)f\left(\frac{\tau+1}{2}\right) = -f(\tau).$$

Proof. The operator U preserves the space of functions on $X_0(2)$. Furthermore $U(f)$, considered as a map $X_0(2) \to \mathbb{P}^1$, has degree at most 2, and hence $U(f) - 24f - 2^{11}f^2$, if non-zero, is a function $X_0(2) \to \mathbb{P}^1$ of degree at most 4. Hence one can check that this function is identically zero by computing the first few terms of its q-expansion and verifying that they are zero. The second identity follows similarly, or directly from the product formulas:

$$\begin{aligned} f(\tau)f(\tau+\tfrac{1}{2}) &= q\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})^{24}} \times (-q)\prod_{n=1}^{\infty}(1+(-q)^n)^{24} \\ &= -q^2\prod_{n=1}^{\infty}\frac{(1-q^{2n-1})^{24}(1+q^{2n})^{24}}{(1-q^{2n-1})^{24}} = -q^2\prod_{n=1}^{\infty}(1+q^{2n})^{24} = -f(2\tau). \end{aligned}$$ □

Using Lemma 1, we may recursively determine $U(f^k)$ for all positive k. To do this, we observe (by definition) that

$$2U(f^k) = f\left(\frac{\tau}{2}\right)^k + f\left(\frac{\tau+1}{2}\right)^k.$$

Thus, if $X_k := U(f^k)$, then, multiplying out, one sees that the X_k satisfy the recurrence relation:

$$X_{k+2} - \left(f\left(\frac{\tau}{2}\right) + f\left(\frac{\tau+1}{2}\right)\right)X_{k+1} + f\left(\frac{\tau}{2}\right)f\left(\frac{\tau+1}{2}\right)X_k = 0 \quad (k \geqslant 0).$$

Moreover, from Lemma 1, we may evaluate the coefficients of this recurrence to conclude that $X_0 = 1$, $X_1 = 24f + 2^{11}f^2$ and, for $k \geqslant 2$,

$$X_k = U(f^k) = (48f + 2^{12}f^2)U(f^{k-1}) + fU(f^{k-2}).$$

In particular, we note that $U(f^k)$ is a polynomial in f with integer coefficients and of degree at most $2k$. These results are in Emerton's thesis [Eme98] and apparently are originally due to Kolberg [Kol61].

Definition 1. Define integers $s_{i,j}$, $i, j \in \mathbb{Z}_{\geqslant 0}$, by

$$U(f^j) = \sum_{i=0}^{\infty} s_{i,j} f^i.$$

Note that $s_{i,j} = 0$ for $i > 2j$. We also note that $s_{i,j} = 0$ for $j > 2i$, by comparing the coefficients of q^i in the definition of $s_{i,j}$. We have $s_{0,0} = 1$, $s_{1,1} = 24$, $s_{2,1} = 2^{11}$, $s_{1,2} = 1$, and $s_{i,j} = 0$ in all other cases with $0 \leqslant i \leqslant 1$ or $0 \leqslant j \leqslant 1$.

Lemma 2. *The integers $s_{i,j}$ satisfy the recurrence relation:*

$$s_{i,j} = 48s_{i-1,j-1} + 2^{12}s_{i-2,j-1} + s_{i-1,j-2} \quad (i, j \geqslant 2).$$

Moreover, for $i, j \geqslant 1$ and $i \leqslant 2j$, $j \leqslant 2i$ we have an equality:

$$s_{i,j} = \frac{3j(i+j-1)!\, 2^{8i-4j-1}}{(2i-j)!(2j-i)!}.$$

Proof. The recurrence for $s_{i,j}$ follows directly from the recurrence for $U(f^k)$. The explicit formula also satisfies this same recurrence and moreover equals $s_{i,j}$ for $s_{i,1}$, $s_{i,2}$, $s_{1,j}$ and $s_{2,j}$. This suffices to prove the second equality. □

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The constant function 1 is an eigenform for U with eigenvalue 1. Thus to determine the spectral theory of U on $\mathbf{M}_0$ it suffices to work with the cuspidal subspace

$$\mathbf{S}_0 := \mathbf{S}_0(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q}_2; 2^{-1/2})$$

of q-expansions with zero constant term. In the p-adic setting, *cuspidal* generalises the notion of vanishing at the single cusp ∞; thus certain Eisenstein series (such as the twin form of the usual Eisenstein series E_{2k}) are considered cuspidal. Lemma 2 provides us with an explicit description of the action of U on $\mathbf{S}_0$. This allows us to gain fine control over the spectrum of U on $\mathbf{S}_0$.

Before we begin the proof of Theorem 2, we recall some elementary facts about continuous endomorphisms of Banach spaces. If M is a Banach space over $\mathbb{Q}_p$, and $\{e_1, e_2, e_3, \dots\}$ is a countable subset of M, then we say that $\{e_1, e_2, e_3, \dots\}$ is *an orthonormal Banach basis for M* if

(a) $|e_i| = 1$ for all i,

(b) every $m \in M$ can be written uniquely as $m = \sum_{i \geqslant 1} a_i e_i$ for a sequence $a_i \in \mathbb{Q}_p$ such that $a_i \to 0$ as $i \to \infty$, and

(c) if m, a_i are as above, then $|m| = \max_i |a_i|$.

If M is a Banach space and $\{e_1, \dots\}$ is an orthonormal Banach basis for M, and if $\phi : M \to M$ is a continuous $\mathbb{Q}_p$-linear map, then we define *the matrix of ϕ* to be the collection $(c_{i,j})_{i,j \geqslant 1}$ such that $\phi(e_j) = \sum_j c_{i,j} e_i$. The collection $(c_{i,j})$ has the following two properties:

(i) for all j, $\lim_{i \to \infty} c_{i,j} = 0$;

(ii) there exists some $C \in \mathbb{R}$ such that $|c_{i,j}| \leqslant C$ for all i, j.

Conversely, given a collection $(c_{i,j})_{i,j \geqslant 1}$ satisfying (i) and (ii) above, there is a unique continuous linear map $\phi : M \to M$ with matrix $(c_{i,j})$. Composition of linear maps corresponds to multiplication of matrices using the usual formula, which one easily checks to converge because of (i) and (ii) above.

Set $r = 1/2$, let $\mathbf{M}_0$ denote the 2-adic Banach space of 2^{-r}-overconvergent 2-adic modular forms of weight 0 and tame level 1, equipped with the supremum norm, and let $\mathbf{S}_0$ denote its cuspidal subspace. We prove in the Appendix that a Banach basis for $\mathbf{M}_0$ is $\{1, g, g^2, g^3, \dots\}$ with $g = 2^6 f$; we consider $\mathbf{M}_0$ as being equipped with this basis once and for all. Moreover, $\mathbf{S}_0$ has a natural basis given by $\{g, g^2, g^3, \dots\}$. We can write the matrix of the operator U on $\mathbf{S}_0$ as $(u_{i,j})_{i,j \geqslant 1}$, where

$$u_{i,j} = 2^{6j-6i} s_{i,j} = \frac{3j(i+j-1)!\, 2^{2i+2j-1}}{(2i-j)!(2j-i)!}$$

(and where we interpret this as being zero if $i > 2j$ or $j > 2i$). Let $A = (a_{i,j})$, $B = (b_{i,j})$ and $D = (d_{i,j})$ $(i, j \geqslant 1)$ be respectively the lower triangular, upper triangular and diagonal matrices defined as follows:

$$a_{i,j} = \frac{2^{2i-2j} i!^2 (2j)!^2 (2j+i-1)!}{(2i)!(i-j)!\, j!(i+j)!(2j-i)!(3j-1)!}, \quad 2j \geqslant i \geqslant j, \quad 0 \text{ otherwise};$$

$$b_{i,j} = \frac{j}{i} \frac{2^{2j-2i} j!^2 (2i)!^2 (2i+j-1)!}{(2j)!(j-i)!\, i!(j+i)!(2i-j)!(3i-1)!}, \quad 2i \geqslant j \geqslant i; \quad 0 \text{ otherwise};$$

and

$$d_{i,i} = \frac{2^{4i+1}(3i)!^2 i!^2}{3(2i)!^4}.$$

Note the symmetry in these formulas. One has $i b_{i,j} = j a_{j,i}$, and $a_{i,i} = b_{i,i} = 1$.

Lemma 3. *The matrices defined by A, B and D satisfy (i) above, and all have coefficients in $\mathbb{Z}_2$ so also satisfy (ii) above. Moreover $A \equiv B \equiv \mathrm{Id} \mod 2$.*

Proof. That A, B and D satisfy (i) is clear because if $i > 2j$ then $a_{i,j} = b_{i,j} = d_{i,j} = 0$. Thus it suffices to prove that $A \equiv B \equiv \mathrm{Id} \mod 2$ and that D has coefficients in $\mathbb{Z}_2$. Recall that $v_2((2n)!) = n + v_2(n!)$ and from this we see that $v_2(d_{i,i}) = 1 + 2v_2((3i)!/i!) > 0$, so D has entries in $\mathbb{Z}_2$. Next we deal with A. Because $a_{i,i} = 1$ for all i, it suffices to prove that the valuation of $a_{i,j}$ is positive for $2j \geqslant i > j \geqslant 1$. We write:

$$a_{i,j} = 6ij\left(\frac{(2j)!}{2^j j!}\right)^2\left(\frac{2^i i!}{(2i)!}\right)^2\left(\frac{(2i-1)!}{(i+j)!}\right)\left(\frac{(2j+i-1)!}{(3j)!}\right)\binom{j}{i-j}.$$

Again using that $v_2((2n)!) = n + v_2(n!)$, we see that for $i > j$ the right-hand side is clearly 6 times a product of terms in $\mathbb{Z}_2$, and so lies in $2\mathbb{Z}_2$.

For B the argument is similar: Since $b_{i,i} = 1$ for all i, it suffices to prove that the valuation of $b_{i,j}$ is positive for $2i \geqslant j > i$. We write:

$$b_{i,j} = \frac{j}{i}a_{j,i} = 6j^2\left(\frac{(2i)!}{2^i i!}\right)^2\left(\frac{2^j j!}{(2j)!}\right)^2\left(\frac{(2j-1)!}{(i+j)!}\right)\left(\frac{(2i+j-1)!}{(3i)!}\right)\binom{i}{j-i},$$

and again observe that the right-hand side is 6 multiplied by a product of terms all of which lie in $\mathbb{Z}_2$. □

Hence the matrices A, B and D all define continuous endomorphisms of $\mathbf{S}_0$, which we also call A, B and D.

Lemma 4. *We have $ADB = U$.*

Proof. It suffices to show that

$$u_{i,j} = \sum_k a_{i,k} d_{k,k} b_{k,j}.$$

The right-hand side of this equation becomes, after expanding out and simplifying,

$$u_{i,j}\frac{4i!^2 j!^2(2i-j)!(2j-i)!}{(2i)!(2j)!(i+j-1)!}\sum_k \frac{(2k+i-1)!\,k(2k+j-1)!}{(i-k)!(i+k)!(j-k)!(j+k)!(2k-i)!(2k-j)!}.$$

Hence it suffices to prove that

$$\frac{(2i)!(2j)!(i+j-1)!}{4i!^2 j!^2(2i-j)!(2j-i)!} = \sum_k \frac{(2k+i-1)!\,k(2k+j-1)!}{(i-k)!(i+k)!(j-k)!(j+k)!(2k-i)!(2k-j)!}.$$

This identity, however, follows from classical results; for example, we derive it from a three-term specialisation of Dougall's ${}_7F_6$ summation formula (see [Har40, (7.2.3)]). First note that, by symmetry, we may assume that $i \leqslant j$. To force the summation to start at $k = 0$, we let $k = i - n$. Let $(a)_n = (a)(a+1)\cdots(a+n-1)$. After repeated application of the formal relations

$$(a)_n := \frac{(a+n-1)!}{(a-1)!}, \quad (-a)_n = \frac{a!(-1)^n}{(a-n)!}, \quad (-a)_n(-a+1/2)_n = \frac{(2a)!}{2^{2n}(2a-2n)!}$$

to transform our sum into hypergeometric form, the required identity becomes the following:

$$\begin{aligned}{}_7F_6&\left(\begin{matrix}(-i)/2, (1-i)/2, (j-2i)/2, (j-2i+1)/2, 1-i, -2i, -i-j\\(1-3i)/2, (2-3i)/2, (-2i+1-j)/2, (-2i+2-j)/2, -i, j-i+1\end{matrix}; 1\right)\\&= \frac{3(2i)!^2(2j)!(i+j-1)!(j-i)!(j+i)!}{4i!\,j!^2(2j-i)!(3i)!(2i+j-1)!}.\end{aligned}$$

The smallest integer in the numerator is $i/2$ or $(i-1)/2$, and we consider each case separately. Dougall's summation formula expresses this hypergeometric sum as a mélange of rising factorials that eventually simplify to the required answer. □

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An alternative method for proving our identity would be via the automated 'creative telescoping' of Zeilberger. See for example [ZE90], where Zeilberger proves Dougall's summation formula in one (rather long) line; the very short proof there specialises to a proof of the identity we require. (Note, however, that the statement of the theorem in [ZE90] contains a typographical error: the $(-1-a-b-c-d)_n$ term in the denominator should be $(-1-a+b+c+d)_n$ and the proof should be modified similarly.)

LEMMA 5. *The Newton polygon of $U = ADB$ is the same as the Newton polygon of D.*

Proof. We use only the fact that A and B are both congruent to the identity modulo 2, and that D is integral, diagonal and compact. Because B is congruent to the identity mod 2, it has an inverse. Note that, by § 5, Corollaire 2 of [Ser62], ADB has the same Newton polygon as $B(ADB)B^{-1} = BAD$, so it suffices to prove that CD has the same Newton polygon as D, for any matrix C congruent to the identity modulo 2.

If $X = (x_{i,j})_{i,j\geqslant 1}$ is a matrix, and $r_1, r_2, \ldots, r_n$ are distinct positive integers, then by the $n \times n$ *principal minor* of X associated to these integers we mean the determinant of the $n \times n$ matrix formed from the r_ith rows and columns of X, $1 \leqslant i \leqslant n$. If X is the matrix associated to a compact morphism, then the Newton polygon of X is the lower convex hull of the points $(n, \Sigma_n) \in \mathbb{R}^2$, where Σ_n is the valuation of the sum of all $n \times n$ principal minors of M.

Firstly, note that, if d is any $n \times n$ principal minor of the diagonal matrix D, then $(n, v_2(d))$ lies on or above the Newton polygon of D. Secondly, note that, for each (n, Σ_n) that lies at a vertex of the Newton polygon of D, there is a unique $n \times n$ minor with maximal valuation. Both of these facts are easily verifiable using the fact that D is diagonal. Next note that if $r_1, r_2, \ldots, r_n$ are distinct positive integers then the principal minors of D and CD associated with these integers have the same valuation, because the $n \times n$ minor of CD associated to these integers is just the product of the minor associated to C (which is a unit) and the minor associated to D. Hence all principal minors of D and CD have the same valuation and now it is easy to check that this forces the Newton polygons of C and D to be the same. $\square$

Theorem 2 follows immediately from the above lemma.

3. Extensions and generalisations

3.1 What is special about the function f?

There are many ways to parameterise a (p-adic) disc, but the choice of f that we made led to the simple formulas which enabled us to prove results about slopes. One possible reason why this f was a good choice is that the basis defined by powers of f behaves well with respect to a certain pairing, which we define below. Recall our function g defining an isomorphism of $X_0(2)$ with the projective line. Let w denote the Atkin–Lehner involution on $X_0(2)$.

LEMMA 6. *For $n \in \mathbb{Z}$ we have $w^* g^n = g^{-n}$.*

Proof. It suffices to prove the lemma for $n = 1$. Since $\Delta(-1/\tau) = \tau^{12}\Delta(\tau)$, we see that

$$w^* g = g(-1/(2\tau)) = \frac{2^6\Delta(-1/\tau)}{\Delta(-1/(2\tau))} = \frac{2^6\tau^{12}\Delta(\tau)}{2^{12}\tau^{12}\Delta(2\tau)} = \frac{1}{g}. \qquad \square$$

Let X denote the rigid affinoid annulus in $X_0(2)$ defined by $X = \{x \in X_0(2) : |g(x)| = 1\}$. This is the width-zero annulus 'in the middle' of the supersingular annulus in $2^{-6} < |g| < 2^6$ in $X_0(2)$, and the Atkin–Lehner involution w induces an involution $X \to X$. If η is a holomorphic one-form on X then we can write $\eta = (\sum_n a_n g^n)\, dg$ and we define $\int_\infty \eta := a_{-1}$. Similarly we can write $\eta = (\sum_n b_n (1/g)^n)\, d(1/g)$ and we define $\int_0 \eta := b_{-1}$. An easy check shows that $\int_0 \eta + \int_\infty \eta = 0$.

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Definition 2. Let $\langle\ ,\ \rangle$ denote the following bilinear form on $\mathbf{M}_0$:

$$\langle \alpha, \beta \rangle = \int_\infty w^*\alpha \cdot d\beta.$$

Lemma 7. *The bilinear form $\langle\ ,\ \rangle$ is symmetric.*

Proof. Since w^* swaps the two cusps, we see that

$$\langle \alpha, \beta \rangle = \int_0 \alpha \cdot d(w^*\beta).$$

Since $\int_0 + \int_\infty = 0$ we see that

$$\langle \alpha, \beta \rangle = -\int_\infty \alpha \cdot d(w^*\beta).$$

In particular,

$$\langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle = \int_\infty w^*\alpha \cdot d\beta + \beta \cdot d(w^*\alpha) = \int_\infty d((w^*\alpha)\beta) = 0. \qquad \square$$

Lemma 8. *The basis $\{g^k\}_{k=0}^\infty$ is an orthogonal basis for $\mathbf{M}_0$, with respect to $\langle\ ,\ \rangle$.*

Proof. By Lemma 6 we have $w^*g^k = g^{-k}$. Hence

$$\langle g^m, g^n \rangle = \int_\infty g^{-m} \cdot n g^{n-1}\, dg = \int_\infty n g^{n-m} \cdot \frac{dg}{g} = n \cdot \delta_{m,n}. \qquad \square$$

Thus powers of g behave nicely with respect to this pairing. On the other hand, this pairing behaves nicely with respect to U:

Theorem 3. *U is self-adjoint with respect to $\langle\ ,\ \rangle$.*

Proof. It is enough to show this for any pair g^j, g^i. We see that

$$\langle Ug^j, g^i \rangle = \sum_{k=0}^\infty u_{k,j} \langle g^k, g^i \rangle = u_{i,j} \langle g^i, g^i \rangle.$$

Now from the above calculation and our explicit evaluation of $u_{i,j}$, this is equal to

$$\frac{2^{2i+2j-1} 3ij(i+j-1)!}{(2i-j)!(2j-i)!} = \langle g^j, Ug^i \rangle,$$

since the penultimate expression is symmetric. $\square$

It is natural to ask whether this theorem is a special case of a more general phenomenon.

3.2 Weights other than zero

Our results in this section are unfortunately much more incomplete. We may relate the action of U in weight 0 to the action of U in weight k by 'Coleman's trick', namely, a judicious application of the identity $U(gV(h)) = hU(g)$. In our case we again obtain explicit formulas for matrix entries of U acting on $\mathbf{S}_k$. Take for example the case $k = -12m$ with $m \geqslant 0$ an integer. Define

$$h_k = \frac{\Delta(2\tau)^m}{\Delta(\tau)^{2m}} = \Delta(2\tau)^{-m} f^{2m}.$$

By Coleman's trick we now observe that

$$U(h_k f^j) = U(\Delta(2\tau)^{-m} f^{j+2m}) = \Delta(\tau)^{-m} U(f^{j+2m}) = h_k f^{-m} U(f^{j+2m}).$$

In particular, with respect to the basis $\{h_k g, h_k g^2, h_k g^3, \dots\}$, U in weight $-12m$ can be given explicitly by the matrix $(2^{-6m} u_{i+m,j+2m})_{i,j\geqslant 1}$. Since for $m \in \mathbb{N}$, $-12m$ is dense in $4\mathbb{Z}_2$, if one could prove Conjecture 2 in the case $k = -12m$ then by a continuity argument one could prove it for all k congruent to 0 mod 4, and then by using the theta operator one should be able to deal with the case $k \equiv 2 \bmod 4$ as well. However, although we have a factorisation of the form $U = ADB$, the matrices A and B (and BA) are unfortunately not integral, and new methods seem to be required.

3.3 Primes other than $p = 2$

Our methods rely strongly on the fact that $X_0(p)$ has genus 0. Presumably our results can be extended to other primes with this property, with perhaps a corresponding increase in combinatorial difficulty. On the other hand, our techniques fail, or at least must be greatly modified, when the genus of $X_0(p)$ is greater than 0, even though similar results seem to be true. For example, the following conjecture appears to be compatible with the conjectures in [Buz04].

Conjecture 3. *Let $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots$ be the non-zero eigenvalues (with multiplicities) of U on $\mathbf{S}_0(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q}_{11}; 11^{-r})$ for some rational r with $11/12 > r > 0$. Then the slope (that is, the 11-adic valuation) of λ_n is given by the following formula:*

$$v_{11}(\lambda_n) = v_{11}\left(\frac{[(6n+1)/5]![(6n+4)/5]!}{[n/5]![n/5]!}\right) + \sum_{k=1}^{4}\left[\frac{n+k}{5}\right],$$

where $[x]$ denotes the largest integer less than or equal to x.

4. Appendix. Overconvergent forms at small level

For this Appendix we work with a general level and a general prime p. For want of a reference, we explain how to extend the theory of overconvergent p-adic modular forms to cases where the level structure is too coarse for the resulting moduli problem to be rigid. The motivation is that there is a growing theory of 'explicit' computations with p-adic modular forms, where typically both the level and the prime are small (for example, this paper). On the other hand, at several points in theoretical papers on the subject, the hypothesis is made that the level in which one is working is at least 5, or that $p \geqslant 3$, for convenience (lifting the Hasse invariant, for example).

One problem at small level is that there is, as far as we know, no reference for the theory of 'rigid analytic stacks', so we proceed by the usual low-level *ad hoc* methods. Here is an overview of the idea: Take the level structure one is interested in, and a sufficiently small auxiliary Galois level structure with Galois group G (for example, a full level M structure for some large prime M). The product level structure is sufficiently fine to imply the existence of a compact curve X representing a moduli problem on generalised elliptic curves, and X is equipped with an action of G. One can form the quotient curve X/G, which is not in general the solution to a moduli problem, but which is a coarse moduli space. One cannot always form an appropriate sheaf ω on these quotient curves, because ω does not always descend from X (problems at elliptic points, for example), but one can still define modular forms as G-invariant sections of tensor powers of ω on X. Next one has to check that the rigid analytic subspaces that one is interested in are all G-invariant, which comes down to checking that they have an intrinsic definition that only depends on the underlying elliptic curve and not the level structure.

We now formulate everything rigorously. Let U_N denote the compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ consisting of matrices congruent to the identity modulo N. If U is an arbitrary compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, then we define the *level* of U to be the smallest positive integer N such that $U_N \subseteq U$.

Let Γ be a compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ and let p be a prime that does not divide the level of Γ. We shall give the main definitions in the theory of overconvergent p-adic modular forms for Γ.

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We say that a level structure Γ is 'sufficiently small' if it satisfies the following two conditions:

(i) The identity element is the only element of Γ which is of finite order and is conjugate in $\mathrm{GL}_2(\mathbb{A}_f)$ to an element of $\mathrm{GL}_2(\mathbb{Q})$.

(ii) If C denotes the surjective $\widehat{\mathbb{Z}}$-module homomorphisms $(\widehat{\mathbb{Z}})^2 \to \widehat{\mathbb{Z}}$, equipped with its natural right action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, then -1 operates without fixed points on C/Γ.

These properties are used in the following way. Property (i) implies that there will be no elliptic points in the associated moduli space, or, more precisely, that the associated moduli problem on elliptic curves is rigid and hence representable [KM85, Appendix to § 4]. Property (ii) implies that there will be no irregular cusps on the compactification of the representing object [KM85, § 10.13 and Proposition 10.13.4]. We remark that $\Gamma = U_M$ for any $M \geqslant 3$ satisfies properties (i) and (ii). However, perhaps the most common modular curves that one sees are the curves $X_0(N)$, corresponding to the compact open subgroup Γ of matrices in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ which are upper triangular mod N, and property (i) always fails for these Γ, since $-\mathrm{Id} \in \Gamma$.

Let us assume initially that Γ is sufficiently small. Because Γ satisfies (i), the associated moduli problem on elliptic curves is representable over $\mathbb{Z}_p$, by a smooth affine curve $Y(\Gamma)$, equipped with a universal elliptic curve $\pi : \mathcal{E}(\Gamma) \to Y(\Gamma)$. Set $\omega := \pi_*\Omega^1_{\mathcal{E}(\Gamma)/Y(\Gamma)}$. Then ω is an invertible sheaf on $Y(\Gamma)$. Because Γ satisfies (ii), the natural compactification $X(\Gamma)$ of $Y(\Gamma)$ is also the solution to a moduli problem (that of parameterising generalised elliptic curves with level structure; see [DR71, III.6], although we shall not use this in what follows). Next note that the sheaf ω extends in a natural way to an invertible sheaf on $X(\Gamma)$ (see [KM85, Proposition 10.13.14]). We define a classical modular form of level Γ and weight k, defined over $\mathbb{Q}_p$, to be a global section of $\omega^{\otimes k}$ on the generic fibre of $X(\Gamma)$.

If R is a $\mathbb{Z}_p$-algebra then we denote by $X(\Gamma)_R$ the base change of $X(\Gamma)$ to R. The special fibre $X(\Gamma)_{\mathbb{F}_p}$ of $X(\Gamma)$ is a smooth proper geometrically connected curve, and has finitely many supersingular points. Let P denote a supersingular point and say P is defined over the finite field $\mathbb{F}$. Let W denote the Witt vectors of $\mathbb{F}$, and let K denote the field of fractions of W. If $X(\Gamma)^{\mathrm{an}}$ denotes the rigid space over K associated to $X(\Gamma)$ then there is a reduction map from $X(\Gamma)^{\mathrm{an}}$ to $X(\Gamma)_{\mathbb{F}}(\overline{\mathbb{F}})$, and the pre-image U of P is isomorphic to an open disc. The completed local ring of $X(\Gamma)_W$ at P is a W-algebra isomorphic non-canonically to a power series ring $W[[t]]$ in one variable; let us fix one such isomorphism. Then t can be thought of as giving an isomorphism from U to the rigid analytic open unit disc. Hence if $r \in \mathbb{Q}_{\geqslant 0}$ then we can talk about the open subdisc $\{x : |t(x)| < p^{-r}\}$ of U. These subdiscs in general depend on the fixed isomorphism between the completed local ring and the power series ring, but if $r < 1$ then an easy calculation shows that they are independent of such choices; the point that we have chosen to be the centre of U is a K-point and the other K-points in U all have distance at most $1/p$ from our chosen centre, because K is an unramified extension of $\mathbb{Q}_p$. (See § 3 of [Buz03] for more details of this construction, or § 2 of [BT99].)

The universal formal deformation of the elliptic curve $E_0/\mathbb{F}$ corresponding to P is an elliptic curve over the completed local ring of $X(\Gamma)_W$ at P, and hence can be regarded via our fixed isomorphism as an elliptic curve $E/W[[t]]$. Fix a basis η of $H^0(E, \Omega^1_{E/W[[t]]})$, so

$$H^0(E, \Omega^1_{E/W[[t]]}) = W[[t]] \cdot \eta.$$

The Hasse invariant can be thought of as a mod p section of the $(p-1)$th tensor power of this module, and hence an element $A(t)\eta^{\otimes(p-1)}$ of $\mathbb{F}[[t]] \cdot \eta^{\otimes(p-1)}$. It is well known that the Hasse invariant has a simple zero at every supersingular elliptic curve, which translates into the fact that $A(t)$ is a uniformiser of $\mathbb{F}[[t]]$. This provides the bridge between our point of view and that of Katz. For example, Katz's analysis of the p-divisible group associated to an elliptic curve and its relation to a lifting of the Hasse invariant [Kat73, § 3.7] shows that, for a $\overline{K}$-point $u \in U$ corresponding to an elliptic curve E_u, if $|t(u)| > 1/p$ then $|t(u)|$ is independent of all choices we have made, and depends only on the isomorphism class of E_u (and not on the level structure).

Assume from now on that $0 \leqslant r < 1$, and that $\mathbb{F}$ is sufficiently large so that all the supersingular points in $X(\Gamma)_{\mathbb{F}_p}$ are defined over $\mathbb{F}$. Choose parameters t as above for every supersingular point, and define $X(\Gamma)_{K,\geqslant p^{-r}}$ to be the rigid space over K which is the complement of the open discs $\{x : |t(x)| < p^{-r}\}$ as above, as P ranges over all supersingular points of $X(\Gamma)_{\mathbb{F}}$. If u is a $\overline{K}$-valued point of $Y(\Gamma)_K$ then let E_u denote the fibre of the universal elliptic curve above x. So E_x is an elliptic curve defined over $K(x)$. Now Katz's arguments show that, if E_x has good supersingular reduction and if $|t(x)| > p^{-1}$, then $|t(x)| = |t(\sigma x)|$ for any $\mathbb{Q}_p$-automorphism of the field $\overline{K(x)}$. Hence for $r \in \mathbb{Q}$ with $0 \leqslant r < 1$ the rigid space $X(\Gamma)_{K,\geqslant p^{-r}}$ is the base extension to K of a rigid subspace $X(\Gamma)_{\geqslant p^{-r}}$ of $X(\Gamma)$ defined over $\mathbb{Q}_p$.

Say $r \in \mathbb{Q}$ with $0 \leqslant r < 1$. We define a p^{-r}-overconvergent modular form of level Γ and weight k, defined over $\mathbb{Q}_p$, to be a section of $\omega^{\otimes k}$ on $X(\Gamma)_{\geqslant p^{-r}}$.

Now let Γ be an arbitrary (not necessarily sufficiently small) compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, and let p be a prime not dividing the level of Γ. Choose a prime $M > 2$ dividing neither p nor the level of Γ, and let Γ' denote $\Gamma \cap U_M$. Then Γ' is sufficiently small in the sense above, and so all of the definitions above apply to Γ'. Furthermore, Γ' is normal in Γ; let G denote the quotient group. Then G is finite (in fact G is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$) and G acts on $X(\Gamma)$ and ω. Moreover, because $|t(x)|$ (notation as above) only depends on the elliptic curve E_x and not any level structure, $X(\Gamma')_{\geqslant p^{-r}}$ is G-invariant if $0 \leqslant r < 1$.

This motivates the following definitions. We define $X(\Gamma)$ to be quotient of $X(\Gamma')$ by the finite group G; note that $X(\Gamma')$ is a projective curve, so taking quotients is not a problem. In practice, one can form the quotient in the following manner: $Y(\Gamma')$ is affine, and is hence of the form $\mathrm{Spec}(R)$, R a ring with an action of G. One defines $Y(\Gamma) = \mathrm{Spec}(R^G)$ and then compactifies. Note that the sheaf ω will probably not in general descend to $X(\Gamma)$. However, we can still define an r-overconvergent modular form of level Γ as being a G-invariant element of $H^0(X(\Gamma')_{\geqslant p^{-r}}, \omega^{\otimes k})$, if $0 \leqslant r < 1$. Sometimes these spaces are zero for trivial reasons, for example if Γ contains -1 and if k is odd. If they are not zero, then they are always infinite-dimensional.

One needs to check that these definitions are independent of the auxiliary choice of Γ'. This is not too difficult: if Γ'_1 and Γ'_2 are two choices for Γ' with Galois groups G_1 and G_2 then one sets $\Gamma' := \Gamma'_1 \cap \Gamma'_2$ and checks that both definitions for $X(\Gamma)$ coincide with the quotient of $X(\Gamma')$ by Γ/Γ', and so on.

One may check without too much difficulty that the weight 0 r-overconvergent forms of level Γ are just the functions on $X(\Gamma)_{\geqslant p^{-r}}$; this comes from the fact that one can form quotients of affinoids by finite groups by looking at invariants, and compatibility of this with the analytification functor.

One useful result is that if $\Gamma \subseteq \Delta$ both have level prime to p then the pre-image of $X(\Delta)_{\geqslant p^{-r}}$ under the canonical (forgetful functor) map from $X(\Gamma)$ to $X(\Delta)$ is $X(\Gamma)_{\geqslant p^{-r}}$. Only a little harder is the fact that if $\gamma\Gamma\gamma^{-1} \subseteq \Delta$ and Γ and Δ have level prime to p, and $\gamma_p = 1$, then the same is true for the map $X(\Gamma) \to X(\Delta)$ induced by γ. This is because $|t(x)|$ depends only on the underlying p-divisible group of the elliptic curve.

The above approach is good for theoretical purposes, but is too abstract in general to be of much computational use. We now show how to use these ideas to make the claims of this paper rigorous, thus turning the argument in this paper from a formal one into a rigorous one. We set $\Gamma = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $p = 2$, and write $X_0(1)$ for $X(\Gamma)$. We now explicitly evaluate $X_0(1)_{\geqslant p^{-r}}$ for $0 \leqslant r < 3/4$.

It is well known that the j-invariant gives an isomorphism $X_0(1) \to \mathbb{P}^1$ defined over $\mathbb{Z}_2$. Let us set $M = 3$ (notation as above), so $\Gamma' = U_3$ and $Y(3) := Y(\Gamma')$ is the modular curve over $\mathbb{Z}_2$ parameterising elliptic curves equipped with two points of order 3 generating the 3-torsion of the

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curve. The generic fibre of this curve is not geometrically connected, but this does not matter. The classical theta series $\theta := \sum_{a,b\in\mathbb{Z}} q^{a^2+ab+b^2} = 1 + 6q + 6q^3 + \cdots$ is a modular form of level Γ' (in fact it has level $\Gamma_1(3)$) and by the q-expansion principle it is a lift of the mod 2 Hasse invariant. The special fibre of $X(3) := X(\Gamma')$ has two geometric fibres, both of genus 0, both defined over $\mathbb{F} := \mathbb{F}_4$, and both with one $\mathbb{F}$-valued supersingular point. Let P denote one of these supersingular points, and define W, K and U as above. Choose an isomorphism ι of the complete local ring of $X(3)_W$ at P with $W[[t]]$, and let $E/W[[t]]$ denote the universal formal deformation of the elliptic curve corresponding to the point P. Fix a basis η of $H^0(E, \Omega^1_{E/W[[t]]})$, so

$$H^0(E, \Omega^1_{E/W[[t]]}) = W[[t]] \cdot \eta.$$

Now if f is any modular form over $W(\overline{\mathbb{F}})$ of level Γ' and any weight k, then $f(E) = h\eta^{\otimes k}$ for some $h \in W[[t]]$. We now think of η as being fixed, and identify a modular form f with the corresponding function h as above. If $u \in U$ then we define $|f(u)| = |h(u)|$ and note that this is independent of the choice of η. Moreover, if $|f(u)| > 1/2$ then this value is also independent of the choice of ι.

Via this fixed isomorphism, θ can be regarded as an element of $W[[t]]$ and, because θ lifts the Hasse invariant, we know that θ mod 2 in $\mathbb{F}[[t]]$ will be of the form $ut + O(t^2)$ with $u \neq 0$. Now consider the classical level 1 Eisenstein series $E_4 = 1+240(\sum_{n\geqslant 1} \sigma_3(n)q^n)$. Note that the q-expansion of θ is congruent to 1 mod 2, and hence θ^4 has q-expansion congruent to 1 mod 8. In particular $\theta^4 \equiv E_4$ mod 8. This congruence can be thought of as a congruence of elements of $W[[t]]$. Now for $u \in U$ with $|E_4(u)| > 1/8$ we see that $|\theta(u)|^4 = |E_4(u)|$ and hence $|\theta(u)| > 2^{-3/4} > 2^{-1}$. So $|t(u)| = |\theta(u)| = |E_4(u)|^{1/4}$. Conversely, if $|t(u)| > 2^{-3/4}$ then $|E_4(u)| = |t(u)|^4 > 1/8$. We conclude that if $0 \leqslant r < 3/4$ then $X(3)_{\geqslant 2^{-r}}$ is the subregion of $X(3)$ where $|E_4| > 2^{-4r}$. Moreover, because $j = (E_4)^3/\Delta$, and $|\Delta(u)| = 1$ for all $u \in U$ (as u corresponds to an elliptic curve with good reduction), we see that for $0 \leqslant r < 3/4$, we have that $X(3)_{\geqslant 2^{-r}}$ is the region defined by $|j| \geqslant 2^{-12r}$. We have proved the following proposition.

Proposition 1. *If $p = 2$ and $0 \leqslant r < 3/4$ then $X_0(1)_{\geqslant p^{-r}}$ is the subdisc of the j-line defined by $|j| \geqslant 2^{-12r}$.*

We now let g denote the modular function $2^6\Delta(2z)/\Delta(z)$. Recall that this is an isomorphism $X_0(2)_{\mathbb{Q}_2} \to \mathbb{P}^1$. Now one checks that $64/j = g/(4g + 1)^3$ and hence the map $X_0(2)_{\mathbb{Q}_2} \to X_0(1)_{\mathbb{Q}_2}$ induced by the forgetful functor sends the region $\{|g| \leqslant 1\}$ to the region $\{|j| \geqslant |64|\}$. Moreover, this map preserves q-expansions, and induces an isomorphism between these two regions (one can write down an inverse, for example, to see this). We deduce that the disc $\{|g| \leqslant 1\}$ in $X_0(2)$ is isomorphic to $X_0(1)_{\geqslant 2^{-1/2}}$, and that the functions $\{1, g, g^2, g^3, \dots\}$ are a Banach basis for $2^{-1/2}$-overconvergent level 1 weight 0 modular forms. Furthermore, because g vanishes at infinity, the functions $\{g, g^2, g^3, \dots\}$ form a basis for the space of $2^{-1/2}$-overconvergent tame level 1 cusp forms of weight 0.

Finally, we say a word about other weights $k \equiv 0$ mod 12. Write $k = -12m$ and define $h_k := \Delta(q^2)^m/\Delta(q)^{2m}$. Then h_k is a meromorphic section of $\omega^{\otimes k}$ on $X_0(2)$ and a computation of q-expansions shows that it is non-vanishing at infinity. It hence defines a section of $\omega^{\otimes k}$ on $X(3)_{\geqslant 2^{-1/2}}$ which is $G := \mathrm{GL}_2(\mathbb{F}_3)$-invariant and non-vanishing, and hence a trivialisation of $\omega^{\otimes k}$ on this rigid space. It is now easy to show that the G-invariant sections of $\omega^{\otimes k}$ are exactly the sections of the form $h_k s$ for s a function on $X_0(1)_{\geqslant 2^{-1/2}}$ and this establishes all the claims about explicit bases of overconvergent modular forms in this paper.

Remark. Matthew Emerton points out to us that this Appendix contains some overlap with [Eme01].

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