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Abstract

We give a method for constructing a Legendrian representative of a knot in S^3 which realizes its maximal Thurston–Bennequin number under a certain condition. The method utilizes Stein handle decompositions of D^4, and the resulting Legendrian representative is often very complicated (relative to the complexity of the topological knot type). As an application, we construct infinitely many knots in S^3 each of which yields a reducible 3-manifold by a Legendrian surgery in the standard tight contact structure. This disproves a conjecture of Lidman and Sivek.

1. Introduction

The maximal Thurston–Bennequin number, denoted by $\overline{tb}$, of a knot in S^3 is an important invariant in low-dimensional topology. For example, a 4-manifold represented by a knot with framing less than $\overline{tb}$ admits a Stein structure ([Eli90a], cf. [Gom98]), and such Stein 4-manifolds have many applications to low-dimensional topology, e.g. exotic smooth structures, contact 3-manifolds and 4-genera of knots (cf. [GS99, OS04]). Here recall that $\overline{tb}(K)$ of a knot K in S^3 is the maximal value of the Thurston–Bennequin number $tb(\mathcal{K})$ of a Legendrian representative $\mathcal{K}$ of K in the standard tight contact structure on S^3. There are several invariants which give good upper bounds for $\overline{tb}$ of knots (e.g. [AM97, LM98, FT97, OS03, Pla04, Pla06, Ras10, Shu07, Ng05, Ng12]). By contrast, it is generally difficult to find a Legendrian representative of a knot realizing an upper bound for $\overline{tb}$ when the crossing number of the knot is large. Hence, determining $\overline{tb}$ is a difficult problem in general.

In this paper, we give a method for constructing a Legendrian representative of a knot in S^3 which realizes its maximal Thurston–Bennequin number under a certain condition. The method utilizes Stein handle decompositions of D^4, and the resulting Legendrian representative (in the front diagram of S^3) is often very complicated (relative to the complexity of the topological knot type). One can easily construct various examples of knots for which this method effectively works, and it seems difficult to determine their $\overline{tb}$ by other methods.

As an application of our method, we discuss reducible Legendrian surgeries. A long-standing open problem in Dehn surgery theory is to determine framed knots in S^3 which produce reducible 3-manifolds. The cabling conjecture [GS86] asserts a complete characterization of such framed knots, and there are many related studies (see [Gre15] and the references therein). Recently, Lidman and Sivek [LS16] gave an interesting new approach to this problem from contact topology. Here we recall basic facts. The Legendrian surgery along a Legendrian knot $\mathcal{K}$ in the standard

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tight contact structure on S^3 is topologically the Dehn surgery along $\mathcal{K}$ with the contact -1 (i.e. $tb(\mathcal{K})-1$) framing, and any integral Dehn surgery along a knot K in S^3 with framing less than $\overline{tb}(K)$ can be realized as a Legendrian surgery along a Legendrian representative of K in the standard contact structure. Applying Eliashberg's theorem [Eli90b] on splittings of Stein 4-manifolds with reducible boundary 3-manifolds, Lidman and Sivek proved the following theorem.

Theorem 1.1 (Lidman and Sivek [LS16]). *For a knot K in S^3 with $\overline{tb}(K) \geqslant 0$, any Dehn surgery along K with coefficient less than $\overline{tb}(K)$ is irreducible.*

In other words, any Legendrian surgery along a knot K with $\overline{tb}(K) \geqslant 0$ in the standard contact structure on S^3 yields an irreducible 3-manifold. Moreover, they conjectured that this result holds without the assumption $\overline{tb}(K) \geqslant 0$.

Conjecture 1.2 (Lidman and Sivek [LS16]). A knot in S^3 never yields a reducible 3-manifold by a Legendrian surgery in the standard tight contact structure.

This conjecture has the following supporting evidence. The cabling conjecture asserts that a framed knot yielding a reducible 3-manifold is the (p,q)-cable of a knot with pq-framing, and the standard cabling construction (cf. [Gol15, § 5]) only gives a Legendrian representative of the cable knot with $tb \leqslant pq$.

Here we disprove this conjecture by applying the aforementioned method.

Theorem 1.3. *There exist infinitely many knots in S^3 each of which yields a reducible 3-manifold by a Legendrian surgery in the standard tight contact structure. Furthermore, each knot K can be chosen so that the surgery coefficient is arbitrarily less than $\overline{tb}(K)$.*

In fact, we give a general method for constructing counterexamples. As an example, we will discuss the $(n,-1)$ cable $K_{m,n}$ of the ribbon knot K_m in Figure 3 for $n \geqslant 2$ and $m \leqslant -4n+3$. Although the standard cabling construction merely gives an estimate $\overline{tb}(K_{m,n}) \geqslant -2n+1$ (see Figure 5 for a representative realizing this estimate), our method determines the explicit value $\overline{tb}(K_{m,n}) = -1$ (Proposition 4.2), implying the above theorem. Indeed, our method yields the very complicated representative of $K_{m,n}$ in Figure 19 which realizes $\overline{tb}(K_{m,n})$. Here the notations and tangles A_n and B_n in the diagram are given in Figures 1 and 18. We hope our method is useful for finding a new phenomenon in contact and symplectic topology.

2. Stein handlebody and notation

In this section, we recall basic definitions and properties. We also introduce our notations, some of which are different from the standard ones.

2.1 Stein handlebody

We briefly review basics of Stein handlebodies. For details, see [Gom98] and [GS99]. For basics of contact topology and Legendrian knots, readers can consult [OS04]. Throughout this paper, we assume that a handlebody is four dimensional, compact, connected and oriented.

Recall that a *1-handlebody* (respectively *2-handlebody*) is a handlebody which consists of 0- and 1-handles (respectively 0-, 1- and 2-handles). We call a handlebody a *Stein handlebody* if it is constructed from a 1-handlebody $\natural_n S^1 \times D^3$ $(n \geqslant 0)$ by attaching 2-handles along a Legendrian

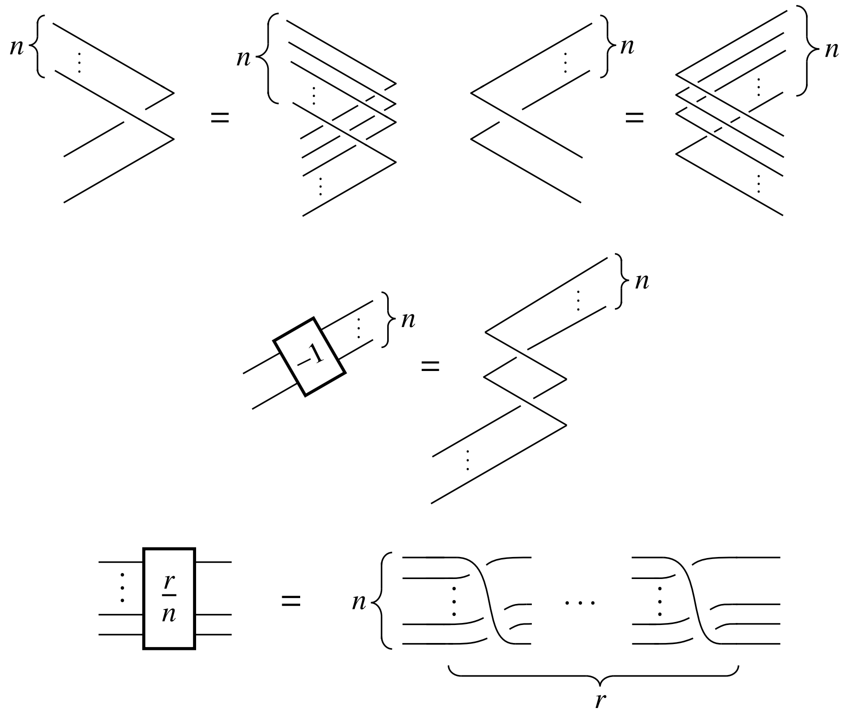

Figure 1. Notations on Legendrian versions of twists ($n \geqslant 1$, $r \geqslant 1$).

link in the Stein fillable contact structure on the boundary $\#_n S^1 \times S^2$ such that the framing of each Legendrian knot is -1 relative to the framing induced from the contact plane (i.e. contact -1 framing). According to a result of Eliashberg [Eli90a] (cf. [Gom98]), any Stein handlebody admits a Stein structure, extending the Stein structure on the 0-handle D^4. We note that each $\#_n S^1 \times S^2$ ($n \geqslant 0$) admits a unique Stein fillable contact structure up to isotopy [Eli92]. In the rest of this paper, a Legendrian link in $\#_n S^1 \times S^2$ ($n \geqslant 0$) means the one with respect to the Stein fillable contact structure. By a result of Gompf [Gom98] (cf. [GS99]), one can draw a Legendrian link in the boundary $\#_n S^1 \times S^2$ of a 1-handlebody using Gompf's *Legendrian link diagram in standard form*. In particular, we can draw a handlebody diagram of a Stein handlebody.

The *Thurston–Bennequin number* $tb(\mathcal{K})$ of a Legendrian knot $\mathcal{K}$ in the boundary $\#_n S^1 \times S^2$ of a 1-handlebody is defined to be the difference between the contact framing and the 0-framing. Here recall that the 0-*framing* of a knot in S^3 (i.e. the boundary of a 0-handle) is defined to be the Seifert framing (i.e. the one induced from a Seifert surface), and recall that the 0-*framing* of a knot in the boundary of a 1-handlebody is defined to be the Seifert framing induced from the dotted circle notation of the 1-handlebody. Consequently, if a knot bounds a Seifert surface in $\#_n S^1 \times S^2$, then the 0-framing coincides with the framing induced from the surface. Note that a knot in $\#_n S^1 \times S^2$ bounds a surface if and only if the knot is null-homologous. For details of Gompf's standard form diagram and calculation of the Thurston–Bennequin number in $\#_n S^1 \times S^2$, we refer to [Gom98] and [GS99]. For the definition of 0-framings, see [GS99].

Let $\mathcal{K}$ be a Legendrian knot in S^3, and let $g_4(\mathcal{K})$ denote the 4-genus of $\mathcal{K}$ (i.e. the minimal genus of a smoothly embedded surface in D^4 which bounds $\mathcal{K}$). One can estimate $g_4(\mathcal{K})$ as follows. By attaching a 2-handle to D^4 along $\mathcal{K}$ with framing $tb(\mathcal{K})-1$, we have a Stein 4-manifold. Since any Stein 4-manifold can be embedded into a closed minimal complex surface of general type with $b_2^+ > 1$ [LM97], applying the adjunction inequality [FS95, KM94, MST96, OS00] for this closed 4-manifold together with Gompf's Chern class formula [Gom98], we obtain the following adjunction inequality, where $r(\mathcal{K})$ denotes the rotation number of $\mathcal{K}$.

Theorem 2.1 [LM97, AM97, LM98]. $tb(\mathcal{K}) + |r(\mathcal{K})| \leqslant 2g_4(\mathcal{K}) - 1$.

Note that this holds even for the genus-zero case (cf. [GS99, OS04]), unlike the version for general closed 4-manifolds.

2.2 Notations and definitions

Here we introduce our notations and definitions. Beware that some of our definitions are different from the standard ones.

The 4-manifold represented by a framed knot in S^3 means the 4-manifold obtained from D^4 by attaching a 2-handle along the framed knot. For a Legendrian knot diagram, left- and right-handed twists are abbreviated as shown in Figure 1. For a knot K in the boundary of a 1-handlebody, a Legendrian knot $\mathcal{K}$ in the boundary is called a *Legendrian representative* of K if $\mathcal{K}$ satisfies the conditions below.

- In the case where K is homologically trivial, $\mathcal{K}$ is smoothly isotopic to K.
- In the case where K is homologically non-trivial, $\mathcal{K}$ is smoothly isotopic to K without sliding $\mathcal{K}$ 'over' any 1-handle. More explicitly, this condition is stated as follows. Consider the dotted circle notation of the 1-handlebody. The condition is that $\mathcal{K}$ is isotopic to K by an isotopy of S^3 fixing the disks bounded by the dotted circles. Note that this condition does not allow slidings over the dotted circles.

We use this narrow definition to define the maximal Thurston–Bennequin number for a homologically non-trivial knot. The *maximal Thurston–Bennequin number* $\overline{tb}(K)$ of a knot K in the boundary of a 1-handlebody is defined to be the maximal value of $tb(\mathcal{K})$ of a Legendrian representative $\mathcal{K}$ of K.

Lemma 2.2. *For any knot K in the boundary of a 1-handlebody, $\overline{tb}(K)$ is a finite number.*

Proof. In the case where K is null-homologous, this claim immediately follows from the adjunction inequality for general Stein 4-manifolds. In the case where K is homologically non-trivial, we can check this claim as follows. Consider a Legendrian representative $\mathcal{K}$ of K in the boundary of a 1-handlebody. By altering the diagram of $\mathcal{K}$ as shown in Figure 2, we obtain a Legendrian knot $\mathcal{K}'$ in S^3. Let K' denote the smooth knot type of $\mathcal{K}'$. Due to our narrow definition of a Legendrian representative, we easily see that K' is independent of the choice of $\mathcal{K}$. Furthermore, as seen from the diagram, $tb(\mathcal{K}') = tb(\mathcal{K}) + \alpha$ for some constant α which depends only on K. Since $\overline{tb}$ of a knot in S^3 is finite, this fact shows that $\overline{tb}(K)$ is also finite. □

We remark that, if we change our narrow definition of a Legendrian representative to the natural one, then there are many examples of homologically non-trivial knots with $\overline{tb} = \infty$.

3. The method

We give a method for constructing a Legendrian representative of a knot realizing its maximal Thurston–Bennequin number. Before the method, we note that a knot $\widetilde{K}$ in the boundary of the sub 1-handlebody X_1 of a 2-handlebody X represents a knot in the boundary ∂X, since we may regard $\widetilde{K}$ as a knot in ∂X after attaching 2-handles of X to X_1.

Now let K be a knot in S^3. The method proceeds as follows. We do not claim that this procedure is always applicable.

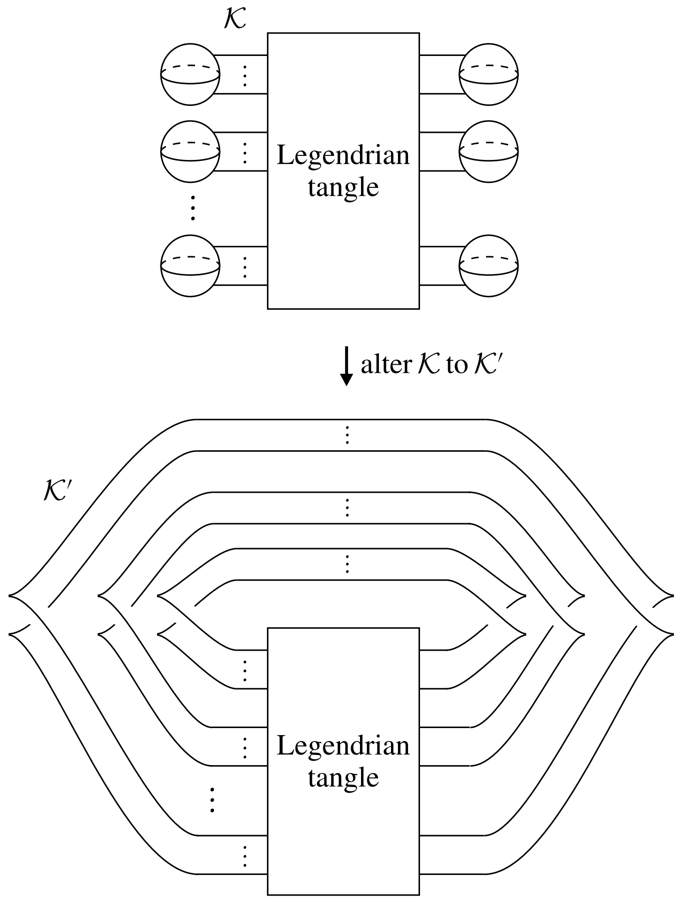

Figure 2. Alter a Legendrian knot $\mathcal{K}$ in $\#_n S^1 \times S^2$ to a Legendrian knot $\mathcal{K}'$ in S^3.

Step 1. Find a 2-handlebody X diffeomorphic to D^4 such that $K \subset \partial X (\cong S^3)$ is represented by a *good* knot $\widetilde{K}$ in the boundary of the sub 1-handlebody X_1 of X.

Here we say that a knot $\widetilde{K}$ in ∂X_1 is *good* if we can draw a Legendrian representative of $\widetilde{K}$ in ∂X_1 realizing $\overline{tb}(\widetilde{K})$. For example, torus knots are good knots in this sense. (We define torus knots in $\#_n S^1 \times S^2$ as cables of the unknot in $\#_n S^1 \times S^2$, similarly to the S^3 case. Note that an unknot in $\#_n S^1 \times S^2$ is a knot bounding a disk.) Experimentally, the method is effective when we choose a null-homologous knot as $\widetilde{K}$.

Let L be the link in ∂X_1 which consists of the attaching circles of the 2-handles of X.

Step 2. By ignoring the link L, first isotope $\widetilde{K}$ to its Legendrian representative $\widetilde{\mathcal{K}}$ in ∂X_1 which realizes $\overline{tb}(\widetilde{K})$, and then keep track of the position of the link L in ∂X_1 by this isotopy. Next, fixing the position of the Legendrian knot $\widetilde{\mathcal{K}}$, isotope L to its Legendrian representative $\mathcal{L}$ so that the framings of 2-handles of X coincide with the contact -1 framing of the Legendrian representative $\mathcal{L}$. Now X is a Stein handlebody. By attaching the 2-handles of X to X_1 along $\mathcal{L}$, we regard $\widetilde{\mathcal{K}}$ as a Legendrian knot (denoted by $\mathcal{K}$) in $\partial X \cong S^3$. Here the contact structure on ∂X is the one induced from the Stein structure on X. Since S^3 has a unique Stein fillable contact structure up to isotopy, $\mathcal{K}$ gives a Legendrian representative of K in the standard tight contact structure on S^3.

We remark a simple sufficient condition that the resulting Legendrian representative $\mathcal{K}$ realizes $\overline{tb}(K)$: if $\widetilde{K}$ bounds a surface of genus g in X_1 satisfying $2g-1=tb(\widetilde{\mathcal{K}})$, then $\overline{tb}(K)=tb(\mathcal{K})(=tb(\widetilde{\mathcal{K}}))$ due to the adjunction inequality. For example, this condition holds if $\widetilde{K}$ is a positive torus knot in ∂X_1. Of course, one can also use other upper bounds of $\overline{tb}$ (cf. § 1) to see whether $\mathcal{K}$ realizes $\overline{tb}(K)$.

Remark 3.1. (1) (Construction of various examples.) One can easily construct various examples of knots for which this method produces Legendrian representatives realizing $\overline{tb}$ as follows. Construct a 2-handlebody X diffeomorphic to D^4, and put a null-homologous knot $\widetilde{K}$ on the boundary ∂X_1 of the sub 1-handlebody of X satisfying $\overline{tb}(\widetilde{K})=2g_{X_1}(\widetilde{K})-1$. Here $g_{X_1}(\widetilde{K})$ denotes the minimal genus of a smoothly embedded surface in X_1 bounded by $\widetilde{K}$. For example, any positive torus knot in ∂X_1 satisfies this condition of $\widetilde{K}$. Now let K be a knot in $\partial X\cong S^3$ represented by $\widetilde{K}$. This process corresponds to Step 1 of the method. If $\overline{tb}$ of the attaching circle of each 2-handle in ∂X_1 is sufficiently larger than the framing of the 2-handle, then one can clearly apply Step 2. By the assumption on $\widetilde{K}$, the resulting Legendrian knot gives a Legendrian representative of K realizing $\overline{tb}(K)$. By using this construction, we can construct many counterexamples to Conjecture 1.2. See the next section.

(2) (Variant of the method.) Although we required that we (can) draw a Legendrian representative of $\widetilde{K}$ realizing $\overline{tb}(\widetilde{K})$ in ∂X_1, the method without this condition is still effective for finding a good lower bound of $\overline{tb}(K)$. See Subsection 4.5 for an example, which also gives counterexamples to Conjecture 1.2.

Remark 3.2. Regarding the Legendrian representative $\mathcal{K}$ obtained by Step 2, beware that $tb(\mathcal{K})$ may not be the same value as $tb(\widetilde{\mathcal{K}})$. This is because the 0-framing of a knot in S^3 and that of a knot in the boundary of a 1-handlebody are defined differently. One can easily calculate $tb(\mathcal{K})$ from $tb(\widetilde{\mathcal{K}})$ by checking the difference of 0-framings induced from S^3 and the boundary of the 1-handlebody X_1. In particular, if $\widetilde{K}$ is null-homologous in ∂X_1, then $tb(\mathcal{K})$ is equal to $tb(\widetilde{\mathcal{K}})$.

Since $\mathcal{K}$ is given as a Legendrian knot on the boundary of the Stein handlebody X, one might wish to find a representative in the front diagram of S^3.

Step 3 (Optional). Slide $\mathcal{K}$ over the 2-handles of X so that the resulting knot does not go over any 1-handle and that it is Legendrian preserving $tb(\mathcal{K})$. Then cancel all 1-handles of X with 2-handles. The resulting Legendrian knot gives a Legendrian representative of K in the front diagram of S^3 realizing $\overline{tb}(K)$.

This process often yields a very complicated Legendrian diagram. Note that this step is not necessary for determining $\overline{tb}$.

4. Example

We demonstrate the method using knots obtained in [Yas15], and we prove Theorem 1.3. Let us recall that, for a knot K in S^3, the (p,q)-cable $C_{p,q}(K)$ of K is defined to be a knot in S^3 which is a simple closed curve in the boundary $\partial\nu(K)$ of the tubular neighborhood $\nu(K)$ of K representing the class $p[K']+q[\alpha]$ in $H_1(\partial\nu(K);\mathbb{Z})$. Here α is the positively oriented meridian of K, and K' is the 0-framing of K induced from a Seifert surface of K.

For an integer m, let K_m be the ribbon knot in Figure 3, where the box denotes the $m-1$ right-handed full twists. For an integer $n\geqslant 2$, let $K_{m,n}$ be the $(n,-1)$ cable of K_m. These knots

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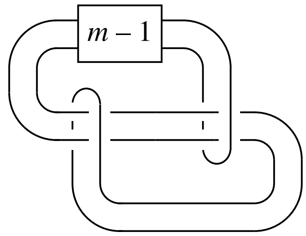

Figure 3. K_m.

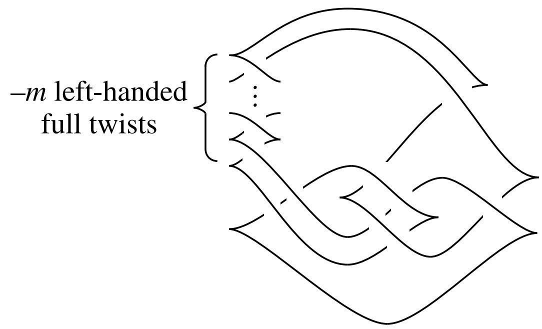

Figure 4. A Legendrian representative of K_m with $tb = -1$ ($m \leqslant -1$).

were constructed in [Yas15], and their maximal Thurston–Bennequin numbers were determined for $m \geqslant 0$ and $n \geqslant 2$ using rulings and a cabling formula. In this paper, we discuss $\overline{tb}(K_{m,n})$ for $m < 0$ using the method introduced in § 3. We remark that the cabling formula of $\overline{tb}$ obtained in [Yas15] does not work in this case.

4.1 Estimate of $\overline{tb}$ by standard construction

To see the effectiveness of our method, we here estimate $\overline{tb}(K_{m,n})$ by the standard cabling construction of a Legendrian representative (cf. [Gol15, § 5]). Before discussing the $(n, -1)$ cable $K_{m,n}$, we discuss its companion K_m. Since K_m bounds a disk in D^4, the adjunction inequality shows that $\overline{tb}(K_m) \leqslant -1$. On the other hand, for $m \leqslant -1$, we can easily check that K_m is isotopic to the Legendrian knot $\mathcal{K}_m$ with $tb = -1$ shown in Figure 4. Therefore, this Legendrian representative of K_m realizes $\overline{tb}(K_m) = -1$.

We draw n copies of the front diagram of $\mathcal{K}_m$, each of which is slightly shifted to the vertical direction. Inserting $(n-1)/n$ right-handed full twists to the resulting diagram appropriately, we obtain the Legendrian representative of $K_{m,n}$ in Figure 5. Calculating the number of left cusps and the writhe, one can easily check that tb of this representative is $-2n + 1$. Thus, the standard construction merely shows that $\overline{tb}(K_{m,n}) \geqslant -2n+1$. It seems difficult to realize a larger Thurston–Bennequin number by modifying this representative.

4.2 Step 1

Now we apply our method to $K_{m,n}$. We give necessary definitions to proceed with Step 1 of the method. For an integer m, let $Z^{(m)}$ be the 4-manifold shown in Figure 6. Since the 2-handle goes over the 1-handle geometrically once after isotopy, $Z^{(m)}$ is diffeomorphic to D^4. We identify the boundary $\partial Z^{(m)}$ with S^3 via this diffeomorphism. Let $\widetilde{K}_m$ and $\widetilde{K}_{m,n}$ be the unframed knots in $\partial Z^{(m)} = S^3$ given by Figures 7 and 8, respectively.

To apply Step 1, we show the lemma below.

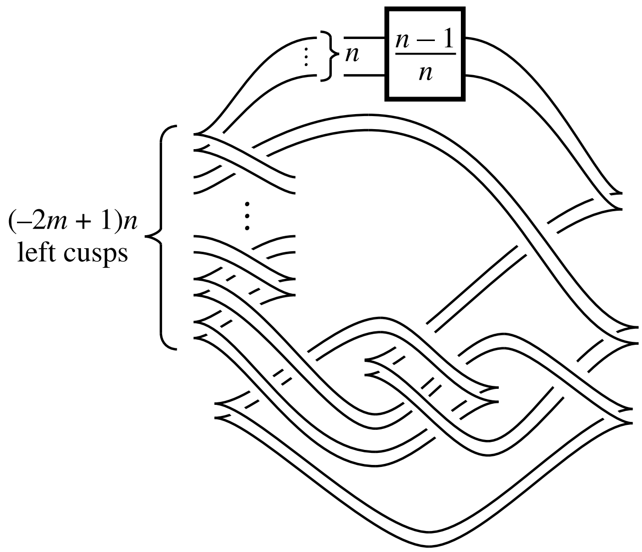

FIGURE 5. A Legendrian representative of $K_{m,n}$ with $tb = -2n + 1$ ($m \leqslant -1$).

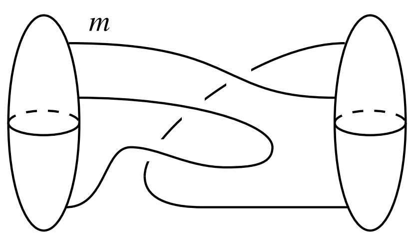

FIGURE 6. The handlebody $Z^{(m)}$ which is diffeomorphic to D^4.

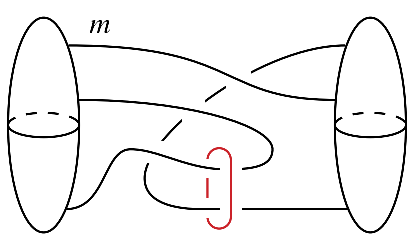

FIGURE 7. (Colour online) The knot $\widetilde{K}_m$ in $\partial Z^{(m)}$.

LEMMA 4.1. *For integers m, n with $n \geqslant 2$, the knots K_m and $K_{m,n}$ are isotopic to the knots $\widetilde{K}_m$ and $\widetilde{K}_{m,n}$ in $\partial Z^{(m)}$, respectively.*

Proof. By Figure 9, we see that $\widetilde{K}_m$ is isotopic to K_m. Figures 7 and 8 and the definition of a cable knot show that $\widetilde{K}_{m,n}$ is the $(n, -1)$ cable of $\widetilde{K}_m$. Hence, $\widetilde{K}_{m,n}$ is isotopic to $K_{m,n}$. □

We regard $\widetilde{K}_{m,n}$ as a knot in the boundary of the sub 1-handlebody $Z_1^{(m)}$ of $Z^{(m)}$. Then $\widetilde{K}_{m,n}$ is clearly an unknot in the boundary $\partial Z_1^{(m)}$, and we know a Legendrian representative of an unknot realizing $\overline{tb}$. Therefore, we have finished Step 1.

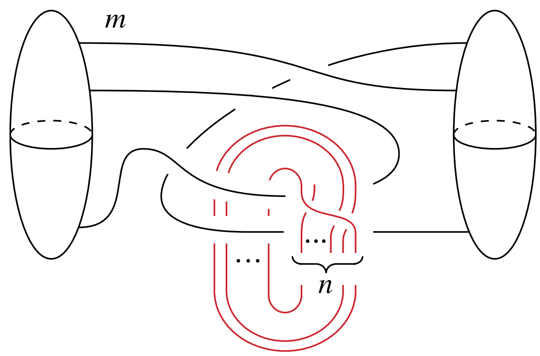

FIGURE 8. (Colour online) The knot $\widetilde{K}_{m,n}$ in $\partial Z^{(m)}$.

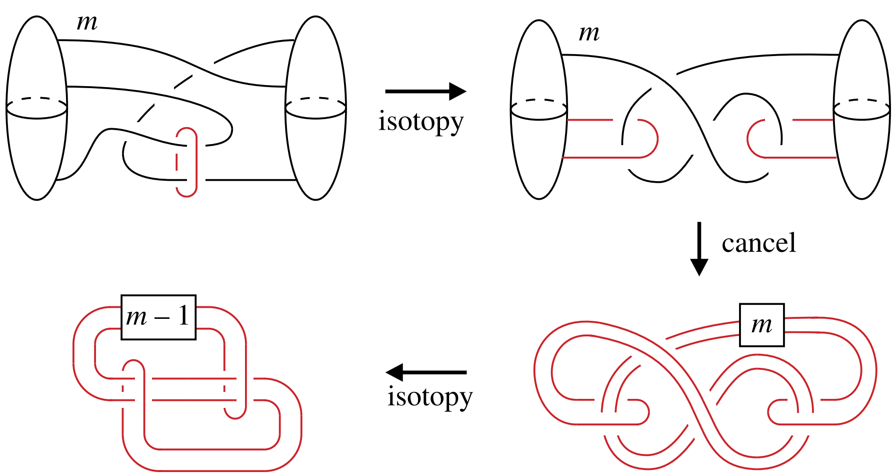

FIGURE 9. (Colour online) Diagrams of the knot K_m in S^3.

4.3 Step 2

Next we apply Step 2. We note that, if the framing m of the 2-handle of $Z^{(m)}$ is a sufficiently large negative number, then we can obviously achieve this step. We first isotope $\widetilde{K}_{m,n}$ to its Legendrian representative realizing $\overline{tb}$ in $\partial Z_1^{(m)}$, and then we keep track of the 2-handle of $Z^{(m)}$ as shown in Figure 10. By putting the 2-handle into a Legendrian position, we obtain the Legendrian representative of $\widetilde{K}_{m,n}$ in $\partial Z_1^{(m)}$ shown in Figure 11 for $m \leqslant -4n+3$. Note that $Z^{(m)}$ is now a Stein handlebody and that $tb(\widetilde{K}_{m,n}) = -1$. The Legendrian representative of $\widetilde{K}_{m,n}$ thus gives a Legendrian representative of $K_{m,n}$ in the boundary $\partial Z^{(m)}$ of the Stein handlebody, since $\widetilde{K}_{m,n}$ represents $K_{m,n}$ in $\partial Z^{(m)}$.

Since $\widetilde{K}_{m,n}$ bounds a disk in $Z_1^{(m)}$, the adjunction inequality shows that $\overline{tb}(K_{m,n}) \leqslant -1$. Therefore, the Legendrian representative of $K_{m,n}$ in Figure 11 realizes $\overline{tb} = -1$. Note that this value of $\overline{tb}$ is equal to the one induced from the front diagram of S^3 (see Remark 3.2). This completes Step 2, and the proposition below follows.

PROPOSITION 4.2. *$\overline{tb}(K_{m,n}) = -1$ for $n \geqslant 2$ and $m \leqslant -4n+3$.*

Now we can easily prove Theorem 1.3.

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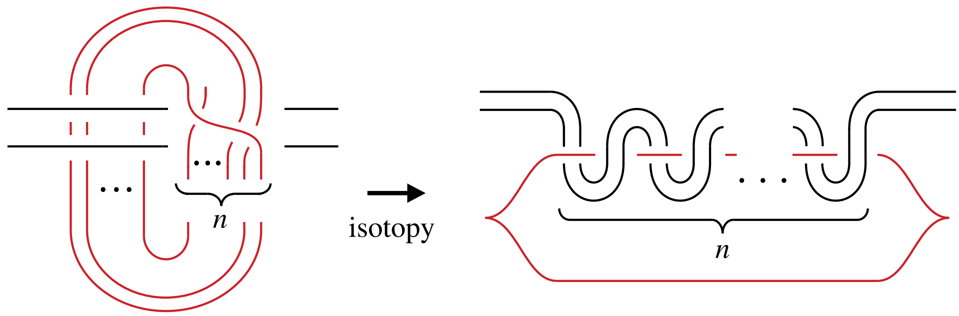

FIGURE 10. (Colour online) Local isotopy.

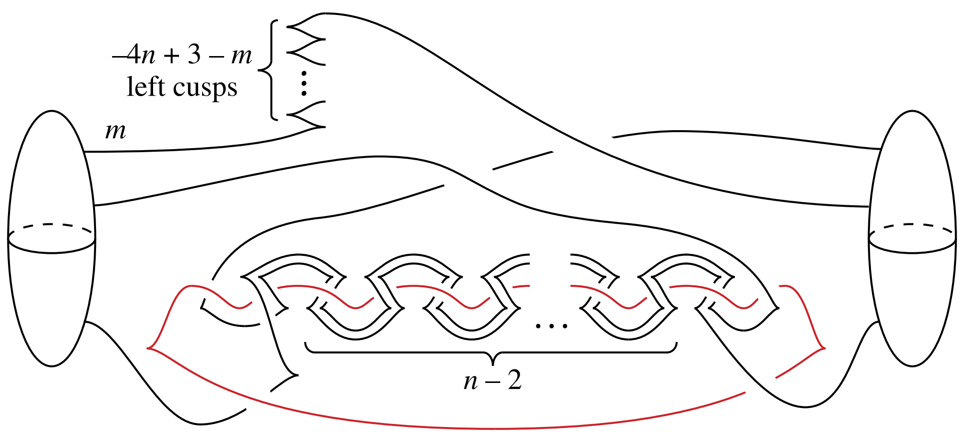

FIGURE 11. (Colour online) Legendrian representative of $K_{m,n}$ with $tb = -1$ in the Stein handlebody diagram of $Z^{(m)}$ ($m \leqslant -4n + 3$).

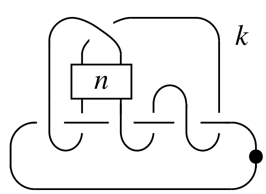

FIGURE 12. $X_{n,k}$.

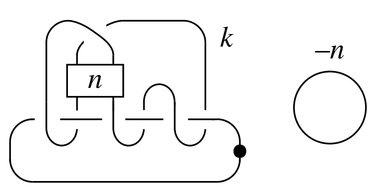

FIGURE 13. $Y_{n,k}$.

Proof of Theorem 1.3. For integers n and k, let $X_{n,k}$ and $Y_{n,k}$ be the 4-manifolds in Figures 12 and 13, respectively. Note that $\partial X_{n,k}$ is a homology 3-sphere, since $X_{n,k}$ is contractible. By [Yas15], the 4-manifold represented by $K_{m,n}$ with $-n$-framing is diffeomorphic to $Y_{n,m+4n}$ for $n \geqslant 2$. The $-n$-surgery along $K_{m,n}$ thus yields the 3-manifold $\partial Y_{n,m+4n}$, which is clearly diffeomorphic

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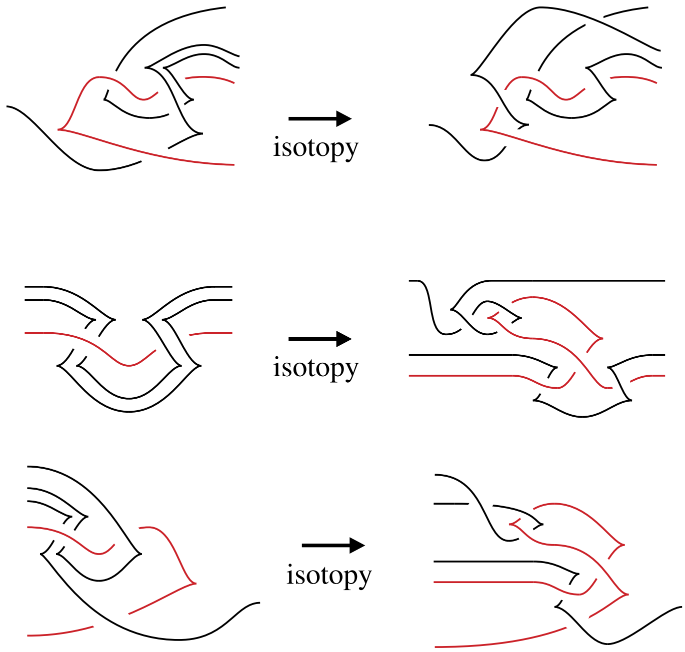

Figure 14. (Colour online) Isotopies fixing the end points.

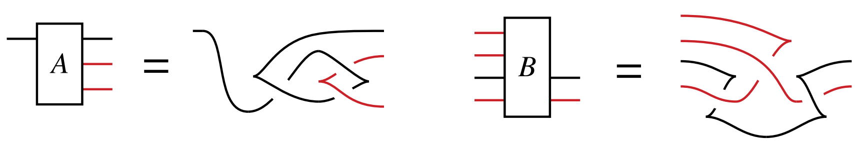

Figure 15. (Colour online) Definition of tangles A and B.

to the connected sum $\partial X_{n,m+4n}\#L(n,1)$. Here $L(n,1)$ denotes the lens space given by $-n$-surgery along the unknot, following the convention in contact topology. The knot $K_{m,n}$ thus yields a reducible 3-manifold by $-n$-surgery for $n \geqslant 2$, since $\partial X_{n,m+4n}$ is not diffeomorphic to S^3 [Yas15]. Here recall that the r-surgery along the (p,q)-cable of a non-trivial knot in S^3 yields a reducible 3-manifold if and only if $r = pq$ [GL87, Theorem 3]. Thus, $K_{m,n}$ is not isotopic to $K_{m,n'}$ if $n \neq n'$. Therefore, by Proposition 4.2, the infinite family of knots $\{K_{m,n} \mid n \geqslant 2,\ m \leqslant -4n+3\}$ satisfies the desired conditions. □

Remark 4.3. We can construct many other counterexamples by using the construction in Remark 3.1. Indeed, if we construct X and $\widetilde{K}$ so that $\widetilde{K}$ is the unknot and that K is the $(n,-1)$-cable of a non-trivial knot in S^3, then $\overline{tb}(K) = -1$, and K yields a reducible 3-manifold by $-n$-surgery, giving a counterexample to Conjecture 1.2.

4.4 Step 3

Finally we apply Step 3 to obtain a Legendrian representative of $K_{m,n}$ realizing $\overline{tb}$ in the front diagram of S^3. We first apply local isotopies in Figure 14 to the Stein handlebody diagram of $Z^{(m)}$ and the knot $K_{m,n}$. Note that these isotopies preserve tb of $K_{m,n}$ and the 2-handle. To simplify the diagram, we use tangles A and B defined in Figure 15. The resulting diagram of $K_{m,n}$ and $Z^{(m)}$ is shown in the first diagram of Figure 16. Now we can easily isotope the 2-handle

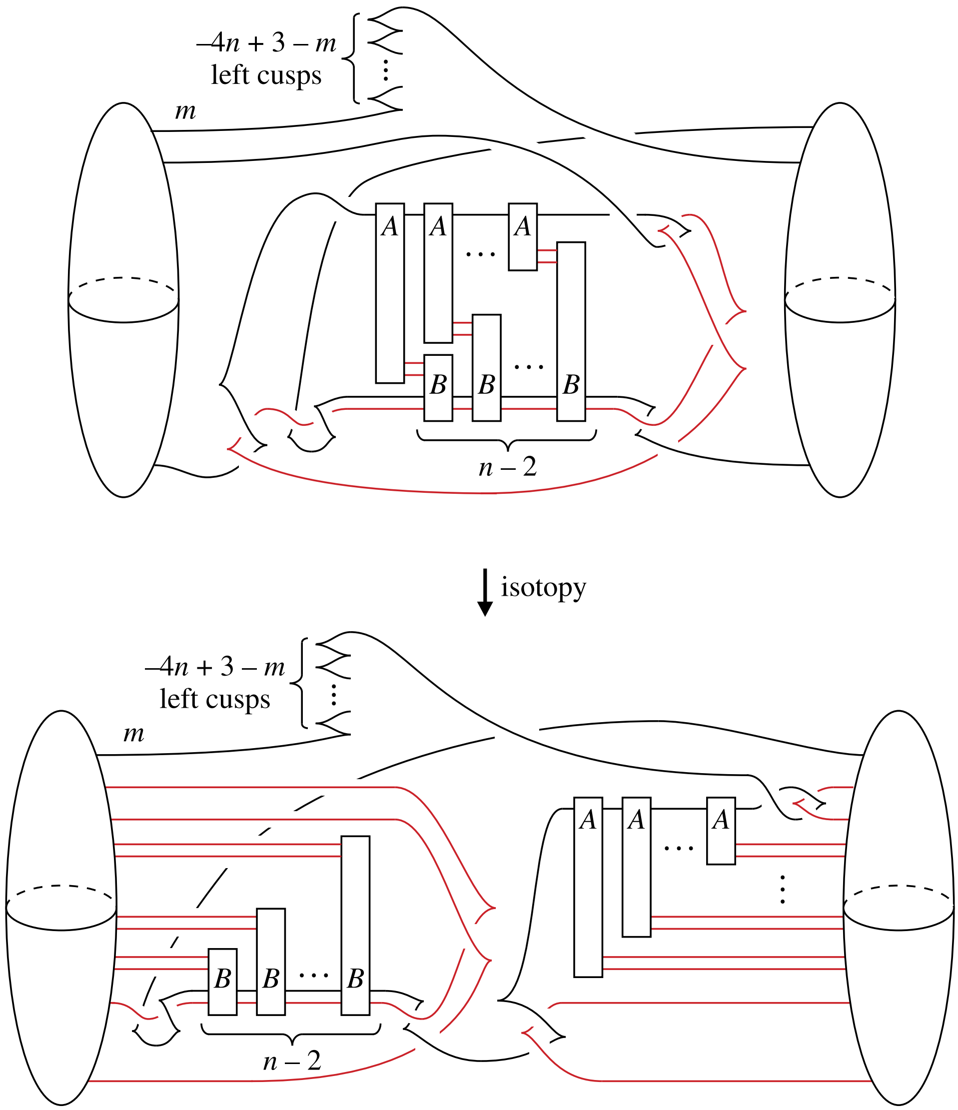

FIGURE 16. (Colour online) Legendrian representatives of $K_{m,n}$ in Stein handlebody diagrams of $Z^{(m)}$ ($n \geqslant 2$ and $m \leqslant -4n+3$).

and $K_{m,n}$ so that the 2-handle goes over the 1-handle geometrically once and that tb of these knots do not change. The resulting diagram is given in the second diagram of Figure 16 (this isotopy can be easily seen by ignoring $K_{m,n}$ and tangles A and B).

Next we slide $K_{m,n}$ over the 2-handle so that $K_{m,n}$ does not go over the 1-handle (after suitable isotopy) and that $tb(K_{m,n})$ does not change. More specifically, we slide $K_{m,n}$ at the right-most part of Figure 16 as shown in Figure 17, where the framings of the 2-handle are the contact -1 framings. We can easily check that this operation preserves $tb(K_{m,n})$ by counting the numbers of positive crossings, negative crossings and left cusps. Clearly, we can isotope the resulting $K_{m,n}$ preserving tb so that it does not go over the 1-handle. Now we can cancel (erase) the canceling pair of 1- and 2-handles. The resulting diagram is given in Figure 19, where we use the tangles A_n, B_n defined in Figure 18. Therefore, this diagram gives a Legendrian representative of $K_{m,n}$ realizing $\overline{tb} = -1$ in the front diagram of S^3. We have thus completed Step 3.

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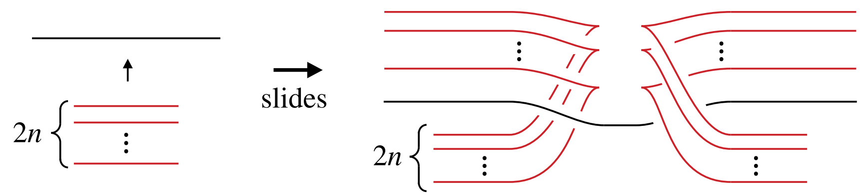

Figure 17. (Colour online) Sliding of $K_{m,n}$ over the 2-handle.

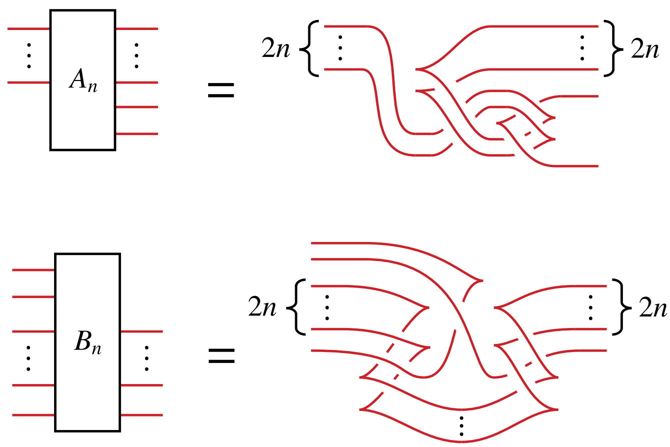

Figure 18. (Colour online) Definition of tangles A_n and B_n.

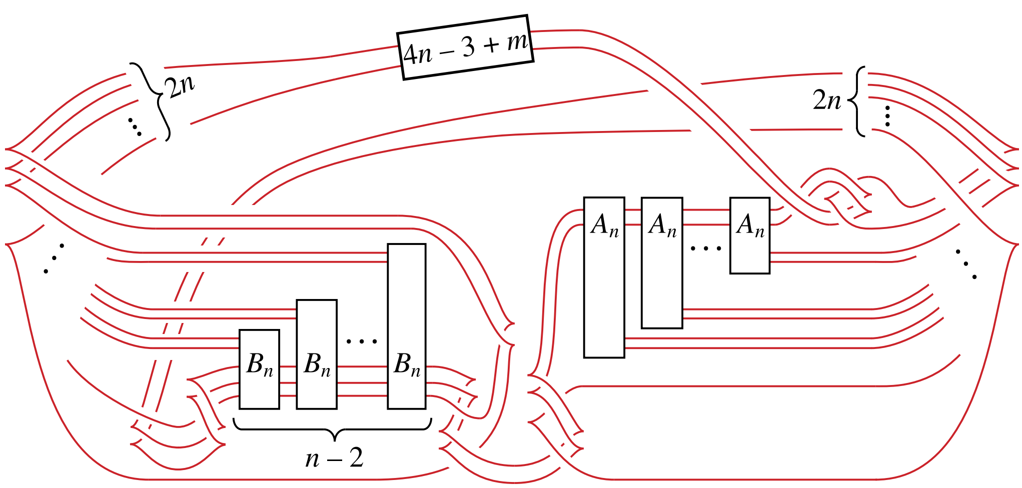

Figure 19. (Colour online) Legendrian representative of $K_{m,n}$ realizing $\overline{tb} = -1$ ($n \geqslant 2$ and $m \leqslant -4n+3$).

4.5 Variant of the method

As we mentioned in Remark 3.1, our method is also effective for finding a good lower bound of $\overline{tb}$ by minor modification. We demonstrate this using the knot $K_{m,n}$ with $n \geqslant 2$ and $m \leqslant -2n-1$.

Recall that $K_{m,n}$ is isotopic to the unframed knot $\widetilde{K}_{m,n}$ in the boundary of the handlebody $Z^{(m)}$ shown in Figure 8. We regard $\widetilde{K}_{m,n}$ as a knot in the boundary of the sub 1-handlebody $Z_1^{(m)}$

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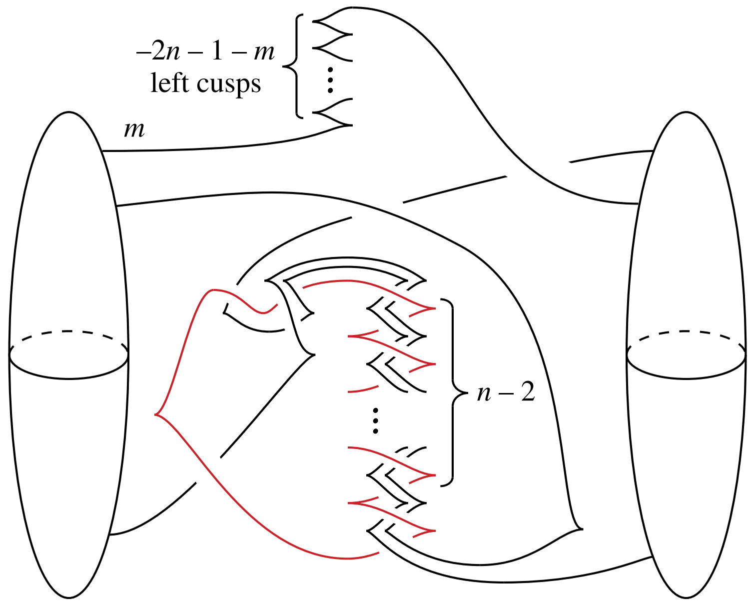

Figure 20. (Colour online) Legendrian representative of $K_{m,n}$ with $tb = -n+1$ in the Stein handlebody diagram of $Z^{(m)}$ ($n \geqslant 2$ and $m \leqslant -2n-1$).

of $Z^{(m)}$. Since $\widetilde{K}_{m,n}$ is an unknot in $\partial Z_1^{(m)}$, we can isotope $\widetilde{K}_{m,n}$ to its Legendrian representative with $tb = -n+1$. We then isotope the 2-handle of $Z^{(m)}$ to its Legendrian representative fixing the representative of $\widetilde{K}_{m,n}$. The resulting diagram of $\widetilde{K}_{m,n}$ and $Z^{(m)}$ is shown in Figure 20. Clearly, this diagram gives a Stein handle decomposition of $Z^{(m)}$, and thus the Legendrian representative of $\widetilde{K}_{m,n}$ gives a Legendrian representative of $K_{m,n}$ with $tb = -n+1$ in the Stein fillable contact structure on $\partial Z^{(m)} \cong S^3$. Hence, the proposition below follows.

Proposition 4.4. *$\overline{tb}(K_{m,n}) \geqslant -n+1$ for $n \geqslant 2$ and $m \leqslant -2n-1$.*

It seems difficult to obtain this estimate without using a handlebody diagram of D^4. We remark that we can also draw a Legendrian representative with $tb = -n+1$ in the front diagram of S^3, similarly to Step 3.

Remark 4.5. In [Yas15], we discussed the following problem. 'Assume that a framed knot in S^3 represents a 4-manifold admitting a Stein structure. Is the framing less than the maximal Thurston–Bennequin number of the knot?' Since we proved in [Yas15] that the 4-manifold represented by $-n$-framed $K_{m,n}$ admits a Stein structure for $n \geqslant 2$ and $m \leqslant -2n-1$, it is natural to ask if the framing $-n$ is less than $\overline{tb}(K_{m,n})$. (We showed the existence of a Stein structure by checking that this 4-manifold is diffeomorphic to the boundary connected sum of two compact Stein 4-manifolds.) The above proposition tells that the framing is indeed less than $\overline{tb}(K_{m,n})$, giving supporting evidence for the above problem.

Characterizing an unknot is a natural question in knot theory, and various characterizations are known. Here we propose the following question as a potential characterization given by maximal Thurston–Bennequin numbers. Recall that $C_{p,q}(K)$ denotes the (p,q)-cable of a knot K in S^3.

Question 4.6. If a knot K in S^3 satisfies $\overline{tb}(C_{p,-1}(K)) = -1$ for any positive integer p, is K the unknot?

We remark that, for each positive integer N, Proposition 4.2 implies the existence of a non-trivial knot K satisfying $\overline{tb}(C_{p,-1}(K)) = -1$ for any positive integer $p \leqslant N$.

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