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Torsion topological groups with minimal open sets

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Torsion topological groups with the additional property that the intersection of open sets is open are considered and their topological structure is determined:

1) the topology is uniquely determined by a normal subgroup;
2) each group is uniquely an extension of an indiscrete group by a discrete group;
3) the topology may be changed within limits without changing the dual group.

An A-space is a topological space where the intersection of open sets is open ([1], [5], [8]). For a point x in an A-space X, U_x, the minimal open set containing x, is the intersection of all open sets containing x. Topological groups and semi-topological groups are defined as in [3]. G_e will denote the component of the identity e.

LEMMA 1. *Let G be an A-space semi-topological group with $a, b \in G$. Then $aU_b = U_{ab} = U_a b$.*

Proof. $U_a b$ is open and contains ab so $U_{ab} \subset U_a b$. Also $U_b \subset a^{-1}U_{ab}$ so $aU_b \subset U_{ab}$. Thus $U_{ab} = aU_b$ and likewise $U_{ab} = U_a b$.

The next lemma is an extension of the results in [6].

LEMMA 2. *Let G be a semi-topological torsion group with an*

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A-space topology. Then U_e *is an open and closed normal subgroup, has the indiscrete topology induced on it, and is* G_e . *Further,* G *is a topological group whose minimal open sets are the cosets of* U_e .

Proof. If $g \in U_e$ then $gU_e = U_g \subset U_e$ thus $U_e U_e \subset U_e$. But $U_e U_e$ is open so that $U_e \subset U_e U_e$ and $U_e U_e = U_e$. Since g has finite order, g^{-1} is a power of g so $g^{-1} \in U_e$. Therefore U_e is a subgroup. Since $U_e x = U_x = xU_e$, $xU_e x^{-1} = U_e$ and so U_e is normal.

Suppose V is a proper open subset of U_e containing g . Then $g^{-1}V \subset U_e$ is an open subset of U_e containing e with $g^{-1}V \neq U_e$. This contradicts the minimality of U_e so V cannot exist and so U_e is indiscrete. Since U_e is open it is also closed [2, 5.5]. Since U_e is connected, $U_e \subset G_e$. But $G_e - U_e$ is open since U_e is closed. Thus the connectedness of G_e implies $U_e = G_e$. The homogeneity of G shows the cosets of U_e are a base for the topology. By [2, 4.5] G is a topological group.

LEMMA 3. *Let* H *be a normal subgroup of a group* G . *Define a subset of* G *to be open iff it is a union of* H*-cosets. Then* G *is a topological group with an* A*-space topology and* $G_e = H$.

Proof. Since H-cosets are disjoint and exhaust G they form a base for a topology. In this topology H is the component of the identity. H with the induced topology is an indiscrete topological group. To show G is a topological group we verify $\mu : G \times G \to G$ given by $\mu(g, k) = g^{-1}k$ is continuous. It suffices to show $\mu(U_g \times U_k) = U_{g^{-1}k} = g^{-1}kH$ since $g^{-1}kH$ is the smallest open set containing $g^{-1}k$. Let $gh \in gH = U_g$. Then

$$\mu(gh \times U_k) = (gh)^{-1}U_k = h^{-1}g^{-1}(kH) = g^{-1}k(h'H) = g^{-1}kH = U_{g^{-1}k} .$$

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Thus

$$\mu\left(U_g \times U_k\right) = \mu\left(gH \times U_k\right) = \bigcup_{h \in H} \mu\left(gh \times U_k\right) = U_{g^{-1}k} \, .$$

From the above it is clear that any finite topological space X which is homogeneous can carry a group structure making it a topological group. For example if X has p minimal open sets with q points in each, then X can be given the group structure of $Z_p \oplus Z_q$ or Z_{pq} .

Lemmas 2 and 3 prove the following theorem which extends [2, 4.21 (b)] to A-spaces. It also gives information on the number of topologies possible on a finite set, a problem considered in [4]. As in [6], the corollary gives an exact answer to a special case of the problem considered in [7].

THEOREM 4. *There is a bijective correspondence between normal subgroups of a torsion group* G *and the* A*-space topologies on* G *giving a topological group.*

COROLLARY 5. *A finite topological group has exactly* 2^r *open subsets where* r *is the index of the largest connected subgroup.*

Proof. A base for the topology is given by the r disjoint open subsets which are the cosets of the largest connected subgroup, the component of the identity. This result was implicit in [6].

Let G be an A-space torsion topological group (not assumed abelian). Let S denote the circle group with the usual topology. The dual of G (or character group) is defined by

$$G^* = \mathrm{Hom}_c(G, S) = \text{the set of continuous homomorphisms from } G \text{ to } S \, .$$

G^* with the compact-open topology and the abelian group structure defined by $(f_1+f_2)g = f_1(g)f_2(g)$ is a topological group ([3, Section 44] or [2, Section 23]). Note that here $f : G \to S$ is continuous iff f is constant on the cosets of G_e .

THEOREM 6. *Any* A*-space torsion topological group* G *can be uniquely expressed as the extension of an indiscrete normal subgroup* N *by a discrete group* D *. Furthermore if* D *is finitely generated,* G^* *is topologically isomorphic to* D *made abelian.*

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Proof. Clearly $N = G_e$ and $D = G/G_e$ gives such an extension. Conversely, if $1 \to N \to G \to D \to 1$ is exact and D is discrete, N must be open. Thus $G_e \subset N$. If $G_e \neq N$ then N is not indiscrete, so $G_e = N$.

The exactness of $G_e \to G \xrightarrow{p} G/G_e \to 1$ gives the exactness of $1 \to \mathrm{Hom}_c(G/G_e, S) \xrightarrow{p^*} \mathrm{Hom}_c(G, S) \to \mathrm{Hom}_c(G_e, S)$ where p^* is a continuous homomorphism. Since G_e is indiscrete, $\mathrm{Hom}_c(G_e, S) = 1$. Thus p^* is a continuous isomorphism. Since p is also a compact mapping, p^* is an open mapping and thus is a topological isomorphism. If $D = G/G_e$ is finitely generated, $\mathrm{Hom}_c(D, S)$ is a discrete group topologically isomorphic to $\mathrm{Hom}(D, S)$. But since S is abelian, $\mathrm{Hom}(D, S) \approx \mathrm{Hom}(D/D', S) \approx D/D'$. Thus G^* is topologically isomorphic to D made abelian.

If G is a Hausdorff abelian topological group, the duality theorem [2, 24.8] shows G^{**} is topologically isomorphic to G. This theorem is seen not to be true if Hausdorff separation is not required. In fact the following result shows how the topology on G can be changed without changing G^*. Examples showing that the condition $G'H = G'K$ is not necessary are easy to construct.

THEOREM 7. *Let G_1 and G_2 be the torsion group G with the A-space topology determined by the normal subgroups H and K respectively. Then $G'H = G'K$ is a sufficient condition for G_1^* to be topologically isomorphic to G_2^*.*

Proof. The natural projection $p : D \to D/D'$ gives a continuous isomorphism $p^* : \mathrm{Hom}_c(D/D', S) \to \mathrm{Hom}_c(D, S)$. The function $q^* : \mathrm{Hom}_c(D, S) \to \mathrm{Hom}_c(D/D', S)$ given by $q^*(f)(dD') = f(d)$ is well defined and the inverse to p^*. To show q^* is continuous, consider the open set in $\mathrm{Hom}_c(D/D', S)$, $(KD', V) = \{h : D/D' \to S \mid h(KD') \subset V\}$ where V is open in S and $KD' = \{kD' \mid k \in K \subset D\}$ is compact and hence a finite set in D/D'. Then $q^{*-1}(KD', V) = (K, V)$ is a basic open set in

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$\mathrm{Hom}_c(D, S)$. So q^* is continuous and p^* is a homeomorphism. Thus G_1^* and G_2^* are topologically isomorphic to $\mathrm{Hom}\left(G/H/(G/H)', S\right)$ and $\mathrm{Hom}\left(G/K/(G/K)', S\right)$ respectively. However as discrete groups $G/H/(G/H)' \approx G/H/G'H/H \approx G/G'H$. So if $G'K = G'H$ it follows that G_1^* is topologically isomorphic to G_2^* .

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