This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to W. O. J. Moser, University of Manitoba, Winnipeg, Manitoba.

RIGHT AND LEFT ORTHOGONALITY

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Let V be a vector space over an arbitrary field F. In V a bilinear form

$$f: \; x, y \rightarrow (x, y)$$

is given. If f is symmetric $[(x, y) \equiv (y, x)]$ or skew-symmetric $[(x, y) + (y, x) \equiv 0]$, then

$$(1) \qquad (x, y) = 0 \leftrightarrow (y, x) = 0.$$

Thus right and left orthogonality coincide. It is well known that (1) implies conversely that f is either symmetric or skew-symmetric in V. We wish to give a simple proof of this result.

If $(x, x) = 0$ for all $x \in V$, then

$$(x, y) + (y, x) = (x + y, x + y) - (x, x) - (y, y) = 0.$$

Thus f is then skew-symmetric. From now on we may assume that there exists a vector v such that

$$(2) \qquad (v, v) \neq 0.$$

We wish to show that f then is symmetric.

Let a, b be any two vectors in V. Assuming (1) we have to show

$$(3) \qquad (a, b) = (b, a).$$

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Case (i). $(a,a) \neq 0$. Put

$$c = (a,b) \cdot a - (a,a) \cdot b.$$

Then

$$(a,c) = (a,b) \cdot (a,a) - (a,a) \cdot (a,b) = 0.$$

Hence by (1)

$$0 = (c,a) = (a,b) \cdot (a,a) - (a,a) \cdot (b,a) = (a,a) \Big((a,b) - (b,a) \Big).$$

Since $(a,a) \neq 0$, this yields (3).

By (2), we now have in particular

(4) $$(a,v) = (v,a) \quad \text{and} \quad (b,v) = (v,b).$$

Case (ii). $(a,a) = 0$. We first show there exists an element

(5) $$\lambda \in F, \ \lambda \neq 0$$

such that

(6) $$(\lambda a + v, \lambda a + v) \neq 0.$$

By (4),

$$(\lambda a + v, \lambda a + v) = \lambda^2 \cdot (a,a) + 2\lambda \cdot (a,v) + (v,v) = 2\lambda \cdot (a,v) + (v,v).$$

If $(a,v) = 0$ or if the characteristic of F is two, $\lambda = 1$ will satisfy (5) and (6). Let $2 \cdot (a,v) \neq 0$. Then F contains at least three elements and there is a $\lambda \in F$ such that

$$\lambda \neq 0, \ \lambda \neq -\frac{(v,v)}{(a,v)}.$$

Thus this λ satisfies (5) and (6).

By case (i), (6) implies

$$(\lambda a + v, b) = (b, \lambda a + v)$$

or

$$\lambda \cdot (a,b) + (v,b) = \lambda \cdot (b,a) + (b,v).$$

By (4) and (5), this implies (3).

Formula (1) permits a second interpretation. If U is any subspace of V, f determines two new subspaces

$$*U = \{x \mid (x, y) = 0 \text{ for all } y \in U\}$$

and

$$U* = \{y \mid (x, y) = 0 \text{ for all } x \in U\}.$$

Thus e.g. $*U$ consists of all the vectors that are left orthogonal to U. Then (1) is readily seen to be equivalent to

$$*U = U* \text{ for all } U. \tag{7}$$

Thus (7) holds true if and only if f is either symmetric or skew-symmetric.

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