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A NOTE ON FINITE GROUPS GENERATED BY THEIR SUBNORMAL SUBGROUPS

A. BALLESTER-BOLINCHES[1] AND L. M. EZQUERRO[2]

[1] *Departament d'Àlgebra, Universitat de València, C/Dr. Moliner 50, 46100 Burjassot, València, Spain* (Adolfo.Ballester@uv.es)

[2] *Departamento de Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía, 31006 Pamplona, Spain* (ezquerro@unavarra.es)

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Abstract Following the theory of operators created by Wielandt, we ask for what kind of formations $\mathfrak{F}$ and for what kind of subnormal subgroups U and V of a finite group G we have that the $\mathfrak{F}$-residual of the subgroup generated by two subnormal subgroups of a group is the subgroup generated by the $\mathfrak{F}$-residuals of the subgroups.

In this paper we provide an answer whenever U is quasinilpotent and $\mathfrak{F}$ is either a Fitting formation or a saturated formation closed for quasinilpotent subnormal subgroups.

Keywords: subnormal subgroups; quasinilpotent groups; Fitting classes; formations

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1. Introduction

All groups considered in this paper are finite.

One of the significant properties of subnormal subgroups is that the nilpotent residual of the subgroup generated by two subnormal subgroups of a group is the subgroup generated by the nilpotent residuals of the subgroups. This is a consequence of an elegant theory of operators created by Wielandt for proving results on permutability of subnormal subgroups (see Chapter 10 of [**8**] or Chapter 4 of [**7**]).

It seems natural to wonder whether the above result is not accidental and could be obtained because of a general completeness property of all formations.

Therefore the following general question arises.

General question. *Let $\mathfrak{F}$ be a formation. Consider two subgroups U and V which are subnormal in $\langle U, V\rangle$.*
When do we have $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = \langle U, V\rangle^{\mathfrak{F}}$?

No affirmative general answer is possible. For example, if we consider the saturated formation $\mathfrak{F}$ composed of all finite groups with no epimorphic image isomorphic to Alt(5),

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i.e. the Alt(5)-perfect groups, and the group $G = \mathrm{Sym}(5)$, we have that $G \in \mathfrak{F}$, that is $G^{\mathfrak{F}} = 1$. However, $A = \mathrm{Alt}(5)$ is a normal subgroup of G and $A^{\mathfrak{F}} = A$.

Therefore some other conditions must be imposed.

A first approach is to analyse new requirements for the formation $\mathfrak{F}$: *for what kind of formations is there a universal positive answer?*

A formation $\mathfrak{F}$ is said to have *the Wielandt property for residuals* if the residual of a group generated by two subnormal subgroups is generated by the residuals of the subgroups.

It is not difficult to see that a formation with the Wielandt property for residuals is a Fitting class. The validity of the converse is not known at the time of writing. Some significant achievements appear in [**2**] and [**6**].

Another point of view on the same problem is to seek sufficient properties for the subnormal subgroups to obtain the validity of the Wielandt property, that is to understand the General Question in the following sense: *for what kind of subnormal subgroups is there a universal positive answer?*

This paper can be considered as a continuation of [**2**] and provides, for certain families of formations, some satisfactory results if the quasinilpotency of one of the subnormal subgroups is assumed.

We refer the reader to [**3**] for notation, terminology and results on formations and Fitting classes.

2. Main results

In this section we analyse the consequences of imposing the hypothesis of quasinormality on one of the subnormal subgroups.

Recall that if $\mathfrak{F}$ is a formation and G is a group, then $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of G, that is the smallest normal subgroup of G whose quotient group is in $\mathfrak{F}$. If $\mathfrak{F}$ is a Fitting class, $G_{\mathfrak{F}}$ denotes the $\mathfrak{F}$-radical of G, i.e. the subgroup generated by all subnormal $\mathfrak{F}$-subgroups of G. The radical for the Fitting class composed of all quasinilpotent groups, i.e. the generalized Fitting subgroup, is denoted by $F^*(G)$.

Proposition 2.1. *Let $\mathfrak{F}$ be a saturated formation. If $G = VF^*(G)$, where V is a subnormal subgroup of G, then $V^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$.*

Proof. Let G be a minimal counterexample to the proposition and let us consider a subnormal subgroup V of G of maximal order for which the statement is false. If M is a maximal normal subgroup of G such that $V < M$, by minimality of G, $M = V(F^*(G) \cap M) = VF^*(M)$ and we have $V^{\mathfrak{F}} \leqslant M^{\mathfrak{F}}$. By maximality of V, we have $M^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. Thus $V^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$.

Then V is a maximal normal subgroup of G. Let N be a minimal normal subgroup of G. Since $F^*(G)/N \leqslant F^*(G/N)$, we have that $G/N = (VN/N)F^*(G/N)$. By minimality of G it follows that $V^{\mathfrak{F}}N/N = (VN/N)^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}N/N$.

If $N \leqslant G^{\mathfrak{F}}$, then $V^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$, a contradiction. Hence we can suppose that $G \in \mathfrak{F}$.

Therefore $V^{\mathfrak{F}}N/N = 1$ for every minimal normal subgroup N of G. That is to say that $V^{\mathfrak{F}}$ is contained in every minimal normal subgroup of G.

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If $V^{\mathfrak{F}} = 1$, then G is not a counterexample. Therefore $V^{\mathfrak{F}} = N$ is the unique minimal normal subgroup of G. If G is a primitive group, then $V^{\mathfrak{F}} = F^*(G)$ and then $G = V$, a contradiction. Therefore $V^{\mathfrak{F}} \leqslant \Phi(G)$. In particular $V^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p. Since $G \in \mathfrak{F}$, it follows that $G/C_G(V^{\mathfrak{F}}) \in F(p)$.

Now observe that $V^{\mathfrak{F}}$ is a minimal normal subgroup of V and $V/C_V(V^{\mathfrak{F}}) = G/C_G(V^{\mathfrak{F}})$, and then $V^{\mathfrak{F}}$ is $\mathfrak{F}$-central in V. This is the final contradiction. □

Proposition 2.2. *Let $\mathfrak{F}$ be a saturated formation. The following are equivalent.*

(i) *Whenever $G \in \mathfrak{F}$ and N is a quasinilpotent subnormal subgroup of G, then $N \in \mathfrak{F}$.*

(ii) *Whenever N is a quasinilpotent subnormal subgroup of a group G, then $N^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$.*

(iii) *Whenever U and V are subnormal subgroups of $\langle U, V\rangle$ and U is a quasinilpotent group, then $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle \leqslant \langle U, V\rangle^{\mathfrak{F}}$.*

Proof. (i) ⇒ (ii) Since $NG^{\mathfrak{F}}/G^{\mathfrak{F}}$ is a quasinilpotent subnormal subgroup of $G/G^{\mathfrak{F}} \in \mathfrak{F}$, then $N/(N \cap G^{\mathfrak{F}}) \in \mathfrak{F}$. Therefore $N^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$.

(ii) ⇒ (i) Conversely, if G is a group in $\mathfrak{F}$ and N is a quasinilpotent subnormal subgroup of G, then $N^{\mathfrak{F}} \leqslant G^{\mathfrak{F}} = 1$ implies that $N^{\mathfrak{F}} = 1$ or, equivalently, that $N \in \mathfrak{F}$.

(ii) ⇒ (iii) Denote $J = \langle U, V\rangle$. Notice that, since $J = VF^*(J)$, then $V^{\mathfrak{F}} \leqslant J^{\mathfrak{F}}$. By (ii), we have $U^{\mathfrak{F}} \leqslant J^{\mathfrak{F}}$.

(iii) ⇒ (ii) Conversely, if N is a quasinilpotent subnormal subgroup of a group G, then, since $G = NG$, we have $G^{\mathfrak{F}} = N^{\mathfrak{F}}G^{\mathfrak{F}}$. That is to say that $N^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. □

Definition 2.3. A saturated formation $\mathfrak{F}$ satisfying one, and hence all, of the above conditions is said to be $\mathrm{S_n}(\mathfrak{N}^*)$-closed.

Notice that any saturated formation composed of soluble groups is $\mathrm{S_n}(\mathfrak{N}^*)$-closed. The quasinilpotent soluble groups are exactly the nilpotent ones. A classical result says that, given a saturated formation of soluble groups $\mathfrak{F}$, if a group G belongs to $\mathfrak{F}$ and p is a prime dividing the order of G, then the class of p-groups is contained in $\mathfrak{F}$ (see [**3**, Corollary IV, 4.3]).

On the other hand the saturated formation of the example in § 1, i.e. the saturated formation $\mathfrak{F}$ of the Alt(5)-perfect groups, is not $\mathrm{S_n}(\mathfrak{N}^*)$-closed. In the group $\mathrm{Sym}(5) \in \mathfrak{F}$, the quasinilpotent normal subgroup $\mathrm{Alt}(5) \notin \mathfrak{F}$.

Lemma 2.4. *Let $\mathfrak{F}$ be a saturated formation. Let us consider a group G and U, V subgroups of G such that U, V are subnormal in $\langle U, V\rangle$. Assume that U is a quasinilpotent group. If U and V are in $\mathfrak{F}$, then $\langle U, V\rangle \in \mathfrak{F}$.*

Proof. Let G be a minimal counterexample and choose a pair (U, V) for which the lemma is false and such that $|G : U| + |V|$ is maximal. It is clear that $G = \langle U, V\rangle$. By minimality of G, we have that $G/N \in \mathfrak{F}$ for any non-trivial normal subgroup N of G. Thus, $G \in \mathfrak{F}$. Then $U \leqslant \mathrm{Soc}(G) = F^*(G)$. Since $\mathrm{Soc}(G)$ normalizes V, we have that V is a normal subgroup of G. Therefore $U \leqslant V$ and then $G = V$, a contradiction. □

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Theorem 2.5. *Let $\mathfrak{F}$ be a formation. Whenever U, V are subnormal subgroups of $\langle U, V\rangle$ and U is a quasinilpotent group, then $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = \langle U, V\rangle^{\mathfrak{F}}$, provided $\mathfrak{F}$ is either*

(i) *a Fitting formation, or*

(ii) *a $\mathrm{S_n}(\mathfrak{N}^*)$-closed saturated formation.*

Proof. We have thought it desirable to collect together the arguments common to both situations.

Let G be a counterexample of least order to the theorem and consider the non-empty set $\mathcal{W}(G)$ composed of all pairs (A, B) such that A and B are subnormal subgroups of $\langle A, B\rangle$, A is quasinilpotent, and $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle < \langle A, B\rangle^{\mathfrak{F}}$. Notice that in both situations (i) and (ii) the set $\mathcal{W}(G)$ is non-empty by Proposition 2.2. Choose a pair $(U, V) \in \mathcal{W}(G)$ such that $|G : U| + |V|$ is maximal. Denote $H = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$.

By minimality of G, it is clear that $G = \langle U, V\rangle$. In particular G is not a simple group and $H < G^{\mathfrak{F}}$. Moreover, $1 < G^{\mathfrak{F}}$ and U and V are proper subgroups of G.

Step 1. *If N is a minimal normal subgroup of G, then $G^{\mathfrak{F}} = HN$.* In particular, H is a core-free subgroup of G and H is normal in $G^{\mathfrak{F}}$. Moreover, $G^{\mathfrak{F}}$ is a direct product of isomorphic simple groups.

Let N be a minimal normal subgroup of G. Consider the group $G/N = \langle UN/N, VN/N\rangle$ and, by minimality of G, we deduce that $G^{\mathfrak{F}}N = HN$. In particular H contains no minimal normal subgroup of G: if $N \leqslant H \leqslant G^{\mathfrak{F}}$, then $H = G^{\mathfrak{F}}$, a contradiction. Therefore H is a core-free subgroup of G.

If N is a minimal normal subgroup of G and N is not contained in $G^{\mathfrak{F}}$, then $N \cap G^{\mathfrak{F}} = 1$. This means that $G^{\mathfrak{F}}N = G^{\mathfrak{F}} \times N$. Since $H \leqslant G^{\mathfrak{F}}$, then $H \cap N = 1$. But $G^{\mathfrak{F}}N = HN$ implies that $|G^{\mathfrak{F}}| = |H|$ and then $G^{\mathfrak{F}} = H$, a contradiction. Hence $\mathrm{Soc}(G) \leqslant G^{\mathfrak{F}}$ and $G^{\mathfrak{F}} = HN$ for every minimal normal subgroup N of G. By a well-known theorem of Wielandt [**8**, Satz II, 12.9], $\mathrm{Soc}(G)$ normalizes H. Therefore H is a normal subgroup of $G^{\mathfrak{F}}$.

Since the quotient group $G^{\mathfrak{F}}/H$ is a direct product of isomorphic simple groups, so is $G^{\mathfrak{F}}/\operatorname{core}_G(H) = G^{\mathfrak{F}}$.

Step 2. *$G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p.* Notice first that for every minimal normal subgroup N of G, since $H \cap N$ is normal in N, we have that $N = (H \cap N) \times N^*$, and $G^{\mathfrak{F}} = H \times N^*$ with $N^* \neq 1$. This implies that $H \leqslant C_G(N^*)$. If there exist two minimal normal subgroups N_1, N_2 of G, then $G^{\mathfrak{F}} = H \times N_i^* \leqslant C_G(N_j^*)$, for $i \neq j$. Therefore $N_i^* \leqslant Z(G^{\mathfrak{F}})$ and both N_1 and N_2 are abelian. In other words, if $\mathrm{Soc}(G)$ is not a minimal normal subgroup of G, then $\mathrm{Soc}(G)$ is abelian.

Assume that $N = \mathrm{Soc}(G)$ is non-abelian. Then N is a minimal normal subgroup of G and $C_G(N) = 1$. If N is a direct product of copies of a non-abelian simple group, E say, then $G^{\mathfrak{F}}$ is a direct product of copies of E, by Step 1. If $N < G^{\mathfrak{F}}$, then there exists a copy of E centralizing N, a contradiction. Therefore $G^{\mathfrak{F}} = \mathrm{Soc}(G) = N$. Since $\mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G, we have that $G^{\mathfrak{F}} = F^*(G) = \mathrm{Soc}(G)$. Recall that $U \leqslant F^*(G)$ and then $G = F^*(G)V$. Moreover, $\mathrm{Soc}(G)$ normalizes V by the theorem

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of Wielandt [**8**, Satz II, 12.9] again. Therefore V is normal in G and so $\mathrm{Soc}(G) \leqslant V$. That is to say that $G = V$, a contradiction.

Therefore $\mathrm{Soc}(G)$ is abelian. Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p. By Step 1 we deduce that $G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p.

Notice that since $\mathrm{Soc}(G) \leqslant G^{\mathfrak{F}}$, we have that $\mathrm{Soc}(G)$ and $F(G)$ are p-groups.

Step 3. *U is a cyclic p-group.* Suppose that $U = PM$, with P, M two maximal normal subgroups of U. Then $G = \langle P, \langle M, V\rangle\rangle$. Notice that $|G : P| > |G : U|$. Both P and M are quasinilpotent subnormal subgroups of G. Since $|G : P| + |\langle M, V\rangle| > |G : U| + |V|$, we have $G^{\mathfrak{F}} = \langle P^{\mathfrak{F}}, \langle M, V\rangle^{\mathfrak{F}}\rangle$. But $|G : M| + |V| > |G : U| + |V|$, and then $\langle M, V\rangle^{\mathfrak{F}} = \langle M^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$. By minimality of G we have that $U^{\mathfrak{F}} = \langle P^{\mathfrak{F}}, M^{\mathfrak{F}}\rangle$. Hence $G^{\mathfrak{F}} = \langle P^{\mathfrak{F}}, \langle M^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle\rangle = H$, a contradiction. Hence U is a single-headed quasinilpotent subgroup.

Single-headed quasinilpotent groups are either perfect or cyclic q-groups for q a prime.

Assume that U is perfect. Then $G = UV$, by a result of Wielandt [**8**, Satz II, 10.10]. Since U is single-headed and $U \not\leqslant V$, then V normalizes U by Satz II, 7.8 of [**8**]. That is to say that U is normal in G. Thus $U^{\mathfrak{F}}$ is normal in G. Since H is core-free in G, it follows that $U^{\mathfrak{F}} = 1$. Therefore $U \in \mathfrak{F}$.

Let us consider a proper normal subgroup N of G such that $V \leqslant N$. Then $N = V(U \cap N)$. By minimality of G, we have $N^{\mathfrak{F}} = V^{\mathfrak{F}}(U \cap N)^{\mathfrak{F}}$. If $V < N$, then, by maximality of the pair (U, V), we deduce that $G^{\mathfrak{F}} = N^{\mathfrak{F}}U^{\mathfrak{F}}$. Therefore $G^{\mathfrak{F}} = V^{\mathfrak{F}}(U \cap N)^{\mathfrak{F}}U^{\mathfrak{F}} = V^{\mathfrak{F}}U^{\mathfrak{F}}$, a contradiction. Thus we can assume that V is a maximal normal subgroup of G. The subgroup $V^{\mathfrak{F}}$ is normal in G. Hence $V^{\mathfrak{F}} = 1$ or, equivalently, $V \in \mathfrak{F}$.

(i) If $\mathfrak{F}$ is a Fitting class, we deduce that $G \in \mathfrak{F}$, i.e. $G^{\mathfrak{F}} = 1$, a contradiction.

(ii) If $\mathfrak{F}$ is saturated, by Lemma 2.4 we deduce that $G \in \mathfrak{F}$, i.e. $G^{\mathfrak{F}} = 1$, a contradiction.

Therefore U is a cyclic q-group for q a prime. If $q \neq p$, then $U \leqslant O_q(G) \leqslant F(G) = O_p(G)$, a contradiction. Thus, $p = q$.

Let us now analyse each one of the cases. We start by assuming that $\mathfrak{F}$ is a Fitting formation.

If $p \notin \operatorname{char}\mathfrak{F}$, then $U^{\mathfrak{F}} = U$. Therefore $U \leqslant G^{\mathfrak{F}}$. This implies that U normalizes $V^{\mathfrak{F}}$. But then $V^{\mathfrak{F}}$ is a normal subgroup of G and then $V^{\mathfrak{F}} = 1$ by Step 1. Thus $V \in \mathfrak{F}$. But $V = V_{\mathfrak{F}} \leqslant N_G(U^{\mathfrak{F}}) = N_G(U)$. Then U is normal in G. This implies that $U^{\mathfrak{F}}$ is normal in G. Since $U^{\mathfrak{F}} \leqslant H$ and H is a core-free subgroup of G, we have that $U^{\mathfrak{F}} = 1$. Then $U \in \mathfrak{F}$, a contradiction.

Thus $p \in \operatorname{char}\mathfrak{F}$. Then $U^{\mathfrak{F}} = 1$, i.e. $U \in \mathfrak{F}$. Then $U_{\mathfrak{F}} = U$. Therefore $U \leqslant N_G(V^{\mathfrak{F}})$. It follows that $V^{\mathfrak{F}}$ is normal in G. As above, this implies that $V \in \mathfrak{F}$. Hence U and V are in $\mathfrak{F}$. Since $\mathfrak{F}$ is a Fitting class, it follows that $G \in \mathfrak{F}$. Hence $G^{\mathfrak{F}} = 1$. This is the final contradiction.

Now we study the other case. We suppose in the sequel that $\mathfrak{F}$ is a $\mathrm{S_n}(\mathfrak{N}^*)$-closed saturated formation.

We will prove first that $p \notin \operatorname{char}\mathfrak{F}$.

Assume that $p \in \operatorname{char} \mathfrak{F}$. Then $G^{\mathfrak{F}} \in \mathfrak{F}$.

Suppose that $VG^{\mathfrak{F}} = G$. Let M be a maximal normal subgroup of G such that $V < M$. Then $M = V(G^{\mathfrak{F}} \cap M)$. By minimality of G, we have that $M^{\mathfrak{F}} = V^{\mathfrak{F}}$, since $G^{\mathfrak{F}} \cap M$ is a p-group. Now, by maximality of the pair (U, V), we have that $G^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, M^{\mathfrak{F}} \rangle = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}} \rangle$, a contradiction. Therefore $V = M$, a maximal normal subgroup of G. Therefore $V \in \mathfrak{F}$. Since U and V are in $\mathfrak{F}$, it follows by Lemma 2.4 that $G \in \mathfrak{F}$, a contradiction. Hence $VG^{\mathfrak{F}}$ is a proper subgroup of G.

Then by minimality of G we have that $(VG^{\mathfrak{F}})^{\mathfrak{F}} = V^{\mathfrak{F}}$, since $G^{\mathfrak{F}}$ is quasinilpotent. Suppose that $V < VG^{\mathfrak{F}}$. By maximality of the pair (U, V), we have that

$$G^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, (VG^{\mathfrak{F}})^{\mathfrak{F}} \rangle = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}} \rangle,$$

a contradiction. Therefore $G^{\mathfrak{F}} \leqslant V$.

Let N be a minimal normal subgroup of G. Then $N \leqslant G^{\mathfrak{F}} \leqslant V$. Recall that $F^*(G)$ centralizes N. Since $G = F^*(G)V$, then N is a minimal normal subgroup of V. Notice that, by Theorem 4.6 of [**1**], since $G^{\mathfrak{F}}$ is an abelian group, we have that $G = DG^{\mathfrak{F}}$, where $D \in \operatorname{Nor}_{\mathfrak{F}}(G)$ and $D \cap G^{\mathfrak{F}} = 1$. By Theorem 4.3 of [**1**], if N is $\mathfrak{F}$-central in G, then $N \leqslant D$. But this implies that $N \leqslant G^{\mathfrak{F}} \cap D$, a contradiction. Therefore N is $\mathfrak{F}$-eccentric in G. Since $G = VF^*(G)$, and $F^*(G)$ centralizes N, it follows that N is a minimal normal subgroup of V. Moreover, since N is $\mathfrak{F}$-eccentric in G, then $G/C_G(V^{\mathfrak{F}}) \notin F(p)$. But $V/C_V(V^{\mathfrak{F}}) = G/C_G(V^{\mathfrak{F}})$ and then $V^{\mathfrak{F}}$ is $\mathfrak{F}$-eccentric in V. Hence $N \leqslant V^{\mathfrak{F}}$, a contradiction by Step 1.

Thus $p \notin \operatorname{char} \mathfrak{F}$.

Then $U^{\mathfrak{F}} = U$ and so $U \leqslant G^{\mathfrak{F}}$. Therefore U normalizes $V^{\mathfrak{F}}$ and then $V^{\mathfrak{F}}$ is a normal subgroup of G. But H is core-free in G. Hence $V^{\mathfrak{F}} = 1$, i.e. $V \in \mathfrak{F}$. Then $G = G^{\mathfrak{F}}V$. Let E be an $\mathfrak{F}$-maximal subgroup of G such that $V \leqslant E$. Then $E \in \operatorname{Nor}_{\mathfrak{F}}(G)$ by Theorems 4.5(ii) and 4.6 of [**1**]. Since $G^{\mathfrak{F}}$ is abelian, we have that $G = G^{\mathfrak{F}}E$ and $E \cap G^{\mathfrak{F}} = 1$. Therefore $V = E$ and $G^{\mathfrak{F}} \cap V = 1$.

Since V is subnormal in G, so is a minimal normal subgroup K of V. Now if K is an elementary abelian q-group, for some prime $q \in \operatorname{char} \mathfrak{F}$, then $O_q(G) \neq 1$, a contradiction. If K is the direct product of non-abelian simple groups, then G also has a minimal normal subgroup that is the product of non-abelian simple groups, again a contradiction. □

Theorem 2.6. *Let $\mathfrak{F}$ be either an $\mathrm{S_n}(\mathfrak{N}^*)$-closed saturated formation or a Fitting formation. Let us consider a group G and U and V subgroups of G such that U and V are subnormal in $\langle U, V \rangle$. Assume that U is a quasinilpotent group. Suppose that $\mathfrak{N} \subseteq \mathfrak{F}$. Then U permutes with $V^{\mathfrak{F}}$.*

Proof. Let G be a minimal counterexample and let U be a subgroup of G of minimal order making false the theorem.

Suppose first that U has two different maximal normal subgroups A_i, $i = 1, 2$. It is clear that $U = A_1A_2$. By minimality of U, both subgroups A_i permute with $V^{\mathfrak{F}}$. Hence U permutes with $V^{\mathfrak{F}}$, a contradiction.

Therefore U is a single-headed subnormal subgroup of G. Since U is a quasinilpotent group, we have two cases.

(1) U is perfect. In this case U permutes with $V^{\mathfrak{F}}$, a contradiction.

(2) U is a cyclic p-group for some prime p. Since $\mathfrak{N} \subseteq \mathfrak{F}$ we have that $U \in \mathfrak{F}$. By the above theorem, $G^{\mathfrak{F}} = U^{\mathfrak{F}}V^{\mathfrak{F}} = V^{\mathfrak{F}}$. Therefore $V^{\mathfrak{F}}$ is a normal subgroup of G, a contradiction. □

Corollary 2.7. *In the conditions of the theorem above, $U^{\mathfrak{F}}$ permutes with $V^{\mathfrak{F}}$.*

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