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ON THE COMPLETION OF LATIN RECTANGLES TO SYMMETRIC LATIN SQUARES

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Abstract

We find necessary and sufficient conditions for completing an arbitrary 2 by n latin rectangle to an n by n symmetric latin square, for completing an arbitrary 2 by n latin rectangle to an n by n unipotent symmetric latin square, and for completing an arbitrary 1 by n latin rectangle to an n by n idempotent symmetric latin square. Equivalently, we prove necessary and sufficient conditions for the existence of an $(n-1)$-edge colouring of K_n (n even), and for an n-edge colouring of K_n (n odd) in which the colours assigned to the edges incident with two vertices are specified in advance.

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1. Introduction

An n by n *(partial) latin square* is an n by n array in which each cell contains (at most) one symbol, chosen from an n-set, such that each symbol occurs (at most) once in each row and (at most) once in each column. A partial latin square is said to be *completed to (or embedded in) a latin square* if its empty cells are filled to produce a latin square. There exist many results on completions or embeddings of partial latin squares, see [2, Sections II.1.6 and IV.17]. Perhaps two of the best known relate to the *Evans Conjecture* and *Hall's Marriage Theorem* [4]. In 1960, Evans conjectured that any partial n by n latin square with at most $n-1$ filled cells can be completed to a latin square. The conjecture was proved in 1981 by Smetaniuk [7].

An m by n *latin rectangle* is an m by n array in which each cell contains one symbol, chosen from an n-set, such that each symbol occurs once in each row and at most once in each column. An m by n latin rectangle may be thought of as a partial

n by n latin square in which the first m rows are filled and the remaining cells are empty. An easy corollary of Hall's Theorem is that any m by n latin rectangle can be completed to an n by n latin square.

A latin square is *symmetric* if for $i \neq j$, the symbol in cell (i, j) is also in cell (j, i). Consider the more difficult problem of completing latin rectangles to symmetric latin squares. If $m = 1$, it is trivial to complete any m by n latin rectangle to a symmetric latin square. Just take any n by n symmetric latin square and permute the symbols so that the first row is the 1 by n latin rectangle. However, the question is far from trivial for $m > 1$. In this paper we prove necessary and sufficient conditions under which a 2 by n latin rectangle can be completed to a symmetric latin square; see Theorem 3.3.

A latin square is *unipotent* if every diagonal cell contains the same symbol. If there exists an n by n unipotent symmetric latin square, then n is necessarily even (since the number of occurrences of each symbol, other than the symbol which occurs on the diagonal, must be even). A latin square is *idempotent* if symbol i is in cell (i, i) for all i. If there exists an n by n idempotent symmetric latin square, then n is necessarily odd (since each symbol occurs once on the diagonal and an even number of times not on the diagonal). We also prove necessary and sufficient conditions under which a 2 by n latin rectangle can be completed to a unipotent symmetric latin square (see Theorem 3.1), and necessary and sufficient conditions under which a 1 by n latin rectangle can be completed to an idempotent symmetric latin square (see Theorem 3.2).

These results extend a rich literature of related embedding results. In the containing n by n latin square L, when either symmetry has been required of L, or the diagonal of L has been prescribed in some way, then results in the literature have limited the filled cells to all occur in cells (i, j), where $i, j \leq r$ for some $r < n$. For example, Cruse has settled such an embedding problem in the case where L is required to be symmetric [3]. It turns out that if n is odd then because of the limitation on where the filled cells occur, this result also solves the embedding problem in the case where L is both symmetric and idempotent. The more general result that predetermines the diagonal of L when L is symmetric was later solved by Andersen [1]. The extremely difficult related problem of specifying the diagonal of L, but not requiring L to be symmetric, has yet to be solved. However, it has been solved by Rodger in the case where $n \geq 2r + 1$ [6], and in the idempotent case where $n = 2r$ [5]. In none of the literature to this point has any result been able to deal with the case where the non-diagonal filled cells span the rows or columns of the containing latin square L, and either L is required to be symmetric or the diagonal of L is specified in some way. Such completions are the main focus of this paper.

An *edge colouring* of a graph G is an assignment of colours to the edges of G such that adjacent edges are assigned distinct colours. If k colours are used then the colouring is called a *k-edge colouring*. An n by n symmetric latin square defines a proper edge colouring of K_n as follows: with $\{v_1, v_2, \dots, v_n\}$ being the vertex set of K_n, let the symbol in cell (i, j), $i \neq j$, be the colour assigned to the edge $v_i v_j$. For n even, an n by n unipotent symmetric latin square is equivalent to an $(n-1)$-edge colouring of K_n (the colour corresponding to the symbol that occurs in the diagonal cells is not assigned to any edge). For n odd, an n by n symmetric latin square is equivalent to an n-edge colouring of K_n (each symbol must occur exactly once on the diagonal and the symbol in cell (i, i) is the colour which does not occur at vertex v_i). Our results allow us to prove necessary and sufficient conditions for the existence of an n-edge colouring of K_n in which the colours incident with two particular vertices are specified in advance, see Theorem 3.4 and Theorem 3.5.

The proof of our results will follow from the existence of *symmetric quasi-latin squares* with specified diagonals. We define and construct these in the following section.

2. Symmetric quasi-latin squares

We define a *symmetric quasi-latin square* (SQLS) with symbols $x_0, x_1, \dots, x_n$ to be an n by n array of cells such that

(i) each cell not on the diagonal contains exactly one symbol;
(ii) each cell on the diagonal contains exactly two distinct symbols;
(iii) each of the symbols $x_1, x_2, \dots, x_n$ occurs in exactly two diagonal cells and x_0 occurs in no diagonal cells;
(iv) for $1 \leq i < j \leq n$, cell (i, j) contains the same symbol as cell (j, i); and
(v) each symbol occurs exactly once in each row and exactly once in each column.

It is clear from (iii) and (iv) that the number of occurrences of each symbol is even and so there are no n by n SQLS's when n is odd.

Suppose an SQLS has symbol x in cells $(i_1, i_1), (i_2, i_3), (i_4, i_5), \dots, (i_{k-1}, i_k)$ and symbol y in cells $(i_1, i_2), (i_3, i_4), (i_5, i_6), \dots, (i_{k-2}, i_{k-1}), (i_k, i_k)$ where $i_1, i_2, \dots, i_k$ are distinct. If we put y instead of x in the cells

$$(i_1, i_1), (i_2, i_3), (i_3, i_2), (i_4, i_5), (i_5, i_4), \dots, (i_{k-1}, i_k), (i_k, i_{k-1})$$

and x instead of y in the cells

$$(i_1, i_2), (i_2, i_1), (i_3, i_4), (i_4, i_3), (i_5, i_6), (i_6, i_5), \dots, (i_{k-2}, i_{k-1}), (i_{k-1}, i_{k-2}), (i_k, i_k)$$

then we obtain a new SQLS whose diagonal differs from that of the original SQLS only in that y is in cell (i_1, i_1) instead of x, and x is in cell (i_k, i_k) instead of y. We call such a configuration of cells and symbols an (x, y)-*path* from cell (i_1, i_1) to cell (i_k, i_k).

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The diagonal of an n by n SQLS defines a 2-regular multigraph with n vertices in an obvious way: the vertices are the symbols occurring on the diagonal and uv is an edge if and only if u and v occur together in a diagonal cell (so a 2-cycle results whenever two distinct diagonal cells contain the same pair of symbols). An SQLS whose diagonal defines a 2-regular multigraph F will be called an SQLS(F). If $F = C_{m_1} + C_{m_2} + \cdots + C_{m_k}$ is the vertex disjoint union of k cycles of lengths $m_1, m_2, \ldots, m_k$, then an SQLS(F) with symbols $0, 1, \ldots, n$ ($n = m_1+m_2+\cdots+m_k$), with

- i and $i - m_t + 1$ in cell (i, i) for $i = \sum_{r=1}^{t} m_r$, $t = 1, 2, \ldots, k$; and with
- i and $i + 1$ in cell (i, i) for $i \in \{1, 2, \ldots, n\} \setminus \left\{\sum_{r=1}^{t} m_r : t = 1, 2, \ldots, k\right\}$

is said to be in *standard form*. It is clear that an SQLS(F) exists if and only if an SQLS(F) in standard form exists; either can be obtained from the other by applying a permutation σ to the symbols and a permutation π to the rows and columns. Unless stated otherwise, from here on, all SQLS(F)s will be assumed to be in standard form. We will however rearrange the order of the cycles in F to aid our constructions. We define $\mathscr{F}_n$ to be the class of all 2-regular multigraphs with n vertices.

If $n \equiv 0 \pmod 4$ then let $m = (n-4)/2$ and if $n \equiv 2 \pmod 4$ let $m = (n-2)/2$. We partition the cells of an SQLS into four regions as follows.

- The cells (i, j) with $1 \le i, j \le m$ will be called the *top left*.
- The cells (i, j) with $m + 1 \le i, j \le n$ will be called the *bottom right*.
- The cells (i, j) with $1 \le i \le m$ and $m + 1 \le j \le n$ will be called the *top right*.
- The cells (i, j) with $m + 1 \le i \le n$ and $1 \le j \le m$ will be called the *bottom left*.

Before our main SQLS constructions, we need the following lemma.

LEMMA 2.1. *Suppose $n \ge 6$ and let α be any permutation of $\{1, 2, \ldots, n\}$ with $\alpha(i) \le i - 1$ or $\alpha(i) = i + 1$. There exists a colouring of the edges of the n-cycle $(v_1, v_2, \ldots, v_n)$ with the colours $\{c_1, c_2, \ldots, c_n\}$ such that each edge receives a different colour and such that for $i \in \{1, 2, \ldots, n\}$, c_i, $c_{\alpha(i)}$ and the colours of the two edges incident with v_i are all distinct.*

PROOF. In this proof, the subscripts are reduced modulo n to the residues $1, \ldots, n$. For $i = 1, \ldots, n$, call c_i and $c_{\alpha(i)}$ the colours assigned to the vertex v_i. Let B be a bipartite graph with bipartition $\{E(C_n), \{c_1, \ldots, c_n\}\}$ of the vertex set $V(B)$. Define the edge set $E(B)$ by joining the edge $\{v_i, v_{i+1}\}$ to the colour c_j if and only if c_j is not assigned to either v_i or v_{i+1}. Then clearly $\delta(B) \ge n - 4$. It is sufficient the show that B contains a perfect matching. Suppose there exists a subset $S \subseteq \{c_1, c_2, \ldots, c_n\}$ such that the neighbourhood $N_B(S)$ of S contains fewer vertices than

S. Then $|S| \geq |N(S)| \geq \delta(B) \geq n-4$. If there is a vertex $v \in E(C_n) \setminus N(S)$ then it is adjacent to at least $\delta(B)$ vertices, none of which are in S. Therefore $n = |\{c_1, \ldots, c_n\}| \geq |S| + |N(v)| \geq 2(n-4)$; so $n \in \{6, 8\}$. It is now straightforward to verify that there is no such set S and the result follows by Hall's Theorem [4]. □

The following is well known, but is included for completeness.

LEMMA 2.2. *There exists a* 1*-factorization of* $K_{2x+1} - H$, *where* H *is a hamilton cycle.*

PROOF. The usual 1-factorization $\{F_0, \ldots, F_{2x-1}\}$ of K_{2x+1} defined by

$$F_i = \left\{\{\infty, i\}, \{i-j, i+j\} | 1 \leq j \leq x-1\right\}$$

has the property that $F_0 \cup F_1$ is a hamilton cycle. □

LEMMA 2.3. *Let* $n \geq 6$ *be even, let* $m = (n-2)/2$ *if* $n \equiv 2 \pmod 4$ *and let* $m = (n-4)/2$ *if* $n \equiv 0 \pmod 4$. *If there exists an* SQLS(F_1), $F_1 \in \mathscr{F}_m$, *then there exists an* SQLS$(F_1 + F_2)$ *for all* $F_2 \in \mathscr{F}_{n-m}$.

PROOF. We fill the diagonal cells so that they define $F_1 + F_2$ and complete the top left to an SQLS(F_1). If $n \equiv 2 \pmod 4$, we can fill the remaining cells in the bottom right with symbols chosen from $\{0, 1, 2, \ldots, m\}$, since this is equivalent to finding an $(n-m-1)$-edge colouring of K_{n-m}. If $n \equiv 0 \pmod 4$, then by Lemma 2.1 we can fill the $n-m$ diagonally opposite pairs of cells $\{(m+1, m+2), (m+2, m+1)\}$, $\{(m+2, m+3), (m+3, m+2)\}, \ldots, \{(n-1, n), (n, n-1)\}$, $\{(n, m+1), (m+1, n)\}$ with the symbols $m+1, m+2, \ldots, n$ such that each symbol occurs in exactly one pair of cells. We can then fill the remaining cells in the bottom right with symbols chosen from $\{0, 1, 2, \ldots, m\}$, since this is equivalent to finding an $(n-m-3)$-edge colouring of $K_{(n-m)}$ with the edges of a hamiltonian cycle removed (see Lemma 2.2).

Let R_0 be the 2 by $n-m$ latin rectangle constructed by placing, for $i = 1, 2, \ldots, n-m$, symbol $m+i$ in cell $(1, i)$ and the other symbol from cell $(m+i, m+i)$ of the existing partial SQLS in cell $(2, i)$. If $n \equiv 2 \pmod 4$, then we let $R = R_0$ and if $n \equiv 0 \pmod 4$, we let R be the 4 by $n-m$ latin rectangle defined as follows. Let rows 1 and 2 of R be equal to rows 1 and 2 of R_0. For $i = 1, 2, \ldots, n-m-1$, place the symbol from cell $(m+i+1, m+i)$ of the existing partial SQLS in cell $(3, i)$ and the symbol from cell $(m+1, n)$ in cell $(3, n-m)$. For $i = 2, 3, \ldots, n-m$, place the symbol from cell $(m+i-1, m+i)$ of the existing partial SQLS in cell $(4, i)$ and the symbol from cell $(n, m+1)$ in cell $(4, 1)$. Note that all the symbols in R are chosen from $\{m+1, m+2, \ldots, n\}$.

We can fill the cells in the top right with symbols chosen from $\{m+1, m+2, \ldots, n\}$, since this is equivalent to completing the latin rectangle R to an $(n-m)$ by $(n-m)$

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latin square. We then complete the SQLS$(F_1 + F_2)$ by filling the cell (j, i) with the symbol from cell (i, j) for $1 \le i \le m$ and $m + 1 \le j \le n$. $\square$

LEMMA 2.4. *Let $n = 6$ or $n \ge 10$ be even, let $m = (n-2)/2$ if $n \equiv 2 \pmod 4$ and let $m = (n-4)/2$ if $n \equiv 0 \pmod 4$. Let $2 \le r \le m-2$ or $r = m$ and let $2 \le s \le n-m-2$ or $s = n-m$. If there exists an* SQLS$(F_1 + C_r)$, $F_1 \in \mathscr{F}_{m-r}$, *then there exists an* SQLS$(F_1 + C_{r+s} + F_2)$ *for all* $F_2 \in \mathscr{F}_{n-m-s}$.

PROOF. Let $F = F_1 + C_{r+s} + F_2$ and let $F' = F_1 + C_r + C_s + F_2$. By Lemma 2.3 there exists an SQLS(F'). Note that the diagonal cells of an SQLS(F') differ from those of an SQLS(F) only in cell (m, m), where symbols m and $m - r + 1$ occur instead of symbols m and $m + 1$, and in cell $(m + s, m + s)$, where symbols $m + s$ and $m + 1$ occur instead of symbols $m + s$ and $m - r + 1$.

However, by the construction of the SQLS(F') in Lemma 2.3, we have the freedom to independently permute the rows in the top right (and simultaneously perform the corresponding permutation to the columns in the bottom left) and the symbols $0, 1, 2, \dots, m$ in the bottom right. We will carry out such permutations so that there exists an $(m - r + 1, m + 1)$-path from cell (m, m) to cell $(m + s, m + s)$. We can then interchange symbols $m - r + 1$ and $m + 1$ along this path so that the required an SQLS(F) results.

When $n \equiv 2 \pmod 4$, we permute the symbols $0, 1, 2, \dots, m$ in the bottom right so that symbol $m - r + 1$ occurs in cell $(m + s, t)$ where t is such that cell (m, t) contains the symbol $m + 1$. This ensures that there exists an $(m - r + 1, m + 1)$-path involving cells (m, m), (m, t), $(m + s, t)$, $(m + s, m + s)$, $(t, m + s)$, (t, m).

When $n \equiv 0 \pmod 4$, we permute the symbols $0, 1, 2, \dots, m$ in the bottom right so that symbol $m - r + 1$ occurs in cell $(m + s, t)$ where t is such that the occurrence of symbol $m + 1$ in column t is in the top right. Since $m \ge 4$, this is always possible (note that $m + 1$ occurs m times in the top right, there are $m + 1$ occurrences of the symbols $0, 1, 2, \dots, m$ in each row of the bottom right, and $2m+1 > n-m = m+4$). We then permute the rows in the top right so that symbol $m + 1$ occurs in cell (m, t). Then, as in the $n \equiv 2 \pmod 4$ case, we have an $(m - r + 1, m + 1)$-path involving cells (m, m), (m, t), $(m + s, t)$, $(m + s, m + s)$, $(t, m + s)$, (t, m). $\square$

LEMMA 2.5. *Let $n \equiv 0 \pmod 4$, $n \ge 12$, $m = (n-4)/2$, $3 \le r \le m-2$ or $r = m$, and let $2 \le s \le n - m - 2$ or $s = n - m$. If there exists an* SQLS$(F_1 + C_r)$, $F_1 \in \mathscr{F}_{m-r}$, *then there exists an* SQLS$(F_1 + C_{r-1} + C_{s+1} + F_2)$ *for all* $F_2 \in \mathscr{F}_{n-m-s}$.

PROOF. Let $F = F_1 + C_{r-1} + C_{s+1} + F_2$ and let $F' = F_1 + C_r + C_s + F_2$. By Lemma 2.3 there exists an SQLS(F'). Note that the symbols in the diagonal cells of an SQLS(F') differ from those of an SQLS(F) only

- in cell $(m-1, m-1)$ where symbols $m-1$ and m occur instead of symbols $m-1$ and $m-r+1$;
- in cell (m, m) where symbols m and $m-r+1$ occur instead of symbols m and $m+1$; and
- in cell $(m+s, m+s)$ where symbols $m+s$ and $m+1$ occur instead of symbols $m+s$ and m.

However, by the construction of the SQLS(F'), we have the freedom to independently permute the rows in the top right and the symbols $0, 1, 2, \dots, m$ in the bottom right. We will carry out such permutations so that there exists an $(m, m+1)$-path from cell $(m-1, m-1)$ to cell $(m+s, m+s)$, an $(m-r+1, m+1)$-path from cell $(m-r+1, m-r+1)$ to cell $(m+1, m+1)$, and such that no cell is common to these two paths. By switching the symbols along these two paths and then interchanging symbols $m-r+1$ and $m+1$ throughout the SQLS we will obtain the required SQLS(F).

The $(m, m+1)$-path will involve cells $(m-1, m-1)$, $(m-1, t)$, $(m+s, t)$, $(m+s, m+s)$, $(t, m+s)$ and $(t, m-1)$ where t is some integer in the range $m+1 \le t \le n$. Choose t arbitrarily such that the symbol $m+1$ in column t is in the top right and such that cell $(m+s, t)$ contains one of the symbols $0, 1, 2, \dots, m$. Since $m \ge 4$ implies $n-m \ge 8$, since there are four occurrences of symbol $m+1$ in the bottom right, and since in each row of the bottom right there are two off-diagonal cells which are filled with symbols other than $0, 1, 2, \dots, m$, there are at least two choices for t. Permute the symbols in the bottom right so that m is in cell $(m+s, t)$ and permute the rows in the top right so that symbol $m+1$ is in cell $(m-1, t)$, thus ensuring that the required $(m, m+1)$-path exists.

The $(m-r+1, m+1)$-path will involve cells $(m-r+1, m-r+1)$, $(m-r+1, t')$, $(m+1, t')$, $(m+1, m+1)$, $(t', m-r+1)$ and $(t', m+1)$ where t' is some integer in the range $m+1 \le t' \le n$. We require that $t' \ne t$, the symbol $m+1$ of column t' is in the top right, and that cell $(m+1, t')$ contains one of the symbols $0, 1, 2, \dots, m-1$. As in the case of finding t for the $(m, m+1)$-path, we have at least two choices for t' after excluding the columns in which $m+1$ occurs in the bottom right and the two cells which do not contain one of the symbols $0, 1, 2, \dots, m$. However, we now also need to ensure that $t \ne t'$ and that the cell $(m+1, t')$ does not contain m. But since the two columns that do not contain one of the symbols $0, 1, 2, \dots, m$ in row $m+s$ of the bottom right are not the same as the two columns that do not contain one of the symbols $0, 1, 2, \dots, m$ in row $m+1$ of the bottom right, we can always find a suitable t'. We now permute the symbols, leaving symbol m fixed, in the bottom right so that symbol $m-r+1$ is in cell $(m+1, t')$ and we permute the rows in the top right, leaving row $m-1$ fixed, so that symbol $m+1$ is in cell $(m-r+1, t')$, thus ensuring that the required $(m-r+1, m+1)$-path exists. □

LEMMA 2.6. *Let* $n \equiv 2 \pmod 4$, $n \geq 10$, $m = (n-2)/2$, $2 \leq r \leq m-2$ *or* $r = m$ *and let* $3 \leq s \leq n-m-2$ *or* $s = n-m$. *If there exists an* SQLS$(F_1 + C_r)$, $F_1 \in \mathscr{F}_{m-r}$, *then there exists an* SQLS$(F_1 + C_{r+1} + C_{s-1} + F_2)$ *for all* $F_2 \in \mathscr{F}_{n-m-s}$.

PROOF. Let $F = F_1 + C_{r+1} + C_{s-1} + F_2$ and let $F' = F_1 + C_r + C_s + F_2$. By Lemma 2.3 there exists an SQLS(F'). Note that the symbols in the diagonal cells of an SQLS(F') differ from those of an SQLS(F) only

- in cell (m, m) where symbols m and $m-r+1$ occur instead of symbols m and $m+1$;
- in cell $(m+1, m+1)$ where symbols $m+1$ and $m+2$ occur instead of symbols $m+1$ and $m-r+1$; and
- in cell $(m+s, m+s)$ where symbols $m+s$ and $m+1$ occur instead of symbols $m+s$ and $m+2$.

As in the previous lemma, we have the freedom to permute the rows in the top right and we have an amount of freedom when we fill the cells in the bottom right with the symbols $0, 1, 2, \ldots, m$. We will ensure that there exists an $(m-r+1, m+1)$-path from cell (m, m) to cell $(m+s, m+s)$, an $(m-r+1, m+2)$-path from cell $(m-r+1, m-r+1)$ to cell $(m+2, m+2)$, and such that no cell is common to these two paths. By switching the symbols along these two paths and then interchanging symbols $m-r+1$ and $m+2$ throughout the SQLS we will obtain the required SQLS(F).

The $(m-r+1, m+1)$-path will involve cells (m, m), (m, t), $(m+s, t)$, $(m+s, m+s)$, $(t, m+s)$, (t, m) and the $(m-r+1, m+2)$-path will involve cells $(m-r+1, m-r+1)$, $(m-r+1, t')$, $(m+2, t')$, $(m+2, m+2)$, $(t', m-r+1)$, $(t', m+2)$ where t and t' are integers in the range $m+1 \leq t \leq n$.

There are $n-m-3 = m-1$ columns in the top right that contain both $m+1$ and $m+2$ (the columns that do not contain both $m+1$ and $m+2$ are columns $m+1$, $m+2$ and $m+s$). Since $n \geq 10$, $m \geq 4$, and so we always have at least 3 such columns in the top right. Let these columns be t, t' and t''. We permute the rows in the top right so that symbol $m+1$ is in cell (m, t). This may force symbol $m+2$ into cell (m, t') or into cell (m, t'') but not both, and so we can assume without loss of generality that symbol $m+2$ is not in cell (m, t'). Thus we can permute the rows in the top right (leaving $m+1$ in cell (m, t)) so that symbol $m+2$ is in cell $(m-r+1, t')$.

When constructing the SQLS(F'), we ensure that symbol $m-r+1$ occurs in cells $(m+s, t)$ and $(m+2, t')$. This is possible, as filling in these cells is equivalent to finding an $(n-m-1)$-edge colouring of K_{n-m} (with vertex set $\{m+1, m+2, \ldots, n\}$) and we can ensure by relabeling the vertices that the the edges corresponding to the cells $(m+s, t)$ and $(m+2, t')$ are assigned the same colour. Note that our choice of t and t' ensures that $m+2$, $m+s$, t and t' are distinct. □

LEMMA 2.7. *Let $n \le 12$ be even and let $F \in \mathscr{F}_n$. Then there exists an* SQLS(F) *if and only if $F \ne C_2 + C_2$.*

PROOF. For $n = 2$ and $n = 4$, the only cases are $F = C_2$, $F = C_4$ and $F = C_2 + C_2$. It is easy to see that there is no SQLS$(C_2 + C_2)$. An SQLS(C_2) and an SQLS(C_4) are shown below. For $n = 6$, the result follows by Lemma 2.3 (and the existence of an SQLS(C_2)) if F contains a 2-cycle. For $F = C_6$, we can use Lemma 2.4 with $r = 2$ and $s = 4$. This leaves only the case of an SQLS$(C_3 + C_3)$ which is shown below.

1, 2	0
0	2, 1

1, 2	4	0	3
4	2, 3	1	0
0	1	3, 4	2
3	0	2	4, 1

1, 2	4	5	6	3	0
4	2, 3	6	1	0	5
5	6	3, 1	0	4	2
6	1	0	4, 5	2	3
3	0	4	2	5, 6	1
0	5	2	3	1	6, 4

For the case $n = 8$, the result follows by Lemma 2.3 (and the existence of an SQLS(C_2)) if F contains a 2-cycle. This leaves only an SQLS(C_8), an SQLS$(C_4 + C_4)$ and an SQLS$(C_5 + C_3)$ which are shown below.

1, 2	8	7	0	3	5	6	4
8	2, 3	6	7	4	0	1	5
7	6	3, 4	8	1	2	5	0
0	7	8	4, 5	2	1	3	6
3	4	1	2	5, 6	8	0	7
5	0	2	1	8	6, 7	4	3
6	1	5	3	0	4	7, 8	2
4	5	0	6	7	3	2	8, 1

1, 2	8	7	5	3	4	6	0
8	2, 3	6	7	4	0	5	1
7	6	3, 4	8	1	5	0	2
5	7	8	4, 1	0	2	3	6
3	4	1	0	5, 6	8	2	7
4	0	5	2	8	6, 7	1	3
6	5	0	3	2	1	7, 8	4
0	1	2	6	7	3	4	8, 5

1, 2	8	7	6	3	5	4	0
8	2, 3	6	7	4	1	0	5
7	6	3, 4	8	2	0	5	1
6	7	8	4, 5	0	2	1	3
3	4	2	0	5, 1	8	6	7
5	1	0	2	8	6, 7	3	4
4	0	5	1	6	3	7, 8	2
0	5	1	3	7	4	2	8, 6

For the case $n = 10$, the result follows by Lemma 2.3 (and the existence of an SQLS(C_4)) if F contains a 4-cycle. Also, if F contains a cycle C_x with $x \ge 6$ then the result follows by Lemma 2.4 with $r = 4$ and $s = x - 4$. For $F = C_5 + C_5$ and

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$F = C_5+C_3+C_2$ we use Lemma 2.6 with $r = 4$ and with $s = 6$ and $s = 4$ respectively. This leaves only an SQLS$(C_3 + C_3 + C_2 + C_2)$ and an SQLS$(C_2 + C_2 + C_2 + C_2 + C_2)$ which are shown below.

1, 2	10	9	8	0	7	3	4	5	6
10	2, 1	8	9	7	0	4	3	6	5
9	8	3, 4	10	1	2	5	6	7	0
8	9	10	4, 3	2	1	6	5	0	7
0	7	1	2	5, 6	8	9	10	3	4
7	0	2	1	8	6, 5	10	9	4	3
3	4	5	6	9	10	7, 8	0	1	2
4	3	6	5	10	9	0	8, 7	2	1
5	6	7	0	3	4	1	2	9, 10	8
6	5	0	7	4	3	2	1	8	10, 9

1, 2	10	9	8	0	7	3	4	5	6
10	2, 3	8	9	7	1	4	5	6	0
9	8	3, 1	10	2	0	5	6	7	4
8	9	10	4, 5	1	2	6	3	0	7
0	7	2	1	5, 6	8	9	10	4	3
7	1	0	2	8	6, 4	10	9	3	5
3	4	5	6	9	10	7, 8	0	1	2
4	5	6	3	10	9	0	8, 7	2	1
5	6	7	0	4	3	1	2	9, 10	8
6	0	4	7	3	5	2	1	8	10, 9

For the case $n = 12$, the result follows by Lemma 2.3 (and the existence of an SQLS(C_4)) if F contains a 4-cycle. Also, if F contains a cycle C_x with $x \geq 6$ then the result follows by Lemma 2.4 with $r = 4$ and $s = x - 4$. For $F = C_3 + C_3 + C_3 + C_3$ and $F = C_3 + C_3 + C_2 + C_2 + C_2$, we use Lemma 2.5 with $r = 4$, $s = 2$ and with $F_2 = C_3 + C_3$ and $F_2 = C_2 + C_2 + C_2$ respectively. For $F = C_3 + C_5 + C_2 + C_2$, we use Lemma 2.5 with $r = 4$, $s = 4$ and with $F_2 = C_2 + C_2$. This leaves only an SQLS$(C_5 + C_5 + C_2)$ and an SQLS$(C_2 + C_2 + C_2 + C_2 + C_2 + C_2)$ which are shown in Figure 1. □

LEMMA 2.8. *Let n be an integer and let $F \in \mathscr{F}_n$. Then there exists an* SQLS(F) *if and only if n is even and $F \neq C_2 + C_2$.*

PROOF. The proof is by induction. We have already noted that there do not exist n by n SQLS's for n odd and the result is true for all $F \in \mathscr{F}_n$ with $n \leq 12$ by Lemma 2.7. So assume $n \geq 14$ is even and assume the result is true for all even

1, 2	10	11	12	9	8	4	0	5	6	3	7
10	2, 1	12	11	8	9	0	4	6	5	7	3
11	12	3, 4	10	7	0	6	9	1	8	2	5
12	11	10	4, 3	0	7	5	6	2	1	9	8
9	8	7	0	5, 6	1	2	12	11	3	4	10
8	9	0	7	1	6, 5	2	11	4	12	10	2
4	0	6	5	2	3	7, 8	10	12	11	1	9
0	4	9	6	12	11	10	8, 7	3	2	5	1
5	6	1	2	11	4	12	3	9, 10	0	8	7
6	5	8	1	3	12	11	2	0	10, 9	7	4
3	7	2	9	4	10	1	5	8	7	11, 12	0
7	3	5	8	10	2	9	1	7	4	0	12, 11

1, 2	10	11	12	9	8	4	0	5	3	6	7
10	2, 3	12	11	8	9	0	4	6	5	7	1
11	12	3, 4	10	7	0	6	1	2	8	5	9
12	11	10	4, 5	0	2	1	6	7	9	3	8
9	8	7	0	5, 1	3	2	12	11	4	10	6
8	9	0	2	3	6, 7	5	11	4	12	1	10
4	0	6	1	2	5	7, 8	10	12	11	9	3
0	4	1	6	12	11	10	8, 9	3	7	2	5
5	6	2	7	11	4	12	3	9, 10	1	8	0
3	5	8	9	4	12	11	7	1	10, 6	0	2
6	7	5	3	10	1	9	2	8	0	11, 12	4
7	1	9	8	6	10	3	5	0	2	4	12, 11

FIGURE 1. An SQLS($C_5 + C_5 + C_2$) and an SQLS($C_2 + C_2 + C_2 + C_2 + C_2 + C_2$).

integers less than n. We use Lemmas 2.3–2.6. As before, let $m = (n-2)/2$ if $n \equiv 2 \pmod 4$ and $m = (n-4)/2$ if $n \equiv 0 \pmod 4$. Since $n \geq 14$, $m \geq 6$ and so an SQLS(F') exists for all $F' \in \mathscr{F}_m$.

It is clear that for any $F \in \mathscr{F}_n$, the cycles of F can be arranged in some order $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$ such that $m - 1 \notin \left\{\sum_{i=1}^{t} m_i : t = 1, 2, \ldots, k\right\}$ unless there exists an integer that divides both $m-1$ and n. Similarly, the cycles of any $F \in \mathscr{F}_n$ can be arranged in some order $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$ such that $m+1 \notin \left\{\sum_{i=1}^{t} m_i : t = 1, 2, \ldots, k\right\}$ unless there exists an integer that divides both $m+1$ and n.

When $n \equiv 2 \pmod 4$, $m - 1$ and n are relatively prime, and so the cycles of F can be ordered such that we can use either Lemma 2.3, 2.4 or 2.6 to construct an SQLS(F). Similarly, when $n \equiv 0 \pmod 4$, $m + 1$ and n are relatively prime, and so the cycles of F can be ordered such that we can use either Lemma 2.3, 2.4 or 2.5 to construct an SQLS(F). □

3. Main results

With Lemma 2.8 in hand, we are now ready to prove our main results. Unless stated otherwise, all n by n latin squares will have symbols $1, 2, \dots, n$.

THEOREM 3.1. *Let $n \geq 2$ be an integer. A 2 by n latin rectangle can be completed to a unipotent n by n symmetric latin square if and only if*

(i) *n is even;*
(ii) *the symbol in cell $(1, 2)$ is also in cell $(2, 1)$;*
(iii) *the symbol in cell $(1, 1)$ is also in cell $(2, 2)$; and*
(iv) *the 2 by n latin rectangle cannot be obtained from*

1	2	3	4	5	6
2	1	4	3	6	5

or

1	2	3	5	4	6
2	1	4	6	3	5

or

1	2	3	5	6	4
2	1	4	6	5	3

by any symbol permutation.

PROOF. Conditions (i), (ii) and (iii) are clearly necessary. Suppose a latin rectangle satisfying conditions (i)–(iv) is completed to a 6 by 6 symmetric latin square. By symmetry row 3 contains symbols 3 and 4 in cells (3, 1) and (3, 2). Therefore the occurrence in row 3 of symbols 1 and 2 must be in the columns that already contain symbols 5 and 6; so symbol 1 does not occur in cell (3, 3) and the latin square is not unipotent. Hence condition (iv) is also necessary.

We use Lemma 2.8 to prove sufficiency. To construct the required latin square L, we first place, for $i = 3, 4, \dots, n$ and $t = 1, 2$, the symbol in cell (t, i) of the latin rectangle in cell (i, t) of L. By Lemma 2.8 and condition (iv), there exists an $(n-2)$ by $(n-2)$ SQLS T with diagonal defined by placing, for $i = 3, 4, \dots, n$, the symbols in cells $(1, i)$ and $(2, i)$ of the latin rectangle in cell $(i-2, i-2)$ of T. Let 0 be the symbol not occurring on the diagonal of T and let y and z be the symbols in cells $(1, 1)$ and $(1, 2)$ of L respectively. For $3 \leq j \leq n$, if cell $(i-2, j-2)$ of T contains symbol x then place y, z or x in cell (i, j) of L if $i = j$, $x = 0$, or otherwise respectively. □

THEOREM 3.2. *Let $n \geq 1$ be an integer. A 1 by n latin rectangle can be completed to an idempotent n by n symmetric latin square if and only if*

(i) *n is odd;*
(ii) *the symbol in cell $(1, 1)$ is 1;*

(iii) *for* $i = 2, 3, \ldots, n$, *symbol* i *is not in cell* $(1, i)$; *and*

(iv) *the* 1 *by* n *latin rectangle is not*

1	3	2	5	4

or

1	4	5	2	3

or

1	5	4	3	2

.

PROOF. Conditions (i), (ii) and (iii) are clearly necessary. To see that the latin rectangles given in condition (iv) cannot be completed to a 5 by 5 idempotent symmetric latin square, fill in the first column (by symmetry) and the diagonal, and then observe that symbol 1 is forced to occur in at least two cells of column 2.

Sufficiency follows almost immediately from Theorem 3.1. First, construct a 2 by $n + 1$ latin rectangle R with first row $\infty, 1, 2, \ldots, n$, symbol 1 in cell (2, 1), symbol ∞ in cell (2, 2) and the symbol from cell $(1, i)$ of the given latin rectangle in cell $(2, i + 1)$ for $i = 2, 3, \ldots, n$. By Theorem 3.1, R can be completed to an $n + 1$ by $n + 1$ unipotent symmetric latin square U. The required n by n idempotent symmetric latin square can then be obtained from U by deleting the first row and column, and then replacing the symbol in cell (i, i) with symbol i for all i. □

THEOREM 3.3. *Let* $n \geq 2$ *be an integer. A* 2 *by* n *latin rectangle can be completed to an* n *by* n *symmetric latin square if and only if*

(i) *the symbol in cell* (1, 2) *is also in cell* (2, 1);

(ii) *if* n *is odd, the symbols in cells* (1, 1) *and* (2, 2) *are distinct*; *and*

(iii) *the* 2 *by* n *latin rectangle cannot be obtained from*

1	2	3	4	5
2	3	1	5	4

or

1	2	4	3	5
2	3	5	1	4

or

1	2	4	5	3
2	3	5	4	1

by any symbol permutation.

PROOF. Condition (i) is clearly necessary. If n is odd, then in any n by n symmetric latin square each symbol occurs exactly once on the diagonal and so condition (ii) is also necessary. It is simpler if we deal separately with the cases $n = 2, 3, 4, 5$ and 6. First we make the following observation.

If we apply any permutation π of $\{3, 4, \ldots, n\}$ to the columns of a given 2 by n latin rectangle R (that is, we shift the symbol in cell (i, j) to cell $(i, \pi(j))$ for $i = 1, 2$ and $j = 3, 4, \ldots, n$) and then apply any permutation σ of $\{1, 2, \ldots, n\}$ to the symbols, we obtain a 2 by n latin rectangle R'. If R' can be completed to an n by n symmetric latin square L', then R can be completed a symmetric latin square L. Applying the symbol permutation σ^{-1} to L' and then shifting the symbol in cell (i, j) to cell $(\pi^{-1}(i), \pi^{-1}(j))$ for $3 \leq i, j \leq n$ results in a symmetric latin square L which has R as its first two rows. Hence we need only consider latin rectangles that cannot be obtained from each other by applying such column and symbol permutations.

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The cases $n = 2$ and $n = 3$ are trivial and for $n = 4$, completions of the two latin rectangles that we need to consider are shown below.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

For $n = 5$, we need to consider the following two latin rectangles.

1	2	3	4	5
2	3	1	5	4

1	2	3	4	5
2	3	4	5	1

It is straightforward to check that the first has no completion and so condition (iii) of the theorem is necessary. A completion of the second is shown below.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

For $n = 6$, there are five cases to consider and completions of each of these are shown below.

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	3	4
6	5	2	1	4	3

1	2	3	4	5	6
2	1	4	5	6	3
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	1	4
6	3	5	2	4	1

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	2	3	1
5	6	4	3	1	2
6	4	5	1	2	3

1	2	3	4	5	6
2	3	4	1	6	5
3	4	5	6	1	2
4	1	6	5	2	3
5	6	1	2	3	4
6	5	2	3	4	1

1	2	3	4	5	6
2	3	4	5	6	1
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	1	4
6	1	5	2	4	3

Now assume that $n \geq 7$ is odd. We begin by adding an extra column and placing a new symbol ∞ in cells $(1, n + 1)$ and $(2, n + 1)$. We then switch the symbols in cells $(1, 1)$ and $(1, n + 1)$ and switch the symbols in cells $(2, 2)$ and $(2, n + 1)$ to

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obtain an $(n+1)$ by $(n+1)$ latin rectangle. By Theorem 3.1, this latin rectangle can be completed to a unipotent symmetric latin square. For $i = 1, 2, \ldots, n$, we replace symbol ∞ in cell (i, i) with the symbol from cell $(i, n+1)$. We then delete row $n+1$ and column $n+1$ to obtain the required n by n symmetric latin square.

Now assume that $n \geq 8$ is even. Let x, y and z be the symbols in cells $(1, 1)$, $(2, 2)$ and $(1, 2)$ respectively (it is possible that $x = y$) and let w be a symbol, distinct from x, y and z that is not in the same column as either x or y in the latin rectangle (since $n \geq 8$, such a symbol exists). Suppose w is in cells $(1, r)$ and $(2, s)$. We begin by switching symbols x and w in row 1 and symbols y and w in row 2. The resulting latin rectangle can be completed to a unipotent symmetric latin square by Theorem 3.1. We then switch symbols x and w in rows 1 and r and switch symbols y and w in rows 2 and s to obtain the required symmetric latin rectangle. □

Because of the equivalence between n by n unipotent symmetric latin squares and $(n-1)$-edge colourings of K_n for n even, and the equivalence between symmetric latin squares and n-edge colourings of K_n for n odd, we obtain the following two results as immediate corollaries of Theorem 3.1 and Theorem 3.3 respectively.

THEOREM 3.4. *Let n be even and let u and v be two vertices of K_n. An assignment A of colours to the edges incident with either u or v can be completed to an $(n-1)$-edge colouring of K_n if and only if*

- *A assigns distinct colours to adjacent edges; and*
- *if $n = 6$, each 4-cycle that is assigned colours by A is assigned at least three distinct colours.*

THEOREM 3.5. *Let n be odd and let u and v be two vertices of K_n. An assignment A of colours to the edges incident with either u or v can be completed to an n-edge colouring of K_n if and only if*

- *A assigns distinct colours to adjacent edges;*
- *every colour is assigned either to u or to v; and*
- *if $n = 5$, each 4-cycle that is assigned colours by A is assigned at least three distinct colours.*

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