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Multiplication operators and composition operators with closed ranges

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The characterizations of the closed ranges of the multiplication operators and the composition operators on $L^2(\lambda)$ are reported in this paper.

1. Preliminaries

Let ϕ be a measurable transformation on a σ-finite measure space (X, S, λ) into itself. Then the composition operator C_ϕ, defined as

$$C_\phi f = f \circ \phi \quad \text{for every} \quad f \in L^2(\lambda) ,$$

is a bounded linear transformation on $L^2(\lambda)$. The multiplication operator M_θ induced by an essentially bounded measurable function θ on X is defined by the relation

$$M_\theta f = \theta . f \quad \text{for every} \quad f \in L^2(\lambda) .$$

The purpose of this note is to characterize multiplication operators and composition operators with closed ranges.

If H is a Hilbert space, then $B(H)$ denotes the Banach algebra of all bounded linear operators on H. If A is an element of $B(H)$, then $R(A)$ and $N(A)$ denote the range and the null space of A respectively.

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For a complex-valued measurable function θ on X the set Z^θ is defined by $Z^\theta = X \backslash \{x \in X : \theta(x) = 0\}$.

2. Multiplication operators and composition operators with closed ranges

First we shall give some examples of the multiplication operators and the composition operators with non-closed ranges.

EXAMPLE 1. Let $l^2(N)$ denote the Hilbert space of all square-summable sequences of complex numbers. Let M_θ be the multiplication operator induced by the function θ defined as

$$\theta(n) = \begin{cases} 0 & \text{for } n = 1 \text{ and } n = 2 , \\ 1/n & \text{for } n = 3, 4, \ldots . \end{cases}$$

Then the range of M_θ consists of all sequences $\langle \delta_1, \delta_2, \delta_3, \ldots \rangle$ with $\sum_{n=3}^{\infty} n^2 |\delta_n|^2 < \infty$, and it is dense in $l^2(N_2)$, where

$$l^2(N_2) = \left\{ \{x_n\} : x_1 = x_2 = 0 \text{ with } \sum_{n=3}^{\infty} |x_n|^2 < \infty \right\} ;$$

since it does not contain the sequence $\langle 0, 0, 1/3, 1/4, \ldots \rangle$, it is not closed.

EXAMPLE 2. If $\theta(x) = x$, then M_θ does not have closed range in $L^2[0, 1]$.

EXAMPLE 3. Let N be the set of positive integers and let $0 < a < 1$. Then define λ on N by $\lambda(\{n\}) = a^{2n}$. If ϕ is a function on N defined as $\phi(n) = n/2$ if n is even and $\phi(n) = (n+1)/2$ if n is odd, then C_ϕ is a composition operator on $l^2(\lambda)$, where $l^2(\lambda) = \left\{ \{x_n\} : \sum_{n=1}^{\infty} \lambda(n) |x_n|^2 < \infty \right\}$. For every $n \in N$ let $f^{(n)}$ be the sequence defined by $f^{(n)}(m) = 0$ if $m \le n$ and $f^{(n)}(m) = 1$ if $m > n$.

Then $\left\|C_\phi f^{(n)}\right\|^2/\|f^{(n)}\|^2 = a^{2n}$. This shows that C_ϕ is not bounded below. Since C_ϕ is one-to-one the range of C_ϕ is not closed.

LEMMA 2.1. *Let $A \in B(H)$. Then A has closed range if and only if it is bounded away from zero on $(N(A))^\perp$.*

Proof. The necessary part follows from [*1*, Problem 41], and sufficiency is clear.

COROLLARY. *Every partial isometry has closed range.*

THEOREM 2.1. *Let $M_\theta \in B(L^2(\lambda))$. Then M_θ has closed range if and only if θ is bounded away from zero on Z^θ.*

Proof. Let $X_1 = \{x : \theta(x) = 0\}$ and $X_2 = X\backslash X_1$. Then we can write $L^2(X, S, \lambda) = L^2(X_1, S_1, \lambda) \oplus L^2(X_2, S_2, \lambda)$, where $S_1 = S \cap X_1$ and $S_2 = S \cap X_2$. Here $N(M_\theta) = L^2(X_1, S_1, \lambda)$ and $(N(M_\theta))^\perp = L^2(X_2, S_2, \lambda)$. Now suppose θ is bounded away from zero on Z^θ. Then M_θ is invertible on $(N(M_\theta))^\perp = L^2(X_2, S_2, \lambda)$ [*1*, Problem 52]. Therefore $R(M_\theta) = L^2(X_2, S_2, \lambda)$ is closed.

Since X is σ-finite, we can write $X = \bigcup_{i=1}^{\infty} Y_i$, where $\lambda(Y_i) < \infty$ for all i. There is no loss of generality in assuming that $\lambda(Y_i) = 1$ for all $i \in N$. Now suppose θ is not bounded away from zero on $Z^\theta = X_2$. Then $E_j = \{x : x \in X_2 \text{ and } |\theta(x)| < 1/j\}$ has positive measure and let $F_{j_i} = Y_i \cap E_j$. Now define g_j as

$$g_j = \sum_{n=1}^{\infty} (1/n)\chi_{F_{j_n}} .$$

Then $\|M_\theta g_j\|/\|g_j\| \le 1/j$. Thus M_θ is not bounded below on $(N(M_\theta))^\perp$, and hence, by Lemma 2.1, M_θ does not have closed range.

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LEMMA 2.2. *Let* $A \in B(H)$ *be normal. Then* A *has closed range if and only if* A^n *has closed range for some* $n \in N$.

Proof. Since A is normal, by the Spectral Theorem, A is unitarily equivalent to a multiplication operator M_θ , and hence A^n is unitarily equivalent to M_{θ^n} . Suppose A^n has closed range. Then θ^n is bounded away from zero on Z^{θ^n} , which implies that θ is bounded away from zero on Z^θ . Hence, by Theorem 2.1, A has closed range.

The necessary part follows similarly.

LEMMA 2.3. *Let* $A \in B(H)$. *Then* A *has closed range if and only if* A^*A *has closed range.*

Proof. Sufficiency follows from Theorem 1 [6, p. 205] and Lemma 2.1.

Conversely, suppose A^*A has closed range. We write $A = UP$, where U is partial isometry and $P = \sqrt{A^*A}$ [1, Solution for Problem 105]. Since P is normal and P^2 has closed range, therefore, from Lemma 2.2, P has closed range. The rest of the proof follows from Lemma 2.1.

THEOREM 2.2. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *has closed range if and only if* f_0 *is bounded away from zero on* Z^{f_0} *, where* f_0 *is the Radon-Nikodym derivative of the measure* $\lambda\phi^{-1}$ *with respect to* λ .

Proof. Since $C_\phi^*C_\phi = M_{f_0}$, where $f_0 = d\lambda\phi^{-1}/d\lambda$, [3], the proof follows from Lemma 2.3 and Theorem 2.1.

Let $p = \{p_1, p_2, \ldots\}$ be a sequence of non-zero positive numbers and let

$$l^2(p) = \left\{\{x_n\} : \sum_{n=1}^{\infty} p_n|x_n|^2 < \infty\right\} .$$

Then $l^2(p)$ is a Hilbert space.

COROLLARY. *If* $\inf p = \alpha_1 > 0$ *and* $\sup p = \alpha_2 < \infty$ *then all* *composition operators on* $l^2(p)$ *have closed range.*

Proof.

$$f_0(n) = \lambda\phi^{-1}(n)/\lambda(n) \geq \alpha_1/\alpha_2 = \alpha > 0 \quad \text{if} \quad n \in \phi(N)$$
$$= 0 \quad \text{if} \quad n \in N\backslash\phi(N) .$$

Therefore f_0 is bounded away from zero on Z^{f_0}. Hence C_ϕ has closed range.

COROLLARY. *Every composition operator on* $l^2(N)$ *has closed range.*

EXAMPLE. Let ϕ be the real valued function on the set of real numbers R defined by $\phi(x) = x + 1$ if $x \in (-\infty, 4]$ and $\phi(x) = x + 2$ if $x \in (4, \infty)$. Then C_ϕ is a composition operator on $L^2(-\infty, \infty)$ and

$$f_0(x) = 1 \quad \text{if} \quad x \in (-\infty, 5] \cup (6, \infty) ,$$

and

$$f_0(x) = 0 \quad \text{if} \quad x \in (5, 6] .$$

Hence, by Theorem 2.2, C_ϕ has closed range.

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