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Asymptotic estimate of solutions in a 4th-order parabolic equation with the Frobenius norm of a Hessian matrix

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Abstract. This paper deals with a 4th-order parabolic equation involving the Frobenius norm of a Hessian matrix, subject to the Neumann boundary conditions. Some threshold results for blow-up or global or extinction solutions are obtained through classifying the initial energy and the Nehari energy. The bounds of blow-up time, decay estimates, and extinction rates are studied, respectively.

1 Introduction

In this paper, we study the following 4th-order parabolic problem involving the Frobenius norm of a Hessian matrix:

$$(1.1)\qquad \begin{cases} u_t - \Delta u + \Delta^2 u - 2|\Delta u|^2 + 2|D^2 u|^2 = |u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \dfrac{\partial u}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(1 \le N \le 3)$ is a general bounded domain with smooth boundary, η is the unit outward normal vector on $\partial\Omega$, the initial datum $u_0 \in H_0^2(\Omega)$, the exponent p is a positive constant, T is the maximal existence time of (1.1), and the Frobenius norm of the Hessian matrix is defined as

$$|D^2 u| := \left[\sum_{i,j=1}^{3} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \right]^{\frac{1}{2}}.$$

By direct computation, problem (1.1) can be rewritten as

$$(1.2)\qquad \begin{cases} u_t - \Delta u + \Delta^2 u - \operatorname{div}(2\Delta u \nabla u) + \Delta|\nabla u|^2 = |u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \dfrac{\partial u}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Problem (1.1) or (1.2) could be used to describe the growth of thin surfaces when exposed to molecular beam epitaxy (see [13, 17, 9, 5]). In particular, u can either

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represent the absolute thickness of the film or rather the relative surface height – that is, the deviation of the film height at the point x from the mean film thickness at time t (see [14]); $-\Delta u$ indicates the diffusion due to evaporation-condensation (see [12]); $\Delta^2 u$ indicates capillarity-driven surface diffusion (see [12]); $\Delta|\nabla u|^2 - \operatorname{div}(2\Delta u \nabla u)$ is related to the equilibration of the inhomogeneous concentration of the diffusing particles on the surface (see [1]); the source term u^p denotes the mean deposition flux of the superlinear growth conditions with respect to u at ∞, which could lead to the singularity of solutions or their derivatives to (1.1) or (1.2).

The parabolic equation in (1.1) or (1.2) is a typical equation of the continuum model of motion for the evolution of the film surface height $u(x,t)$:

$$u_t + A_1\Delta u + A_2\Delta^2 u + A_3\operatorname{div}(|\nabla u|^2\nabla u) + A_4\Delta|\nabla u|^2 = f + \eta, \quad x \in \Omega,\ t > 0,$$

where f is the deposition flux and η is the Gaussian random variable which describes the fluctuations in the average deposition flux.

In the case $A_1, A_2 > 0$, $A_3 < 0$ and $A_4 = 0$, Kohn and Yan in [10] considered

$$u_t + \Delta^2 u + \operatorname{div}(2(1 - |\nabla u|^2)\nabla u) = 0, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}^2$ is a square domain. They obtained the decay of energy in time.

In the case $A_1, A_3 < 0$, $A_2 > 0$, and $A_4 = 0$, Liu and Li in [11] studied

$$u_t - \Delta u + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u), \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a square domain. They obtain the sufficient conditions on the global existence, asymptotic behavior, and finite time blow-up of weak solutions, but also show exact descriptions of smallness conditions on the initial data.

In the case $A_1, A_2, A_4 > 0$, and $A_3 = 0$, Winkler in [14] investigated the following equation by using the computational methods

$$u_t + \mu\Delta u + \Delta^2 u + \lambda\Delta|\nabla u|^2 = f(x), \quad x \in \Omega,\ t > 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (1 \leq N \leq 3)$ is a bounded convex domain with smooth boundary, $\mu \geq 0$ and $\lambda > 0$. Under appropriate assumptions on f, the global existence of weak solutions was obtained. Under an additional smallness condition on μ and the size of f, it was shown that there exists a bounded set which is absorbing for (1.3) in some sense for any solution. Blomker and Gugg in [3] also studied the related problem

$$u_t + A_1\Delta u + \Delta^2 u + \Delta|\nabla u|^2 = \eta, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}$ is a bounded interval. The global existence of weak solutions was proved. This result was extended by Blomker et al. in [4] to the parabolic equation

$$u_t + A_1\Delta u + A_2\Delta^2 u + A_4\Delta|\nabla u|^2 = \nu|\nabla u|^2 + \eta, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}$ is a bounded interval and $\nu > 0$.

In the case $A_1, A_2, A_4 > 0$ and $A_3 < 0$, Agélas in [1] dealt with

$$u_t + \nu\Delta u + \nu_2\Delta^2 u - \nu_3\operatorname{div}(|\nabla u|^2\nabla u) + \nu_4\Delta|\nabla u|^2 = \nu_5|\nabla u|^2, \quad x \in \Omega,\ t > 0,$$

where $\Omega = \mathbb{R}^N (N = 1, 2)$. He proved the existence, uniqueness, and regularity of global weak solutions. Moreover, under the condition $\nu_2\nu_3 > \nu_4^2$, the author proved the existence and uniqueness of global strong solutions for sufficiently smooth initial data.

In [6], Escudero dealt with both the initial and initial-boundary value problems for the partial differential equation $u_t + \Delta^2 u = \det(D^2 u)$ posed either on $\mathbb{R}^2$ or on a bounded subset of the plane, where $\det(D^2 u)$ is the determinant of the Hessian matrix $D^2 u$. The author studied the blow-up behavior including the complete blow-up in either finite or infinite time. Moreover, he refined a blow-up criterium that was proved for this evolution equation. The interested authors could find other results in [16, 18] and the papers cited therein.

To our knowledge, the 4th-order parabolic problem (1.1) involving a Frobenius type nonlinearity has been rarely considered before. Moreover, the mean deposition flux of the superlinear growth conditions would play an important role in the property of the solutions, including the existence of blow-up, extinction solutions. Inspired by the works [14, 3], we want to study the threshold results on the initial data with respect to the existence of blow-up, global, and extinction solutions of (1.1) or (1.2). Throughout this paper, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm and by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, respectively. For $u \in H_0^2(\Omega)$ with norm $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$, we define the energy functional and the Nehari functional, respectively,

$$J(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \int_\Omega |\nabla u|^2 \Delta u \, dx - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

$$I(u) := \|\nabla u\|_2^2 + \|\Delta u\|_2^2 + 3\int_\Omega |\nabla u|^2 \Delta u dx - \|u\|_{p+1}^{p+1}, \tag{1.4}$$

which satisfy

$$J(u) := \frac{1}{3}I(u) + \frac{1}{6}\|\nabla u\|_2^2 + \frac{1}{6}\|\Delta u\|_2^2 + \frac{p-2}{3(p+1)}\|u\|_{p+1}^{p+1}.$$

We give the weak solutions of problem (1.1) as follows.

Definition 1.1 Let $T > 0$. A function $u(x,t) \in L^\infty(0,T;H_0^2(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ is the so-called weak solution to (1.1) or (1.2) in $\Omega \times [0,T)$, if $u(x,0) = u_0(x) \in H_0^2(\Omega)$, for any $\varphi(x) \in H_0^2(\Omega)$,

$$\begin{aligned}
&(u_t,\varphi) + (\nabla u,\nabla\varphi) + (\Delta u,\Delta\varphi) - (2|\Delta u|^2,\varphi) + (2|D^2u|^2,\varphi) = (|u|^{p-1}u,\varphi),\\
&\text{or } (u_t,\varphi) + (\nabla u,\nabla\varphi) + (\Delta u,\Delta\varphi) + (2\Delta u\nabla u,\nabla\varphi) + (|\nabla u|^2,\Delta\varphi) = (|u|^{p-1}u,\varphi).
\end{aligned} \tag{1.5}$$

Moreover, there is the relationship for the energy of the weak solutions,

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u) = J(u_0) \quad \text{for a.e. } t \in (0,T). \tag{1.6}$$

Table 1: Complete classification of initial energy.

Initial energy	$J(u_0) - d$	$I(u_0)$	p	$\|u_0\|_2$	*Solution*	*Main Theorems*
Subcritical	$-$	$+$	$p > 2$		*G.E.*	Theorem 3.1
Subcritical	$-, J(u_0) \neq 0$	$-$	$p > 2$		*B.U.*	Theorem 3.2
Subcritical	$J(u_0) < 0$		$p > 2$		*B.U.*	Theorem 6.1
Subcritical	$J(u_0) \leq 0$		$p < 1$	$\|u_0\|_2^{1-p} \geq D_4 D_3^{-1}$	*N.E.*	Theorem 7.2
Subcritical	$J(u_0) \leq 0$		$p = 1$ or $p \geq 2$		*N.E.*	Theorem 7.2
Critical	0	+ or 0	$p > 2$		*G.E.*	Theorem 4.1
Critical	0	$-$	$p > 2$		*B.U.*	Theorem 4.2
Supercritical	$+$	$+$	$p > 2$	$\|u_0\|_2 \leq \lambda_{J(u_0)}$	*G.E.*	Theorem 5.1(i)
Supercritical	$+$	0	$p < 2, N = 1$	$\|u_0\|_2^{1-p} \geq D_2 D_1^{-1}$	*E.*	Corollary 7.2
Supercritical	$+$	$-$	$p > 2$	$\|u_0\|_2 \geq \Lambda_{J(u_0)}$	*B.U.*	Theorem 5.1(ii)

Define the Nehari manifold $\mathcal{N} := \left\{u \in H_0^2(\Omega) \middle| I(u) = 0, \ \|\Delta u\|_2 \neq 0\right\}$. The potential well and its corresponding sets are defined by

$$\mathcal{W} := \left\{u \in H_0^2(\Omega) \middle| I(u) > 0, \ J(u) < d\right\} \cup \{0\}, \quad \mathcal{V} := \left\{u \in H_0^2(\Omega) \middle| I(u) < 0, \ J(u) < d\right\},$$
$$\mathcal{N}_+ := \left\{u \in H_0^2(\Omega) \middle| I(u) > 0\right\}, \quad \mathcal{N}_- := \left\{u \in H_0^2(\Omega) \middle| I(u) < 0\right\},$$

where $d := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in \mathcal{N}} J(u)$ is the so-called depth of the potential well $\mathcal{W}$.

For any $\delta > 0$, we further define the modified functional and the Nehari manifold as

$$I_\delta(u) := \delta\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + 3\int_\Omega |\nabla u|^2 \Delta u dx - \|u\|_{p+1}^{p+1},$$
$$\mathcal{N}_\delta := \left\{u \in H_0^2(\Omega) \middle| I_\delta(u) = 0, \ \|\Delta u\|_2 \neq 0\right\}.$$

The modified potential wells and their corresponding sets are defined respectively by

$$\mathcal{W}_\delta := \left\{u \in H_0^2(\Omega) \middle| I_\delta(u) > 0, \ J(u) < d(\delta)\right\} \cup \{0\},$$
$$\mathcal{V}_\delta := \left\{u \in H_0^2(\Omega) \middle| I_\delta(u) < 0, \ J(u) < d(\delta)\right\}.$$

Here, $d(\delta) := \inf_{u \in \mathcal{N}_\delta} J(u) > 0$ is the potential depth of $\mathcal{W}_\delta$. We also define the open sublevels of J, $J^s := \left\{u \in H_0^2(\Omega) \middle| J(u) < s\right\}$. Furthermore, by the definitions of $J(u)$, $\mathcal{N}$ and J^s, we see that $\mathcal{N}^s := \mathcal{N} \cap J^s \neq \varnothing$ for $\forall s > d$. For any $s > d$, we define

$$\lambda_s := \inf\left\{\|u\|_2 \middle| u \in \mathcal{N}^s\right\}, \quad \Lambda_s := \sup\left\{\|u\|_2 \middle| u \in \mathcal{N}^s\right\}. \tag{1.7}$$

It is clear that λ_s is nonincreasing and Λ_s is nondecreasing with respect to s, respectively.

We summarize the main results through the following table. The abbreviations "*N.E.*," "*E.*," "*B.U.*," and "*G.E.*" denote non-extinction, extinction, blow-up, and global existence of weak solutions of (1.1) or (1.2), respectively. It could be checked that if $J(u_0) < d$, then $I(u_0) \neq 0$.

This paper is arranged as follows. In the next section, we give some important lemmas. Sections 3, 4, and 5 are devoted to the subcritical, the critical, and the

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supercritical energy cases, respectively. Section 6 gives the upper and the lower bounds of blow-up time of weak solutions. In Section 7, we show some results about non-extinction or extinction of weak solutions.

2 Preliminary Lemmas

In this section, we give ten lemmas which play important roles in the proof of the main results.

Lemma 2.1 *Let $p > 2$. For any $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 \neq 0$, we have*

(i) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to +\infty} J(\lambda u) = -\infty$.

(ii) *There exists an unique constant $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*} = 0$. $J(\lambda u)$ is increasing for $0 < \lambda < \lambda^*$, is decreasing for $\lambda^* < \lambda < +\infty$, and takes its maximum at $\lambda = \lambda^*$.*

(iii) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$, and $I(\lambda^* u) = 0$.*

Proof (i) Define the function $j : \lambda \mapsto J(\lambda u)$ for $\lambda > 0$. Then

$$j(\lambda) := J(\lambda u) = \frac{\lambda^2}{2}\|\nabla u\|_2^2 + \frac{\lambda^2}{2}\|\Delta u\|_2^2 + \lambda^3 \int_\Omega |\nabla u|^2 \Delta u \, dx - \frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1}.$$

We obtain $\lim_{\lambda\to 0^+} J(\lambda u) = 0$ and $\lim_{\lambda\to +\infty} J(\lambda u) = -\infty$.

(ii) Elementary calculations imply that

$$j'(\lambda) = \lambda\|\nabla u\|_2^2 + \lambda\|\Delta u\|_2^2 + 3\lambda^2 \int_\Omega |\nabla u|^2 \Delta u dx - \lambda^p \|u\|_{p+1}^{p+1}.$$

Let $k(\lambda) := \lambda^{-2} j'(\lambda)$. After direct calculation, we have

$$k'(\lambda) = -\lambda^{-2}\|\nabla u\|_2^2 - \lambda^{-2}\|\Delta u\|_2^2 - (p-2)\lambda^{p-3}\|u\|_{p+1}^{p+1} < 0.$$

Since $\lim_{\lambda\to 0^+} k(\lambda) = +\infty$, $\lim_{\lambda\to +\infty} k(\lambda) = -\infty$, there exists a unique constant $\lambda^* > 0$ such that $k(\lambda) > 0$ for $0 < \lambda < \lambda^*$, $k(\lambda) < 0$ for $\lambda^* < \lambda < +\infty$, and $k(\lambda^*) = 0$. By $j'(\lambda) = \lambda^2 k(\lambda)$, $I(\lambda u) = \lambda j'(\lambda)$, cases (ii) and (iii) hold. ∎

Lemma 2.2 *Let $p > 2$. The depth d of the potential well $\mathcal{W}$ is positive.*

Proof Employing Hölder's inequality, we obtain

$$-\int_\Omega |\nabla u|^2 \Delta u dx \le \left(\int_\Omega |\nabla u|^4 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\Delta u|^2 \, dx\right)^{\frac{1}{2}} \le B_2^2\|\Delta u\|_2^3.$$

Fix $u \in \mathcal{N}$. Since $p > 2$ and by (1.4),

$$\|\Delta u\|_2^2 \le \|u\|_{p+1}^{p+1} + 3B_2^2\|\Delta u\|_2^3 \le B_1^{p+1}\|\Delta u\|_2^{p+1} + 3B_2^2\|\Delta u\|_2^3,$$

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where B_1 is the optimal constant in the embedding $H_0^2(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and B_2 is the optimal constant in the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. Let

$$B_0 := \inf\left\{x \in (0,+\infty) \mid 1 \le B_1^{p+1} x^{p-1} + 3B_2^2 x\right\}. \tag{2.1}$$

Then $J(u) = \frac{1}{6}\|\nabla u\|_2^2 + \frac{1}{6}\|\Delta u\|_2^2 + \frac{p-2}{3(p+1)}\|u\|_{p+1}^{p+1} + \frac{1}{3}I(u) \ge \frac{1}{6}\|\Delta u\|_2^2 \ge \frac{1}{6}B_0^2 > 0$. Therefore, $d = \inf_{u\in\mathcal{N}} J(u) > 0$. ■

Lemma 2.3 *Suppose $p > 2$ and $u \in H_0^2(\Omega)$. Define $r(\delta) := \inf\{x \in (0,+\infty) \mid \delta \le B_1^{p+1}x^{p-1} + 3B_2^2 x\}$, where B_1, B_2 are defined in (2.1). These are the following results.*

(i) *If $I_\delta(u) < 0$, then $\|\Delta u\|_2 > r(\delta)$. Specially, if $I(u) < 0$, then $\|\Delta u\|_2 > r(1)$.*
(ii) *If $0 \le \|\Delta u\|_2 \le r(\delta)$, then $I_\delta(u) \ge 0$. Specially, if $0 \le \|\Delta u\|_2 \le r(1)$, then $I(u) \ge 0$.*
(iii) *If $I_\delta(u) = 0$, then $\|\Delta u\|_2 = 0$ or $\|\Delta u\|_2 \ge r(\delta)$. Specially, if $I(u) = 0$, then $\|\Delta u\|_2 = 0$ or $\|\Delta u\|_2 \ge r(1)$.*

Proof If $I_\delta(u) < 0$ and $\delta\|\Delta u\|_2^2 < \delta\|\Delta u\|_2^2 + \|\nabla u\|_2^2 \le B_1^{p+1}\|\Delta u\|_2^{p+1} + 3B_2^2\|\Delta u\|_2^3$, then $\|\Delta u\|_2 > r(\delta)$, and hence, cases (i) and (ii) hold. If $I_\delta(u) = 0$, we get $\|\Delta u\|_2 \ge r(\delta)$. If $\|\Delta u\|_2 = 0$, $I_\delta(u) = 0$. ■

Lemma 2.4 (Lemma 2.4 in [7]) *$d(\delta)$ is increasing for $0 < \delta \le 1$, is decreasing for $\delta \ge 1$, and takes its maximum $d = d(1)$ at $\delta = 1$.*

Lemma 2.5 (Lemma 5 in [15]) *Let $p > 2$. Assume $u \in H_0^2(\Omega)$, $0 < J(u) < d$, and $\delta_1 < 1 < \delta_2$ is the two roots of the equation $d(\delta) = J(u)$. Then the sign of $I_\delta(u)$ does not change for $\delta_1 < \delta < \delta_2$.*

Lemma 2.6 (Lemma 8 in [15]) *Let $p > 2$ and assume that u is a weak solution of problem (1.1) in $\Omega \times [0,T)$ with $0 < J(u_0) < d$. Let $\delta_1 < 1 < \delta_2$ be the two roots of the equation $d(\delta) = J(u_0)$.*

(i) *If $I(u_0) > 0$, then $u \in \mathcal{W}_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*
(ii) *If $I(u_0) < 0$, then $u \in \mathcal{V}_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*

Lemma 2.7 *Let $p > 2$. $dist(0,\mathcal{N}) > 0$ and $dist(0,\mathcal{N}_-) > 0$.*

Proof For any $u \in \mathcal{N}$, by the definition of d, we obtain

$$d = \inf_{u\in\mathcal{N}} J(u) \le \frac{1}{6}B_3^2\|\Delta u\|_2^2 + \frac{1}{6}\|\Delta u\|_2^2 + \frac{p-2}{3(p+1)}B_1^{p+1}\|\Delta u\|_2^{p+1},$$

which indicates that $\frac{1}{6}B_3^2\|\Delta u\|_2^2 + \frac{1}{6}\|\Delta u\|_2^2 \ge \frac{d}{2}$ or $\frac{p-2}{3(p+1)}B_1^{p+1}\|\Delta u\|_2^{p+1} \ge \frac{d}{2}$. Then $\|\Delta u\|_2 \ge \left(\frac{3d}{B_3^2+1}\right)^{\frac{1}{2}}$, or $\|\Delta u\|_2 \ge \left(\frac{3d(p+1)}{2(p-2)B_1^{p+1}}\right)^{\frac{1}{p+1}}$, where B_3 is the optimal constant in $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$. Let

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(2.2) $$C_0 := \min\left\{\left(\frac{3d}{B_3^2+1}\right)^{\frac{1}{2}}, \left[\frac{3d(p+1)}{2(p-2)B_1^{p+1}}\right]^{\frac{1}{p+1}}\right\}.$$

Then $\mathrm{dist}(0,\mathcal{N}) = \inf_{u\in\mathcal{N}} \|\Delta u\|_2 \geq C_0 > 0$. For any $u \in \mathcal{N}_-$, we have $\|\Delta u\|_2^2 \leq B_1^{p+1}\|\Delta u\|_2^{p+1} + 3B_2^2\|\Delta u\|_2^3$, which implies $\|\Delta u\|_2 \geq B_0$. Here, B_0 is given in (2.1). Then $\mathrm{dist}(0,\mathcal{N}_-) = \inf_{u\in\mathcal{N}^-} \|\Delta u\|_2 \geq B_0 > 0$. ■

Lemma 2.8 *Let $p > 2$. For any $s > d$, $u \in J^s \cap \mathcal{N}_+$, $\|\Delta u\|_2 < C_3 := (6s)^{\frac{1}{2}}$.*

Proof For any $s > d$ and $p > 2$, $u \in J^s \cap \mathcal{N}_+$, we have

$$s > J(u) = \frac{1}{6}\|\nabla u\|_2^2 + \frac{1}{6}\|\Delta u\|_2^2 + \frac{p-2}{3(p+1)}\|u\|_{p+1}^{p+1} + \frac{1}{3}I(u) > \frac{1}{6}\|\Delta u\|_2^2,$$

which yields $\|\Delta u\|_2 < C_3$. ■

For suitable u and $p > 2$, by using the Gagliardo-Nirenberg inequality, we have

(2.3) $$\|\nabla u\|_4 \leq C_1\|\Delta u\|_2^a\|u\|_2^{1-a}, \quad \|u\|_{p+1} \leq C_2\|\Delta u\|_2^b\|u\|_2^{1-b},$$

where $a := \frac{N+4}{8} \in (0,1)$ and $b := \left(\frac{1}{2} - \frac{1}{p+1}\right)\cdot\frac{N}{2} \in (0,1)$.

Lemma 2.9 *Let $p > 2$. For any $s > d$, λ_s and Λ_s in (1.7) satisfy $0 < K_4 \leq \lambda_s \leq \Lambda_s \leq K_1 < +\infty$, where C_0 is defined in (2.2) and the constants a, b, C_1, C_2 are defined in (2.3); $K_1 := B_4C_3$,*

$$K_2 := \begin{cases} \left[\dfrac{C_0^{2-b(p+1)}}{2C_2^{p+1}}\right]^{\frac{1}{(1-b)(p+1)}}, & p < \frac{8+N}{N}, \\ \left[\dfrac{C_3^{2-b(p+1)}}{2C_2^{p+1}}\right]^{\frac{1}{(1-b)(p+1)}}, & p \geq \frac{8+N}{N}, \end{cases}$$

$$K_3 := \left(\frac{C_3^{1-2a}}{6C_1^2}\right)^{\frac{1}{2(1-a)}}, \quad K_4 := \min\{K_2, K_3\}.$$

Proof For $u \in \mathcal{N}^s$, we have $\frac{1}{6}B_4^{-2}\|u\|_2^2 \leq \frac{1}{6}\|\Delta u\|_2^2 \leq J(u) < s$, where B_4 is the optimal constant in the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$. Then $\|u\|_2 \leq K_1$. By (2.3), we have

$$\begin{aligned}\|\Delta u\|_2^2 &\leq \|u\|_{p+1}^{p+1} + 3\|\nabla u\|_4^2\|\Delta u\|_2 \\ &\leq C_2^{p+1}\|\Delta u\|_2^{b(p+1)}\|u\|_2^{(1-b)(p+1)} + 3C_1^2\|\Delta u\|_2^{2a+1}\|u\|_2^{2(1-a)},\end{aligned}$$

and hence,

(2.4)
$$C_2^{p+1}\|\Delta u\|_2^{b(p+1)}\|u\|_2^{(1-b)(p+1)} \geq \frac{1}{2}\|\Delta u\|_2^2, \quad \text{or} \quad 3C_1^2\|\Delta u\|_2^{2a+1}\|u\|_2^{2(1-a)} \geq \frac{1}{2}\|\Delta u\|_2^2.$$

By the similar proof of Lemma 2.8, we get $\|\Delta u\|_2 < C_3$. Combining with (2.4), we have $\|u\|_2 \geq K_2$ or $\|u\|_2 \geq K_3$. Then $\|u\|_2 \geq K_4 > 0$; hence, $\lambda_s > 0$. ■

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Lemma 2.10 (Lemma 2.1 in [8]) *Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality $\theta''(t)\theta(t)-(1+\beta)\theta'(t)^2 \geq 0$, $t>0$, where $\beta>0$ is a constant. If $\theta(0)>0$ and $\theta'(0)>0$, then there exists $0<t_1<\frac{\theta(0)}{\beta\theta'(0)}$ such that $\theta(t)$ tends to ∞ as $t\to t_1$.*

3 The subcritical case

This section is devoted to the property of weak solution of (1.1) or (1.2) under the case $J(u_0)<d$.

Theorem 3.1 *Let $p>2$. If $J(u_0)<d$ and $I(u_0)>0$, then problem (1.1) admits a global weak solution $u\in L^\infty(0,\infty;H_0^2(\Omega))$ with $u_t\in L^2(0,\infty;L^2(\Omega))$ and $u(t)\in\mathcal{W}$ for $0\leq t<\infty$. Moreover, there exists a constant $\hat{C}>0$ such that $\|u\|_2^2\leq\|u_0\|_2^2e^{-\hat{C}t}$.*

Proof The proof is divided into two steps.

Step 1. Global existence. We would use the Galerkin's approximation with some priori estimates. Let $\{\omega_i(x)\}$ be the orthogonal basis of $H_0^2(\Omega)$. Construct the approximate solutions $u_m(x,t)$ of (1.1), $u_m(x,t):=\sum_{i=1}^m a_{mi}(t)\omega_i(x)$, $m=1,2,\cdots$, $i=1,2,\cdots,m$, which satisfy

$$(3.1)\qquad \begin{aligned}&(u_m',\omega_i)+(\nabla u_m,\nabla\omega_i)+(\Delta u_m,\Delta\omega_i)-(2|\Delta u_m|^2,\omega_i)+(2|D^2u_m|^2,\omega_i)\\&=(|u_m|^{p-1}u_m,\omega_i),\end{aligned}$$

and $u_{0m}:=\sum_{i=1}^m b_{mi}(t)\omega_i(x)\to u_0(x)$ in $H_0^2(\Omega)$ as $m\to+\infty$.

By the standard theory of ODEs (e.g., the Peanos theorem), we deduce that the existence of a local solution to (3.1). Multiplying (3.1) by $a_{mi}'(t)$, summing over i from 1 to m and integrating with respect to t, we have

$$(3.2)\qquad \int_0^t\|u_m'\|_2^2\mathrm{d}\tau+J(u_m(x,t))=J(u_m(x,0)),\quad 0\leq t\leq T.$$

Due to the convergence of $u_{0m}\to u_0(x)$ in $H_0^2(\Omega)$, one has $J(u_m(x,0))\to J(u_0(x))<d$, $I(u_m(x,0))\to I(u_0(x))>0$. Therefore, for sufficiently large m and any $0\leq t<+\infty$, we obtain

$$(3.3)\qquad \int_0^t\|u_m'\|_2^2\mathrm{d}\tau+J(u_m)=J(u_m(x,0))<d,\quad I(u_m(x,0))>0,$$

which implies that $u_m(x,0)\in\mathcal{W}$ for sufficiently large m.

By applying the similar discussion of Theorem 8 in [15], one could show from (3.3) that $u_m(x,t)\in\mathcal{W}$ for large m and $0\leq t<+\infty$. Thus, $I(u_m(x,t))>0$, $J(u_m(x,t))<d$ for all $t\in[0,T]$. Then

$$(3.4)\qquad \frac{1}{6}\|\nabla u_m\|_2^2+\frac{1}{6}\|\Delta u_m\|_2^2+\frac{p-2}{3(p+1)}\|u_m\|_{p+1}^{p+1}<J(u_m(t))<d.$$

In addition, by using (3.2–3.4), we get for some positive constant C,

$$\|u_m'\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{3.5}$$

$$\|u_m\|_{L^\infty(0,T;H_0^2(\Omega))} \le C, \tag{3.6}$$

$$\|u_m\|_{L^\infty(0,T;W_0^{1,2}(\Omega))} \le C, \tag{3.7}$$

$$\|u_m\|_{L^\infty(0,T;L^{p+1}(\Omega))} \le C. \tag{3.8}$$

By the uniform estimates (3.5–3.8), it was seen that the local solutions can be extended globally. Thus, by the standard diagonal method and the Aubin-Lions-Simon theorem, we know there exists a function u and a sequence of $\{u_m\}$ (still by $\{u_m\}$) such that for each $T > 0$, one could obtain

$$u_m' \rightharpoonup u', \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.9}$$

$$u_m \rightharpoonup u, \quad \text{weakly in } L^\infty(0,T;W_0^{1,2}(\Omega)), \tag{3.10}$$

$$u_m \rightharpoonup u, \quad \text{weakly in } L^\infty(0,T;H_0^2(\Omega)), \tag{3.11}$$

$$|u_m|^{p-1}u_m \to |u|^{p-1}u, \quad \text{strongly in } L^{\frac{p+1}{p}}(\Omega \times (0,T)), \tag{3.12}$$

$$|\Delta u_m|^2 \rightharpoonup |\Delta u|^2, \quad \text{weakly star in } L^\infty(0,T;L^1(\Omega)), \tag{3.13}$$

$$|D^2 u_m|^2 \rightharpoonup |D^2 u|^2, \quad \text{weakly star in } L^\infty(0,T;L^1(\Omega)), \tag{3.14}$$

as $m \to +\infty$. Fix $k \in \mathbb{N}$. In order to show the limit function u in (3.9–3.14) is a weak solution of (1.1), we choose a function $v \in C^1([0,T];H_0^2(\Omega))$ defined as $v(x,t) := \sum_{i=1}^{k} l_i(t)\omega_i(x)$, where $\{l_i(t)\}_{i=1}^k$ are arbitrarily given C^1 functions. Taking $m \ge k$ in (3.1), multiplying (3.1) by $l_i(t)$, summing for i from 1 to k, and integrating with respect to t, we obtain

$$\begin{aligned} &\int_0^T (u_m', v) + (\nabla u_m, \nabla v) + (\Delta u_m, \Delta v) - (2|\Delta u_m|^2, v) + (2|D^2 u_m|^2, v)\,\mathrm{d}t \\ &= \int_0^T (|u_m|^{p-1}u_m, v)\mathrm{d}t. \end{aligned} \tag{3.15}$$

Taking $m \to +\infty$ in (3.15) and recalling the convergence yield that

(3.16)
$$\int_0^T (u', v) + (\nabla u, \nabla v) + (\Delta u, \Delta v) - (2|\Delta u|^2, v) + (2|D^2 u|^2, v)\,\mathrm{d}t = \int_0^T (|u|^{p-1}u, v)\mathrm{d}t.$$

Since functions of the form in (3.15) are dense in $L^2(0,T;H_0^2(\Omega))$, (3.16) also holds for all $v \in L^2(0,T;H_0^2(\Omega))$. By the arbitrariness of $T > 0$, we know

$$(u', v) + (\nabla u, \nabla v) + (\Delta u, \Delta v) - (2|\Delta u|^2, v) + (2|D^2 u|^2, v) = (|u|^{p-1}u, v), \quad \text{a.e. } t > 0.$$

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Then u is a global weak solution to problem (1.1). To prove (1.6), we first assume that u was smooth enough that $u_t \in L^2(0,T;H_0^2(\Omega))$. Taking $v = u_t$ in (3.16), it is seen that (1.6) is true. By the density of $L^2(0,T;H_0^2(\Omega))$ in $L^2(\Omega\times(0,T))$, we know (1.6) also holds for weak solutions to (1.1). The existence of global solutions to (1.1) is obtained.

Step 2. Decay rate. Taking $\varphi := u$ in (1.5), we get $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = 2(u_t,u) = -2I(u)$. From Lemma 2.6, we know that $u(x,t)\in \mathcal{W}_\delta$ for $\delta_1<\delta<\delta_2$ and $0<t<\infty$ under the condition $J(u_0)<d$ and $I(u_0)>0$. Thus, $I_{\delta_1}(u)\ge 0$ for $0<t<\infty$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|u\|_2^2}{2} = -I(u) = (\delta_1-1)\|\Delta u\|_2^2 - I_{\delta_1}(u) \le (\delta_1-1)B_4^{-2}\|u\|_2^2,$$

where $B_4>0$ is the best embedding constant from $H_0^2(\Omega)$ to $L^2(\Omega)$ (i.e., $\|u\|_2\le B_4\|\Delta u\|_2$ for $\forall u\in H_0^2(\Omega)$). Consequently, $\|u\|_2^2\le \|u_0\|_2^2 \mathrm{e}^{-\hat{C}t}$ with $\hat{C} := 2B_4^{-2}(1-\delta_1)>0$. ■

Theorem 3.2 *Let $p>2$ and u be a weak solution of (1.2). If $J(u_0)<d$, $J(u_0)\ne 0$ and $I(u_0)<0$, then u blows up at some finite time T in the sense of $\lim_{t\to T}\int_0^t\|u\|_2^2 d\tau = +\infty$.*

Proof We employ the concavity method. Assume on the contrary that u was a global weak solution to (1.2) with $J(u_0)<d$, $J(u_0)\ne 0$, $I(u_0)<0$ and define $F(t) := \int_0^t \|u\|_2^2\mathrm{d}\tau$, $t\ge 0$. Then $F'(t) = \|u\|_2^2$,

$$F''(t) = 2(u,u_t) = -2I(u). \tag{3.17}$$

By (1.6), (1.4), and (3.17), we have

$$F''(t) \ge -6J(u) + \|\Delta u\|_2^2 = -6J(u_0) + 6\int_0^t \|u'\|_2^2\mathrm{d}\tau + \|\Delta u\|_2^2.$$

Noticing that $\frac14(F'(t)-F'(0))^2 = (\int_0^t\int_\Omega uu'\mathrm{d}x\mathrm{d}\tau)^2 \le \int_0^t\|u\|_2^2\mathrm{d}\tau\int_0^t\|u'\|_2^2\mathrm{d}\tau$, we have

$$F''(t)F(t) - \frac32\left(F'(t)\right)^2 \ge B_4^{-2}F(t)F'(t) - 6J(u_0)F(t) - 3F'(0)F'(t). \tag{3.18}$$

In the forthcoming proof, the cases that $J(u_0)<0$ and $0<J(u_0)<d$ will be discussed separately.

(i) If $J(u_0)<0$, then (3.18) implies $F''(t)F(t)-\frac32\left(F'(t)\right)^2 \ge B_4^{-2}F(t)F'(t) - 3F'(0)F'(t)$. Now we will prove that $I(u)<0$ for all $t>0$. Otherwise, there must be a constant $t_0>0$ such that $I(u(t_0))=0$ and $I(u)<0$ for $0\le t<t_0$. From Lemma 2.3 (i, iii), $\|\Delta u\|_2>r(1)$ for $0\le t<t_0$, and $\|\Delta u(t_0)\|_2\ge r(1)$, where $\|\Delta u(t_0)\|_2\ne 0$. In fact, using (1.6) and $J(u_0)<0$, we have $J(u(t_0))<0$. Since $0 = F''(t_0)\ge -6J(u(t_0)) + \|\Delta u(t_0)\|_2^2$, we have $\|\Delta u(t_0)\|_2\ne 0$. Hence, we have $J(u(t_0))\ge d$, which contradicts to (1.6). By (3.17), we get $F''(t)>0$ for $t\ge 0$. Since $F'(0) = \|u_0\|_2^2\ge 0$, $F'(t)>0$, for large t, $F(t)>3B_4^2\|u_0\|_2^2$, we have

$$F''(t)F(t) - \frac32\left(F'(t)\right)^2 > 0. \tag{3.19}$$

(ii) If $0<J(u_0)<d$, then by Lemma 2.6 (ii), we know that $u(x,t)\in\mathcal{V}_\delta$ for $t\ge 0$ and $\delta_1<\delta<\delta_2$, where $\delta_1<1<\delta_2$ are the two roots of $d(\delta)=J(u_0)$. Hence,

$I_{\delta_2}(u) \le 0$ and $\|\Delta u\|_2 \ge r(\delta_2)$ for $t \ge 0$. It follows from (3.18) that for $t \ge 0$, $F''(t) \ge 2(\delta_2 - 1)r^2(\delta_2)$, which shows for all $t \ge 0$ that $F'(t) \ge 2(\delta_2 - 1)r^2(\delta_2)t$ and $F(t) \ge (\delta_2 - 1)r^2(\delta_2)t^2$. Therefore, for sufficiently large t, we have

$$B_4^{-2}F'(t) > 12J(u_0), \quad B_4^{-2}F(t) > 6F'(0).$$

Then we obtain (3.19). Due to $(F^{-\frac{1}{2}}(t))'' = -\frac{1}{2}F^{-\frac{5}{2}}(t)[F(t)F''(t) - \frac{3}{2}(F'(t))^2] < 0$, $F^{-\frac{1}{2}}(t)$ is concave in $(0, +\infty)$. So there exists a positive constant t_0 such that $F(t_0) > 0$, $F'(t_0) > 0$ and $\left(F^{-\frac{1}{2}}(t_0)\right)' < 0$. Since $F(t) > 0, F''(t) > 0$ for $t \ge t_0$ and $F'(t_0) > 0$, one can find $(F^{-1/2})'(t) < 0$ for $t \ge t_0$, and hence, there is a constant $T > t_0$ such that $\lim_{t\to T} F^{-\frac{1}{2}}(t) = 0$; that is, $\lim_{t\to T} \int_0^t \|u(x,\tau)\|_2^2 d\tau = +\infty$. ∎

4 The critical case

For $J(u_0) = d$, the invariance of $\mathcal{W}_\delta$ could not be proved in general. By using approximation, we could prove the global existence of weak solutions.

Theorem 4.1 *Let $p > 2$. If $J(u_0) = d$ and $I(u_0) \ge 0$, then problem (1.2) admits a global weak solution $u \in L^\infty(0,\infty; H_0^2(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$ and $u \in \overline{\mathcal{W}} = \mathcal{W} \cup \partial\mathcal{W}$ for $0 \le t < \infty$. Moreover, if $I(u(x,t)) > 0$ for all $t > 0$, $u(x,t)$ does not vanish and there exist positive constants $\hat{C}_1$ and $\hat{C}_2$ such that $\|u\|_2^2 \le \hat{C}_1 e^{-\hat{C}_2 t}$. If not, problem (1.2) admits a solution that vanishes in finite time.*

Proof Let $\lambda_k = 1 - \frac{1}{k}$, $k = 2, 3, \cdots$. Consider the approximation problems

$$(4.1)\quad \begin{cases} u_t - \Delta u + \Delta^2 u - \operatorname{div}(2\nabla u \Delta u) + \Delta|\nabla u|^2 = |u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0^k(x) := \lambda_k u_0(x), & x \in \Omega. \end{cases}$$

Noticing that $I(u_0) \ge 0$, by Lemma 2.1 (iii), we could deduce that there exists a unique constant $\lambda^* = \lambda^*(u_0) \ge 1$ such that $I(\lambda^* u_0) = 0$. By $\lambda_k < 1 \le \lambda^*$, we get $I(u_0^k) = I(\lambda_k u_0) > 0$ and $J(u_0^k) = J(\lambda_k u_0) < J(u_0) = d$. In view of Theorem 3.1, for each k, problem (4.1) admits a global weak solution $u^k(x,t) \in L^\infty(0,\infty; H_0^2(\Omega)) \cap \mathcal{W}$ with $u_t^k \in L^2(0,\infty; L^2(\Omega))$ for $0 \le t < \infty$ satisfying $\int_0^t \|u_\tau^k\|_2^2 d\tau + J(u^k) = J(u_0^k) < d$. Applying the similar discussion in Theorem 3.1, there exist a subsequence of $\{u^k\}$ and a function u such that u is a weak solution of (1.2) with $I(u) \ge 0$ and $J(u) \le d$ for $0 \le t < \infty$.

Let us discuss the decay rate of $\|u\|_2^2$. First, suppose that $I(u) > 0$ for $0 < t < \infty$; then u does not vanish in finite time. Combining with (3.17), we have $u_t \not\equiv 0$. By (1.6), for small $t_0 > 0$, we have $0 < J(u(t_0)) < J(u_0) = d$. Taking $t = t_0$ as the initial time and by Lemma 2.6, we get $u \in \mathcal{W}_\delta$ for $\delta_1 < \delta < \delta_2$ and $t_0 < t < \infty$. Hence, $I_{\delta_1}(u) \ge 0$ for $t_0 < t < \infty$ and

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u) = (\delta_1 - 1)\|\Delta u\|_2^2 - I_{\delta_1}(u) \le B_4^{-2}(\delta_1 - 1)\|u\|_2^2,$$

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which implies $\|u\|_2^2 \le \|u(t_0)\|_2^2 e^{-2B_4^{-2}(1-\delta_1)(t-t_0)}$. The decay rate holds with $\hat{C}_2 := 2B_4^{-2}(1-\delta_1)$ and $\hat{C}_1 := \|u(t_0)\|_2^2 e^{2B_4^{-2}(1-\delta_1)t_0}$.

Next, suppose there is a positive constant t_1 such that $I(u) > 0$ for $0 < t < t_1$ and $I(u(x,t_1)) = 0$. Obviously, $u_t \not\equiv 0$ for $0 < t < t_1$ and $\int_0^{t_1} \|u_\tau\|_2^2 d\tau > 0$. Applying (1.6) again, we have $J(u(t_1)) = J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 d\tau < J(u_0) = d$. By the definition of d, we know $\|u(t_1)\|_2 = 0$ and $\|u\|_2 \equiv 0$ for all $t \ge t_1$. It is seen that such a function u is a weak solution of (1.2) which vanishes in finite time. ∎

Theorem 4.2 *Let $p > 2$. If $J(u_0) = d$ and $I(u_0) < 0$, then there exists a finite time T such that the solution u blows up at that time T in the sense of $\lim_{t\to T} \int_0^t \|u\|_2^2 d\tau = +\infty$.*

Proof Similarly to the proof of Theorem 4.1, we could get

$$F''(t)F(t) - \frac{3}{2}\left(F'(t)\right)^2 \ge B_4^{-2}F'(t)F(t) - 6J(u_0)F(t) - 3F'(t)F'(0).$$

Since $J(u_0) = d$, $I(u_0) < 0$, by the continuity of $J(u)$ and $I(u)$ with respect to t, there exists a constant $t_0 > 0$ such that $J(u(x,t)) > 0$ and $I(u(x,t)) < 0$ for $0 < t \le t_0$. From $(u_t, u) = -I(u)$, we have $u_t \not\equiv 0$ for $0 < t \le t_0$. Furthermore, we have $J(u(t_0)) = d - \int_0^{t_0} \|u_\tau\|_2^2 d\tau < d$. Taking $t = t_0$ as the initial time and by Lemma 2.6, we know that $u(x,t) \in \mathcal{V}_\delta$ for $\delta_1 < \delta < \delta_2$ and $t > t_0$, where $\delta_1 < 1 < \delta_2$ are the two roots of the equation $d(\delta) = J(u_0)$. Thus, $I_{\delta_2}(u) \le 0$ for $t > t_0$. The rest of the proof is similar to that of Theorem 3.2. ∎

5 The supercritical case

We obtain some results for arbitrarily high initial energy.

Theorem 5.1 *Let $p > 2$ and $J(u_0) > d$.*

(i) *If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \le \lambda_{J(u_0)}$, then the weak solution u of (1.1) exists globally and $u \to 0$ as $t \to +\infty$.*

(ii) *If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \ge \Lambda_{J(u_0)}$, then the weak solution u of (1.1) blows up in finite time.*

Proof Let $T_{\max}$ be the maximal existence time of (1.1). If $T_{\max} = \infty$, we denote the ω-limit set of u_0 as $\omega(u_0) = \bigcap_{t\ge 0} \overline{\{u(\tau) : \tau \ge t\}}^{H_0^2(\Omega)}$.

(i) Assume that $u_0 \in \mathcal{N}_+$ with $\|u_0\|_2 \le \lambda_J(u_0)$. We first claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. If not, there would exist a constant $t_0 \in (0, T_{\max})$ such that $u(t) \in \mathcal{N}_+$ for $0 \le t < t_0$ and $I(u(t_0)) = 0$. It follows from $I(u(t)) = -\int_\Omega u_t u dx$ that $u_t \not\equiv 0$ for $(x,t) \in \Omega \times [0, t_0)$. Using (1.6), we have $J(u(t_0)) < J(u_0)$, which deduces $u(t_0) \in J^{J(u_0)}$. Therefore, $u(t_0) \in J^{J(u_0)} \cap \mathcal{N} = \mathcal{N}^{J(u_0)}$. By the definition of $\lambda_{J(u_0)}$, we have

$$\lambda_{J(u_0)} \le \|u(t_0)\|_2. \tag{5.1}$$

Noticing $I(u(t)) > 0$ for $t \in [0, t_0)$, $\|u(t_0)\|_2 < \|u_0\|_2 \le \lambda_{J(u_0)}$, a contradiction to (5.1). So $u(t) \in \mathcal{N}_+$ and $u(t) \in J^{J(u_0)} \cap \mathcal{N}_+$ for all $t \in [0, T_{\max})$. Lemma 2.8 shows

$\|\Delta u\|_2 < C_3$, $t \in [0, T_{\max})$, so that $T_{\max} = \infty$. Let ω be an arbitrary element in $\omega(u_0)$. We have $J(\omega) < J(u_0)$, $\|\omega\|_2 < \lambda_{J(u_0)}$, which implies $\omega \notin \mathcal{N}^{J(u_0)}$. Recalling the definition of $\lambda_{J(u_0)}$, $\omega(u_0) \cap \mathcal{N} = \varnothing$. Hence, $\omega(u_0) = \{0\}$. Therefore, the weak solution u of problem (1.1) exists globally and $u \to 0$ as $t \to +\infty$.

(ii) Assume that $u_0 \in \mathcal{N}_-$ with $\|u_0\|_2 \geq \Lambda_{J(u_0)}$. We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T_{\max})$. If not, there would be a constant $t_0 \in (0, T_{\max})$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < t_0$ and $I(u(t_0)) = 0$. Similarly to case (i), one has $J(u(t_0)) < J(u_0)$, which implies $u(t_0) \in J^{J(u_0)}$, and $u(t_0) \in \mathcal{N}^{J(u_0)}$. By the definition of $\Lambda_{J(u_0)}$, we have

$$\Lambda_{J(u_0)} \geq \|u(t_0)\|_2. \tag{5.2}$$

However, $I(u(t)) < 0$ for $t \in [0, t_0)$, and we get $\|u(t_0)\|_2 > \|u_0\|_2 \geq \Lambda_{J(u_0)}$, a contradiction with (5.2). So $u(t) \in \mathcal{N}_-$, $t \in [0, T_{\max})$. Suppose $T_{\max} = \infty$; for every $\omega \in \omega(u_0)$, we get $\|\omega\|_2 > \Lambda_{J(u_0)}$, $J(\omega) < J(u_0)$, and hence, $\omega \in J^{J(u_0)}$ and $\omega \notin \mathcal{N}^{J(u_0)}$. Recalling the definition of $\Lambda_{J(u_0)}$, we obtain $\omega(u_0) \cap \mathcal{N} = \varnothing$. Thus, $\omega(u_0) = \{0\}$, a contradiction to Lemma 2.7. Hence, $T_{\max} < +\infty$. ∎

Remark 5.1 For any $M > d$, there exists $u_M \in \mathcal{N}_-$ such that $J(u_M) \geq M$ and the solution of (1.1) blows up in finite time. In fact, by using Theorem 5.1, if the initial data satisfy the following inequality

$$\frac{3(p+1)}{p-2} J(u_0)|\Omega|^{\frac{p-1}{2}} \leq \|u_0\|_2^{p+1} \leq \|u_0\|_{p+1}^{p+1} |\Omega|^{\frac{p-1}{2}},$$

then the solution of (1.1) blows up in finite time.

6 Blow-up time estimates

Theorem 6.1 *For $p > 2$ and $J(u_0) < 0$, the solution $u(x,t)$ blows up at finite time*

$$T_* \leq \frac{\|u_0\|_2^2}{3|J(u_0)|}. \tag{6.1}$$

The upper bound of blow-up rate is given as $\|u\|_2 \leq \left[3 \cdot 2^{-\frac{3}{2}} \eta\right]^{-1} (T_* - t)^{-1}$, where $\eta := |J(u_0)| L^{-\frac{3}{2}}(0)$ and $L(t) := \frac{1}{2}\|u\|_2^2$.

Proof Let $K(t) := -J(u(t))$. Then $L(0) > 0$ and $K(0) > 0$. By (1.6), we get $K'(t) = \|u'\|_2^2 \geq 0$, which implies $K(t) \geq K(0) > 0$ for all $t \in [0, T)$. By (1.4), we have

$$L'(t) = -I(u) \geq -3J(u) = 3K(t). \tag{6.2}$$

Combining (6.2) with the Cauchy inequality, we obtain

$$L(t)K'(t) = \frac{1}{2}\|u\|_2^2 \|u'\|_2^2 \geq \frac{1}{2}(u, u')^2 = \frac{1}{2}(L'(t))^2 \geq \frac{3}{2} L'(t)K(t). \tag{6.3}$$

According to (6.3), $(K(t)L^{-\frac{3}{2}}(t))' = L^{-\frac{5}{2}}(t)(K'(t)L(t) - \frac{3}{2}K(t)L'(t)) \geq 0$. Therefore,

$$0 < K(0)L^{-\frac{3}{2}}(0) \leq K(t)L^{-\frac{3}{2}}(t) \leq \frac{1}{3} L'(t) L^{-\frac{3}{2}}(t) = -\frac{2}{3}(L^{-\frac{1}{2}}(t))'. \tag{6.4}$$

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Integrating (6.4) with respect to t, we have $\eta t \le -\frac{2}{3}(L^{-\frac{1}{2}}(t) - L^{-\frac{1}{2}}(0))$, where $\eta := K(0)L^{-\frac{3}{2}}(0)$, and hence, we obtain that there exists a constant $T_* < +\infty$ such that $\lim_{t\to T_*} L(t) = \infty$ (i.e., the weak solution blows up). Hence, (6.1) holds. Similarly, integrating (6.4) from t to T_*, we have $L(t) \le \left[3\cdot 2^{-1}\eta\right]^{-2}(T_* - t)^{-2}$. ■

Theorem 6.2 *For $p > 2$ and $0 \le J(u_0) < \frac{1}{6}B_4^{-2}\|u_0\|_2^2$, the solution u blows up at finite time $T_* \le \frac{16\|u_0\|_2^2}{B_4^{-2}\|u_0\|_2^2 - 6J(u_0)}$.*

Proof We assert that for any $t \in [0,T)$, $I(u) < 0$. From (1.4), $I(u_0) \le 3J(u_0) - \frac{1}{2}\|\Delta u_0\|_2^2 < 0$. If the assertion was not true, there would be a constant $t_0 \in (0,T)$ such that $I(u) < 0$ and $I(u(t_0)) = 0$. From (6.2), we know $\|u\|_2^2$ is strictly increasing with $t \in [0, t_0)$. Therefore,

$$J(u(t_0)) \le J(u_0) < \frac{1}{6}\|\Delta u_0\|_2^2 \le \frac{1}{6}\|\Delta u(t_0)\|_2^2. \tag{6.5}$$

From the definition of $J(u)$and $I(u(t_0)) = 0$, we have $J(u(t_0)) > \frac{1}{6}\|\Delta u(t_0)\|_2^2$. It contradicts to (6.5). Therefore, we have $I(u) < 0$ for any $t \in [0,T)$, and $\|u\|_2^2$ is strictly increasing with respect to t. For any $T \in (0, T^*)$, $\rho > 0$, $\xi > 0$, we define an auxiliary function

$$G(t) := \int_0^t \|u\|_2^2 d\tau + (T-t)\|u_0\|_2^2 + \rho(t+\xi)^2, \quad t \in [0, T^*].$$

By direct calculation, we have $G'(t) = \|u\|_2^2 - \|u_0\|_2^2 + 2\rho(t+\xi) = \int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau + 2\rho(t+\xi) = 2\int_0^t (u, u_\tau) d\tau + 2\rho(t+\xi)$, and

$$G''(t) \ge -6J(u) + \|\Delta u\|_2^2 + 2\rho \ge -6J(u_0) + 6\int_0^t \|u_\tau\|_2^2 d\tau + \|\Delta u\|_2^2 + 2\rho.$$

It is obvious that $G''(t) > 0$. Then $G'(t)$ is increasing on $[0,T]$ and $G'(t) \ge G'(0) > 0$. This indicates that $G(t)$ is increasing on $[0,T]$. Define

$$f(t) := \left[\int_0^t \|u\|_2^2 d\tau + \rho(t+\xi)^2\right]\left(\int_0^t \|u_\tau\|_2^2\, d\tau + \rho\right) - \left[\int_0^t (u, u_\tau)\, d\tau + \rho(t+\xi)\right]^2.$$

By Cauchy-Schwarz inequality, we have $f(t) \ge 0$. For any $t \in [0,T]$,

$$G(t)G''(t) - \frac{3}{2}(G'(t))^2 \ge G(t)\left[-6J(u_0) + B_4^{-2}\|u_0\|_2^2 - 4\rho\right] \ge 0,$$

where $0 < \rho \le -\frac{3}{2}J(u_0) + \frac{1}{4}B_4^{-2}\|u_0\|_2^2 := \hat{\rho}$. From Lemma 2.10, $T \le \frac{T\|u_0\|_2^2 + \rho\xi^2}{\rho\xi}$. Fixing $\rho_0 \in (0, \hat{\rho}]$, we have $0 < \frac{\|u_0\|_2^2}{\rho\xi} < 1$ for any $\xi \in (\frac{\|u_0\|_2^2}{\rho_0}, +\infty)$. This indicates $T \le \frac{\rho_0\xi^2}{\rho_0\xi - \|u_0\|_2^2}$. The right side of the above formula takes the minimum value at $\xi := \frac{2\|u_0\|_2^2}{\rho_0}$. Then $T \le \frac{4\|u_0\|_2^2}{\rho_0}$. For $\rho_0 \in (0, \hat{\rho}]$, $T < T^* \le \frac{16\|u_0\|_2^2}{B_4^{-2}\|u_0\|_2^2 - 6J(u_0)}$. ■

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Theorem 6.3 *Let $N = 1$. If u blows up in finite time in its $H_0^2(\Omega)$ norm, there exists a positive constant such that*

$$T_0 := \int_{M(0)}^{+\infty} \frac{dM}{\tilde{\alpha} M^p + \tilde{\beta} M} \le T, \tag{6.6}$$

where $M(0) := \|\Delta u_0\|_2^2$, $\tilde{\alpha} := \gamma^{2p}/\mu$, $\tilde{\beta} := 1/\lambda$, and γ is the embedding constant from $H_0^2(\Omega)$ to $L^{2p}(\Omega)$.

Proof Let $M(t) := \|\Delta u\|_2^2$ and $M(0) > 0$. When $N = 1$, problem (1.1) can be simplified to

$$\begin{cases} u_t - \Delta u + \Delta^2 u = |u|^{p-1} u, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{6.7}$$

By computation, we have

$$M'(t) = 2 \int_\Omega (\Delta u - \Delta^2 u + |u|^{p-1} u) \Delta^2 u \mathrm{d}x. \tag{6.8}$$

Let us estimate the integrals in (6.8). We have

$$2 \int_\Omega \Delta^2 u \Delta u \mathrm{d}x \le \frac{1}{\lambda} \int_\Omega |\Delta u|^2 \,\mathrm{d}x + \lambda \int_\Omega (\Delta^2 u)^2 \mathrm{d}x, \tag{6.9}$$

$$2 \int_\Omega |u|^{p-1} u \Delta^2 u \mathrm{d}x \le \frac{1}{\mu} \int_\Omega |u|^{2p} \,\mathrm{d}x + \mu \int_\Omega (\Delta^2 u)^2 \mathrm{d}x, \tag{6.10}$$

with two arbitrary positive constants λ, μ. By replacing (6.9) and (6.10) in (6.8), we obtain

$$M'(t) \le \frac{1}{\mu} \int_\Omega |u|^{2p} \mathrm{d}x + \frac{1}{\lambda} \int_\Omega |\Delta u|^2 \,\mathrm{d}x + (\mu + \lambda - 2) \int_\Omega (\Delta^2 u)^2 \mathrm{d}x. \tag{6.11}$$

Choosing $\lambda + \mu \le 2$, inequality (6.11) reduces to $M'(t) \le \tilde{\alpha} M^p(t) + \tilde{\beta} M(t)$. Then we get (6.6). ∎

7 Extinction and non-extinction

Theorem 7.1 *If $p < 1$ and $N = 1$, then the weak solution u of (1.1) becomes extinct in finite time if the initial data satisfies $\|u_0\|_2^{1-p} \ge D_2 D_1^{-1}$. An upper bound of extinction rate for u is*

$$\begin{cases} \|u\|_2 \le \left[\|u_0\|_2^{1-p} + \frac{1-p}{2} \left(D_2 - D_1 \|u_0\|_2^{1-p} \right) t \right]^{\frac{1}{1-p}}, & 0 < t < T_1, \\ \|u\|_2 = 0, & t \in [T_1, \infty), \end{cases} \tag{7.1}$$

where $T_1 := \frac{2\|u_0\|_2^{1-p}}{(1-p)(D_1 \|u_0\|_2^{1-p} - D_2)}$, $D_1 := B_3^{-2} + B_4^{-2}$, and $D_2 := B_5^{p+1}$.

Proof When $N = 1$, we get (6.7). First, we give the upper bound of extinction rate. Multiplying the equation in (6.7) by u and integrating it over $\Omega \times (t, t+h)$ with $h > 0$ and then dividing the result by h yields that

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(7.2)

$$\frac{1}{h}\int_t^{t+h}\int_\Omega u_\tau u \mathrm{d}x\mathrm{d}\tau + \frac{1}{h}\int_t^{t+h}\|\nabla u\|_2^2\mathrm{d}\tau + \frac{1}{h}\int_t^{t+h}\|\Delta u\|_2^2\mathrm{d}\tau = \frac{1}{h}\int_t^{t+h}\|u\|_{p+1}^{p+1}\mathrm{d}\tau.$$

Let $h \to 0^+$ in (7.2). Using the Lebesgue differentiation theorem in [2], one could obtain

$$H'(t) + 2\|\nabla u\|_2^2 + 2\|\Delta u\|_2^2 = 2\|u\|_{p+1}^{p+1}, \tag{7.3}$$

where $H(t) := \|u\|_2^2$. According to the embedding relationship $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we have $\|u\|_2 \le B_3\|\nabla u\|_2$, $\|u\|_2 \le B_4\|\Delta u\|_2$, $\|u\|_{p+1} \le B_5\|u\|_2$. Then (7.3) can be written $H'(t) + 2D_1H(t) \le 2D_2H^{\frac{p+1}{2}}(t)$, where $D_1 := B_3^{-2} + B_4^{-2}$, $D_2 := B_5^{p+1}$. Defining $\varphi(t) := H^{\frac{1-p}{2}}(t)$, we get the following differential inequality:

$$\varphi'(t) \le (1-p)\left(D_2 - D_1\varphi(t)\right) := \zeta(t). \tag{7.4}$$

It is clear from (7.4) that $\zeta(0) < 0$. Recalling the continuity of $\zeta(t)$, there exists a sufficiently small $T_0 > 0$ such that $\zeta(t) < \frac{\zeta(0)}{2} < 0$ for $0 < t \le T_0$. Then $\varphi'(t) \le \zeta(0)/2$, which implies that

$$\begin{cases} \varphi(t) \le \varphi(0) + \frac{\zeta(0)t}{2}, & 0 < t < T_1, \\ \varphi(t) = 0, & t \ge T_1. \end{cases}$$

Obviously, by the definition of $\zeta(t)$, (7.1) holds. ∎

Corollary 7.1 *If $p < 1$, $N = 1$, and $J(u_0) \le 0$, then the weak solution u of (1.1) becomes extinct in finite time with the initial data satisfying $D_2D_1^{-1} \le \|u_0\|_2^{1-p} \le aD_2D_1^{-1}$. The results of Theorem 7.1 hold, and a lower bound of extinction rate for u is*

$$\begin{cases} \|u\|_2 \ge \left[(\|u_0\|_2^{1-p} - D_4D_3^{-1})e^{D_3t} + D_4D_3^{-1}\right]^{\frac{1}{1-p}}, & 0 < t < T_2, \\ \|u\|_2 = 0, & t \in [T_2, \infty), \end{cases} \tag{7.5}$$

where $T_2 := \frac{1}{D_3}\log\frac{D_4D_3^{-1}}{D_4D_3^{-1} - \|u_0\|_2^{1-p}}$, $D_3 := \frac{1-p}{2}D_1 > 0$, $D_4 := \frac{(2-p)(1-p)}{p+1}D_2 > 0$, *and* $a := \frac{4-2p}{p+1}$.

Proof The proof of a upper bound of extinction rate for u is the same as Theorem 7.1. We only give the proof of a lower bound of extinction rate. Let $H(t) := \|u\|_2^2$. We have

$$\begin{aligned} H'(t) &\ge -6J(u_0) + \|\nabla u\|_2^2 + \|\Delta u\|_2^2 + \frac{2(p-2)}{p+1}\|u\|_{p+1}^{p+1} \\ &\ge (B_3^{-2} + B_4^{-2})H(t) + \frac{2(p-2)}{p+1}B_5^{p+1}H^{\frac{p+1}{2}}(t). \end{aligned} \tag{7.6}$$

By the definition of $\varphi(t)$ in Theorem 7.1, we have $\varphi'(t) \ge D_3\varphi(t) - D_4$; hence, we get (7.5). ∎

Corollary 7.2 *Let $p < 1$, $N = 1$, and $\|u_0\|_2^{1-p} \geq D_2 D_1^{-1}$. If $J(u_0) > d$ and $I(u_0) = 0$, then the results of Theorem 7.1 hold.*

In the following, we give a result on non-extinction of weak solutions.

Theorem 7.2 *Let $J(u_0) \leq 0$. If one of the following conditions holds: (i) $p < 1$ and $\|u_0\|_2^{1-p} > D_4 D_3^{-1}$; (ii) $p = 1$; (iii) $p \geq 2$, then the solution u of (1.1) does not vanish in finite time.*

Proof Let $H(t) := \|u\|_2^2$. In the forthcoming proof, the cases $p < 1$, $p = 1$, and $p \geq 2$ will be discussed separately.

(i) $p < 1$, $\|u_0\|_2^{1-p} > D_4 D_3^{-1}$. By the similar proof of Corollary 7.1, we have

$$\|u\|_2 \geq \left[\left(\|u_0\|_2^{1-p} - D_4 D_3^{-1} \right) e^{D_3 t} + D_4 D_3^{-1} \right]^{\frac{1}{1-p}}.$$

(ii) $p = 1$. By (7.6), we obtain $H'(t) \geq D_5 H(t)$, where $D_5 := B_3^{-2} + B_4^{-2} - B_5^2$. Hence, $\|u\|_2 \geq \|u_0\|_2^2 e^{\frac{D_5}{2} t}$.

(iii) $p \geq 2$. By (7.6), we obtain $H'(t) \geq D_6 H(t)$, where $D_6 := B_3^{-2} + B_4^{-2}$. Hence, $\|u\|_2 \geq \|u_0\|_2^2 e^{\frac{D_6}{2} t}$. ■

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