A NOTE ON B^*-ALGEBRAS

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Let A be a Banach $*$-algebra. We assume neither that A has an identity element, nor that the involution on A is continuous.

A linear functional F on A is *hermitian* if $F(a^*) = \overline{F(a)}$ for all $a \in A$. A linear functional F on A is *positive* if $F(a^*a) \geqq 0$ for all $a \in A$. A linear functional F on A is *representable* if it is hermitian and there exists $M > 0$ such that $|F(a)|^2 \leqq MF(a^*a)$ for all $a \in A$. (This last definition is not that given in Rickart [5], but is equivalent to Rickart's definition in view of the square root lemma of Ford [2].) In general, the set of representable linear functionals is a proper subset of the positive ones.

Grothendieck [3] proved that every continuous linear functional on a B^*-algebra is a linear combination of four representable linear functionals; he also proved, conversely, that a Banach $*$-algebra C, with isometric involution (i.e., with $\|c^*\| = \|c\|$ for all $c \in C$) and such that the dual space of C is spanned by representable linear functionals, has a B^*-norm equivalent to the original norm.

The purpose of this note is to prove this converse result without the assumption of an isometric involution. The proof given here uses the greatest B^*-semi-norm, denoted by $|.|$, and discussed by Rickart [5, Theorem 4.6.9]. The key property of the greatest B^*-semi-norm on A is that for every representable linear functional F on A there exists a constant $M_F > 0$ such that

$$|F(a)| \leqq M_F |a| \quad \text{for all} \quad a \in A. \tag{1}$$

The greatest B^*-semi-norm $|.|$ has recently been discussed by Bonsall [1] and Palmer [4].

The theorem below was proved while the author was a research student under Professor F. F. Bonsall at Edinburgh University.

THEOREM. *Let A be a Banach $*$-algebra such that every continuous linear functional on A is a linear combination of representable linear functionals. Then there exists a B^*-norm on A equivalent to the original norm.*

Proof. Let $S = \{\hat{a} : a \in A, |a| \leqq 1\}$. (Here, $|.|$ denotes the greatest B^*-semi-norm and $\hat{a} \in A^{**}$ is defined, as usual, by $\hat{a}(f) = f(a)$ for all $f \in A^*$.) For each $f \in A^*$ we have

$$f = \sum_{i=1}^{n} \alpha_i F_i$$

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for some representable linear functionals F_i and complex numbers α_i. Hence

$$\begin{aligned} |f(a)| &= \left|\sum_{i=1}^{n} \alpha_i F_i(a)\right| \\ &\leqq \sum_{i=1}^{n} |\alpha_i||F_i(a)| \\ &\leqq \sum_{i=1}^{n} |\alpha_i| M_{F_i} |a|, \quad \text{by} \quad (1). \end{aligned}$$

So

$$|\hat{a}(f)| \leqq \sum_{i=1}^{n} |\alpha_i| M_{F_i} \quad \text{for all} \quad \hat{a} \in S.$$

Hence, by the Principle of Uniform Boundedness, there exists $K > 0$ such that $\|\hat{a}\| \leqq K$ for all $\hat{a} \in S$. From this and the definition of S, we deduce that $\|a\| \leqq K|a|$ for all $a \in A$. Hence $|.|$ is a B^*-norm on A.

Also, from Bonsall [1, Theorem 8], we see that there exists $L > 0$ such that $|a| \leqq L\|a\|$ for all $a \in A$. So $\|a\| \leqq K|a| \leqq KL\|a\|$ for all $a \in A$. Hence $|.|$ is a B^*-norm on A equivalent to $\|.\|$.

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