Can. J. Math., Vol. XL, No. 4, 1988, pp. 833-864

STRONG MORITA EQUIVALENCE FOR HEISENBERG C^*-ALGEBRAS AND THE POSITIVE CONES OF THEIR K_0-GROUPS

JUDITH A. PACKER

Introduction. In **[14]** we began a study of C^*-algebras corresponding to projective representations of the discrete Heisenberg group, and classified these C^*-algebras up to *-isomorphism. In this sequel to **[14]** we continue the study of these so-called Heisenberg C^*-algebras, first concentrating our study on the strong Morita equivalence classes of these C^*-algebras. We recall from **[14]** that a Heisenberg C^*-algebra is said to be of class i, $i \in \{1, 2, 3\}$, if the range of any normalized trace on its K_0 group has rank i as a subgroup of $\mathbf{R}$; results of Curto, Muhly, and Williams **[7]** on strong Morita equivalence for crossed products along with the methods of **[21]** and **[14]** enable us to construct certain strong Morita equivalence bimodules for Heisenberg C^*-algebras. For those of class 2 we are able to prove the following:

PROPOSITION 1.6. *Let β_1, β_2 be irrational numbers, and p_1/q_1, p_2/q_2 rational numbers in lowest terms. Then $H(\beta_1, p_1/q_1)$ is strongly Morita equivalent to $H(\beta_2, p_2/q_2)$ if and only if there exists*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$$

with

$$q_2\beta_2 = \frac{aq_1\beta_1 + b}{cq_1\beta_1 + d}.$$

For class 3 Heisenberg C^*-algebras, the strong Morita equivalence classes can be described as follows:

THEOREM 1.8. *Let $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ be Heisenberg C^*-algebras of class* 3 *where α_i, $\beta_i \in \mathbf{R}$, $i = 1, 2$. Then $H(\alpha_1, \beta_1)$ is strongly Morita equivalent to $H(\alpha_2, \beta_2)$ if and only if there exists*

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z})$$

Received June 26, 1987. This research was partially supported by N.S.F. grant MCS 8211309.

with

$$\alpha_2 = \frac{a\alpha_1 + b\beta_1 + c}{g\alpha_1 + h\beta_1 + i} \quad \textit{and} \quad \beta_2 = \frac{d\alpha_1 + e\beta_1 + f}{g\alpha_1 + h\beta_1 + i}.$$

Class 1 Heisenberg C^*-algebras were shown to be strongly Morita equivalent to the universal rotation algebra in [**14**].

The above ideas can be restated more vividly as follows: One can associate to each element of the real projective plane a Heisenberg C^*-algebra (this correspondence is not one-to-one). With respect to this correspondence the strong Morita equivalence classes Heisenberg C^*-algebras are parametrized by the orbit spaces $GL(3, \mathbf{Z})\backslash\mathbf{R}P^2$, where $GL(3, \mathbf{Z})$ acts on $\mathbf{R}P^2$ viewed as lines in $\mathbf{R}^3$.

By using these strong Morita equivalence bimodules we are able to construct projective modules corresponding to each element in the positive cone of the K_0-groups involved, which we show to consist precisely of those elements having positive dimension under the range of a trace, and [0]. The endomorphism ring for each of the projective modules constructed is itself a matrix algebra over some Heisenberg C^*-algebra. This allows us to prove cancellation for these C^*-algebras of class 2 and 3, using methods suggested by the work of M. Rieffel on cancellation for irrational rotation algebras [**21**]. As in [**21**], the calculation of the range of a faithful, normalized trace on the K_0-group and of projective modules for the $H(\alpha, \beta)$ plays an important role in distinguishing between inequivalent C^*-algebras; in fact the order structure of this range completely determines the strong Morita equivalence classes of Heisenberg C^*-algebras.

We hope that the method used here in the construction of the strong Morita equivalence bimodules and finitely generated projective modules for these C^*-algebras can be used as a model in the attempt to prove similar results for twisted group C^*-algebras corresponding to more general nilpotent discrete groups.

The structure of our work is as follows: In the first section we discuss a constructive method for forming strong Morita equivalence bimodules between Heisenberg C^*-algebras which, when applied together with the isomorphism theorem of [**14**], allows us to determine the strong Morita equivalence classes for all Heisenberg C^*-algebras. In the second section we discuss the positive cone of the K_0-group for Heisenberg C^*-algebras, which we are able to identify as those elements of the K_0-group having strictly positive trace, and [0]. This identification involves the determination of representatives up to stable equivalence of all projective modules for $H(\alpha, \beta)$ and a description of their endomorphism rings. This allows us to prove cancellation for Heisenberg C^*-algebras of class 2 and 3, by the method of Rieffel.

After preparing the first version of this paper we received the preprint

"Projective modules over higher dimensional non-commutative tori", by Marc Rieffel in which among many other results all finitely generated projective modules are constructed and cancellation is proved for "non-rational" non-commutative tori. We would like to thank Professor Rieffel for many helpful remarks, and for showing us the above and other relevant preprints of his work.

1. The classification of Heisenberg C^*-algebras up to strong Morita equivalence. To begin this section we review several relevant facts from [**14**] about Heisenberg C^*-algebras, and crossed products of strong Morita equivalence bimodules for unital C^*-algebras, which will be crucial in our subsequent construction of strong Morita equivalence bimodules for the Heisenberg C^*-algebras. The constructions we will use stem from techniques outlined in [**14**] which are in turn applications of the method of Curto, Muhly, and Williams [**7**] and F. Combes [**5**].

A twisted group C^*-algebra generated by a 2-cocycle on the discrete Heisenberg group taking values in $\mathbf{T}$ is termed a *Heisenberg C^*-algebra.* Such a C^*-algebra is determined by three unitary generators, U, V, and W, which satisfy the relations

$$UV = e^{2\pi i\alpha}VU, \quad WV = e^{2\pi i\beta}VW, \quad \text{and} \quad UW = VWU$$

for some α, $\beta \in \mathbf{R}$. We denote, for fixed α, $\beta \in \mathbf{R}$, the corresponding C^*-algebra by $H(\alpha, \beta)$. The range of any faithful normalized trace on $K_0(H(\alpha, \beta))$ is equal to $\mathbf{Z} + \alpha\mathbf{Z} + \beta\mathbf{Z}$; $H(\alpha, \beta)$ is termed of class i, $i \in \{1, 2, 3\}$, if this subgroup of $\mathbf{R}$ has rank i. The following result from [**14**] classifies the isomorphism types of the Heisenberg C^*-algebras:

THEOREM 1.1. *Let $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ be two Heisenberg C^*-algebras. Then $H(\alpha_1, \beta_1)$ is *-isomorphic to $H(\alpha_2, \beta_2)$ if and only if there exists*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$$

with

$$e^{2\pi i\alpha_2} = e^{2\pi i(a\alpha_1 + b\beta_1)} \quad \textit{and} \quad e^{2\pi i\beta_2} = e^{2\pi i(c\alpha_1 + d\beta_1)}.$$

This theorem contains as corollaries the facts that Heisenberg C^*-algebras of class 1 can be parametrized by

$$\left\{H\left(\frac{1}{d}, 0\right) | d \in \mathbf{N}\right\}$$

and Heisenberg C^*-algebras of class 2 can be parametrized by

$$\left\{H\left(\frac{\alpha}{q}, \frac{p}{q}\right) | \text{ irrational } \alpha \in \left[0, \frac{1}{2}\right],\right.$$

$$\text{rational } \frac{p}{q} \in \left[0, \frac{1}{2}\right] \text{ in the lowest terms}\Big\}.$$

An important tool in the proof of Theorem 1.1 was the construction of strong Morita equivalence bimodules between certain Heisenberg C^*-algebras, using the crossed product techniques of Curto-Muhly-Williams [7] and Combes [5]. Recall that if X is a left module for the C^*-algebra A which is a left A-rigged space, and if $\alpha:G \to \mathrm{Aut}(A)$ is an action of the locally compact group G on A, we say that (A, α, G) is a unitarily covariant system with respect to X if there exists a strongly continuous homomorphism $U:G \to \mathrm{Aut}(X)$ which satisfies

(1) $\langle U_g x, U_g y\rangle = \alpha(g)(\langle x, y\rangle_A)\ \forall g \in G, \forall x, y \in X$

(2) $U_g \circ a \circ U_{g^{-1}} = \alpha(g)(a) \in \mathrm{End}(X), \forall g \in G, \forall a \in A.$

Then the following is true [5], [7], [14]:

THEOREM 1.2. *Let $A - X - B$ be a strong Morita equivalence bimodule for unital C^*-algebras A and B, and suppose that (A, α, G) is a unitarily covariant system with respect to X. Then there exists a continuous action β of G on B such that (B, β, G) is unitarily covariant with respect to X, and the crossed product C^*-algebras $A \times_\alpha G$ and $B \times_\beta G$ are strongly Morita equivalent to one another. The action β is defined by*

$$\beta(g)(b) = U_g(b(U_{g^{-1}})) \in \mathrm{End}_A X \cong B.$$

We will use Theorem 1.2 to establish

LEMMA 1.3. *Let α be irrational and let*

$$\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Set $\rho = (a\alpha + b)/(q\alpha + p)$ and choose $\beta \in \mathbf{R}$. Then $H(\alpha, \beta)$ is strongly Morita equivalent to $H(\rho, -\beta/(q\alpha + p))$.

Proof. We first consider the case where q is odd and $q\alpha + p > 0$. We let A_α represent the rotation algebra corresponding to α, i.e., A_α is generated by unitaries U and V satisfying

$$UV = e^{2\pi i\alpha}VU.$$

Let $G = \mathbf{R} \times \mathbf{Z}/|q|\mathbf{Z}$ and let H and K be the following subgroups of G:

$$H = \{(n, [pn]):n \in \mathbf{Z}\},$$

$$K = \{(n\gamma, [n]):n \in \mathbf{Z}\} \text{ for } \gamma = 1/(q\alpha + p).$$

Rieffel in [21] showed that

$$C^*(K\backslash G, H) = A_\alpha \text{ and } C^*(G/H, K) = A_\rho \quad \text{for } \rho = \frac{(a\alpha + b)}{(q\alpha + p)},$$

and constructed a strong Morita equivalence bimodule between A_α and A_ρ by completing $C_c(G)$, which has a $C_c^*(G/H, K) - C_c^*(K\backslash G, H)$ bimodule structure, to a strong Morita equivalence bimodule $A_\rho - X - A_\alpha$. We recall that for $f(t, [n]) \in C(G)$,

$$f(t, [n])U_\alpha = f(t - 1, [n - p]),$$

$$f(t, [n])V_\alpha = f(t, [n])e^{2\pi i[(t/\gamma - n)/q]}, \quad \text{where } \gamma = \frac{1}{q\alpha + p},$$

$$U_\rho f(t, [n]) = f(t + \gamma, [n + 1]),$$

$$V_\rho f(t, [n]) = e^{2\pi i[(an - t)/q]}f(t, [n]).$$

Define a map $Q:C_c(G) \to C_c(G)$ by

$$Q(f)(t, [n]) = e^{2\pi i g(t)}l([n])f(t + \epsilon, n)$$

where

$$\epsilon = \frac{q}{q\alpha + p}\beta,$$

$$g(t) = \frac{1}{2\gamma q}t^2 + \left((1 - p\gamma)/\gamma q - \frac{1}{2\gamma q}\right)t,$$

$$l([n]) = e^{2\pi i(1 + bq)[(n(n+1))/2q]}.$$

We use the assumption that q is odd here, in order to ensure that l is well defined on $\mathbf{Z}/|q|\mathbf{Z}$. By calculation one can verify that

$$\left.\begin{array}{ll} (1) & Q((Q^{-1}f)V_\alpha) = fe^{2\pi i\beta}V_\alpha \\ (2) & Q((Q^{-1}f)U_\alpha) = fV_\alpha^* U_\alpha \\ (3) & \langle Qf, Qf\rangle_{A_\alpha} = \mathscr{A}_W(\langle f, f\rangle_{A_\alpha}) \end{array}\right\} f \in C_c(G)$$

where $\mathscr{A}_W$ is the automorphism of A_α sending U_α to $V_\alpha^* U_\alpha$ and V_α to $e^{2\pi i\beta}V_\alpha$.

By (3), we have

$$\| \langle Qf, Qf\rangle_{A_\alpha}\| = \| \langle f, f\rangle_{A_\alpha}\| \ \forall f \in C_c(G)$$

so that Q extends to a map on the completion of $C_c(G)$ as a strong Morita equivalence bimodule for A_ρ and A_α, $X(p, q)$. Then $(A_\alpha, \mathscr{A}_W, \mathbf{Z})$ is a unitarily covariant system for $X(p, q)$ and by Theorem 1.2 there exists an action of $\mathbf{Z}$ on A_ρ with $A_\rho \times \mathbf{Z}$ strongly Morita equivalent to

$$A_\alpha \times_{\mathscr{A}_W} \mathbf{Z} = H(\alpha, \beta).$$

A straightforward calculation shows that the action on A_ρ determined by Q is conjugate to the action

$$U_\rho \to V_\rho^* U_\rho, \quad V_\rho \to e^{-2\pi i\beta/(q\alpha + p)}V_\rho.$$

Hence $X_{(p,q)} \times {}_Q\mathbf{Z}$ gives a strong Morita equivalence bimodule between $H(\rho, -\beta/(q\alpha + p))$ and $H(\alpha, \beta)$, where $\rho = (a\alpha + b)/(q\alpha + p)$, where $(q, p) = 1$ and q is odd. If q is even, then a must be odd, since

$$\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Then by the above proof $H(\alpha, \beta)$ is strongly Morita equivalent to

$$H\left(-(q\alpha + p)/(a\alpha + b), -\frac{\beta}{(a\alpha + b)}\right).$$

But $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has as its lower left entry an odd number as well so that

$$H(-(q\alpha + p)/(a\alpha + b), -\beta/(a\alpha + b))$$

is strongly Morita equivalent to

$$H((a\alpha + b)/(q\alpha + p), -\beta/(q\alpha + p)),$$

as desired.

Actually we can combine our observations to produce

LEMMA 1.4. *Let α be irrational and let*

$$\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in GL(2, \mathbf{Z}).$$

Let $\rho = (a\alpha + b)/(q\alpha + p)$ and let $\beta \in \mathbf{R}$. Then $H(\alpha, \beta)$ is strongly Morita equivalent to $H(\rho, \beta/(q\alpha + p))$.

Proof. When

$$\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in SL(2, \mathbf{Z}),$$

Lemma 1.3 shows that $H(\alpha, \beta)$ is strongly Morita equivalent to $H(\rho, -\beta/(q\alpha + p))$. But by Theorem 1.1, $H(\rho, -\beta/(q\alpha + p))$ is *-isomorphic to $H(\rho, \beta/(q\alpha + p))$. Hence $H(\alpha, \beta)$ is strongly Morita equivalent to $H(\rho, \beta/(q\alpha + p))$. Now assume

$$\det\begin{pmatrix} a & b \\ q & p \end{pmatrix} = -1.$$

Then

$$\begin{pmatrix} -a & -b \\ q & p \end{pmatrix} \in SL(2, \mathbf{Z})$$

so that $H(\alpha, \beta)$ is strongly Morita equivalent to

$$H(-(a\alpha + b)/(q\alpha + p), \beta/(q\alpha + p)).$$

But $H(-(a\alpha + b)/(q\alpha + p), \beta/(q\alpha + p))$ is *-isomorphic to

$$H((a\alpha + b)/(q\alpha + p), \beta/(q\alpha + p)).$$

Hence $H(\alpha, \beta)$ is strongly Morita equivalent to $H(\rho, \beta/(q\alpha + p))$ as desired.

Remark 1.5. The strong Morita equivalence bimodule established between $H(\alpha, \beta)$ and $H(\rho, -\beta/(q\alpha + p))$ in Lemma 1.4 is in fact a projective module for $H(\alpha, \beta)$. The projection in $H(\alpha, \beta)$ corresponding to this projective module is the projection in $A_\alpha \times \mathbf{Z}$ corresponding to the initial $A_\rho - A_\alpha$ equivalence bimodule, i.e., the projection in A_α of trace $|q\alpha + p|$.

Lemma 1.4 is valid for any $\beta \in \mathbf{R}$, but we now restrict ourselves to class 2 Heisenberg C^*-algebras corresponding to Anzai's skew-product actions on the torus, i.e., we let $\beta = 0$ (and α be irrational). Lemma 1.4 then gives an alternate proof of Theorem 4.1. of **[12]**, and we can extend it further as follows:

PROPOSITION 1.6. *Let β_1 and β_2 be irrational numbers, and p_1/q_1, p_2/q_2 rational numbers in lowest terms. Then $H(\beta_1, p_1/q_1)$ is strongly Morita equivalent to $H(\beta_2, p_2/q_2)$ if and only if there exists*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}),$$

with

$$q_2\beta_2 = \frac{aq_1\beta_1 + b}{cq_1\beta_1 + d}.$$

Proof. In 2.8 of **[14]** we showed via the crossed product technique that $H(\beta_1, p_1/q_1)$ is strongly Morita equivalent to $H(q_1\beta_1, 0)$ and $H(\beta_2, p_2/q_2)$ is strongly Morita equivalent to $H(q_2\beta_2, 0)$. If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}),$$

we can use Lemma 1.4 to show that $H(q_1\beta_1, 0)$ is strongly Morita equivalent to $H(q_2\beta_2, 0)$, which implies the desired result, by transitivity of strong Morita equivalence. If $H(\beta_1, p_1/q_1)$ is strongly Morita equivalent to $H(\beta_2, p_2/q_2)$, then $H(q_1\beta_1, 0)$ is strongly Morita equivalent to $H(q_2\beta_2, 0)$. Then the tracial argument given in **[12**, 4.1**]** shows that there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$$

with

$$q_2\beta_2 = \frac{aq_1\beta_1 + b}{cq_1\beta_1 + d}.$$

We now move on to the study of class 3 Heisenberg C^*-algebras, so that α and β are irrational and the numbers 1, α, β are linearly independent. We have enough information to develop necessary and sufficient conditions under which two class 3 Heisenberg C^*-algebras are strongly Morita equivalent.

THEOREM 1.7. *Let $H(\alpha_1, \beta_1)$ and $H(\alpha_2, \beta_2)$ be class 3 Heisenberg C^*-algebras where $\alpha_i, \beta_i \in \mathbf{R}$, $i = 1, 2$. Then $H(\alpha_1, \beta_1)$ is strongly Morita equivalent to $H(\alpha_2, \beta_2)$ if, and only if, there exists*

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z})$$

with

$$(*) \quad \begin{aligned} \alpha_2 &= (a\alpha_1 + b\beta_1 + c)/(g\alpha_1 + h\beta_1 + i) \\ \beta_2 &= (d\alpha_1 + e\beta_1 + f)/(g\alpha_1 + h\beta_1 + i). \end{aligned}$$

Proof. We first show sufficiency. We will do this by constructing an equivalence bimodule between the two C^*-algebras in question. This bimodule will provide a projective module over $H(\alpha_1, \beta_1)$ whose corresponding projection has trace $|g\alpha_1 + h\beta_1 + i|$. Suppose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = Q \in GL(3, \mathbf{Z})$$

satisfies equations (*) with respect to (α_1, β_1) and (α_2, β_2). It follows that g, h, i have no common factor so that we can write $g\alpha_1 + h\beta_1 + i$ as $DG\alpha_1 + DH\beta_1 + i$ where $(G, H) = 1$ and $(D, i) = 1$, $D, G, H \in \mathbf{Z}$. Find $g', h', m, n \in \mathbf{Z}$ with $g'G - h'H = 1$ and $im - Dn = 1$. Then set

$$S = \begin{pmatrix} G & H & 0 \\ h' & g' & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} n & 0 & m \\ 0 & 1 & 0 \\ D & 0 & i \end{pmatrix};$$

we claim that $QS^{-1}T^{-1}$ is a matrix of the form

$$\begin{pmatrix} J & K & L \\ M & N & R \\ 0 & 0 & 1 \end{pmatrix}, \quad J, K, L, M, N, R \in \mathbf{Z},$$

which implies that $JN - KM = \pm 1$. One calculates, as follows, that

$$QS^{-1}T^{-1} = Q\begin{pmatrix} g' & -H & 0 \\ -h' & G & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} i & 0 & -m \\ 0 & 1 & 0 \\ -D & 0 & n \end{pmatrix}.$$

We need only compute the bottom row of this product. The first entry in the bottom row is

$$DGig' - DHih' - iD = iD(Gg' - Hh') - iD = 0.$$

The second entry of the bottom row is

$$-DGH + DHG = 0.$$

The third entry of the bottom row is

$$\begin{aligned} -mDg'G + mh'DH + in &= -mD(g'G - h'H) + in \\ &= in - mD = 1. \end{aligned}$$

Thus,

$$QS^{-1}T^{-1} = \begin{pmatrix} J & K & L \\ M & N & R \\ 0 & 0 & 1 \end{pmatrix},$$

as promised. Hence

$$Q = \begin{pmatrix} J & K & L \\ M & N & R \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n & 0 & m \\ 0 & 1 & 0 \\ D & 0 & i \end{pmatrix} \begin{pmatrix} G & H & 0 \\ h' & g' & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 1.1, $H(\alpha_1, \beta_1)$ is *-isomorphic to $H(\alpha', \beta')$ where

$$\alpha' = G\alpha_1 + H\beta_1, \quad \beta' = h'\alpha_1 + g'\beta_1,$$

hence is strongly Morita equivalent to $H(\alpha', \beta')$. By Lemma 1.4, $H(\alpha', \beta')$ is strongly Morita equivalent to $H(\alpha'', \beta'')$ where

$$\alpha'' = (m + n\alpha')/(i + D\beta'), \ \beta'' = \beta'/(i + D\beta').$$

Finally setting $\alpha''' = J\alpha'' + K\beta''$, $\beta''' = M\alpha'' + N\beta''$, we have that $H(\alpha''', \beta''')$ is *-isomorphic to (hence strongly Morita equivalent to) $H(\alpha'', \beta'')$, again by Theorem 1.1. Since (α_2, β_2) satisfies equation (*), it is easy to check that

$$\alpha''' = \alpha_2 \bmod 1 \quad \text{and} \quad \beta''' = \beta_2 \bmod 1,$$

and thus by transitivity of Morita equivalence $H(\alpha_1, \beta_1)$ is strongly Morita equivalent to $H(\alpha_2, \beta_2)$. This shows the sufficiency of condition (*) to produce strong Morita equivalence.

We now check the necessity of the condition, i.e., that if the class 3 C^*-algebra $H(\alpha_1, \beta_1)$ is strongly Morita equivalent to $H(\alpha_2, \beta_2)$ then there exists

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z})$$

with

https://doi.org/10.4153/CJM-1988-037-8 Published online by Cambridge University Press

$$\alpha_2 = (a\alpha_1 + b\beta_1 + c)/(g\alpha_1 + h\beta_1 + i) \bmod 1,$$
$$\beta_2 = (d\alpha_1 + e\beta_1 + f)/(g\alpha_1 + h\beta_1 + i) \bmod 1.$$

Suppose that $H(\alpha_1, \beta_1)$ is strongly Morita equivalent to $H(\alpha_2, \beta_2)$, where both are class 3 Heisenberg C^*-algebras. Let X be the equivalence bimodule and let τ be the (unique) normalized trace on $H(\alpha_1, \beta_1)$. Then τ induces a trace $\mathrm{Ind}_X(\tau)$ on $H(\alpha_2, \beta_2)$. By [**19**, Corollary 2.6]

$$(\mathrm{Ind}_X\tau)(K_0(H(\alpha_2, \beta_2))) = \tau^*(K_0(H(\alpha_1, \beta_1))) = \mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z}.$$

If we denote the normalization of $\mathrm{Ind}_X(\tau)$ on $H(\alpha_2, \beta_2)$ by $n(\mathrm{Ind}_X(\tau))$, it follows that there exists $r \in \mathbf{R}^+$ with

$$\begin{aligned} n(\mathrm{Ind}_X\tau) * (K_0(H(\alpha_2, \beta_2))) &= r(\mathrm{Ind}_X(\tau)) * (K_0(H(\alpha_2, \beta_2))) \\ &= r(\mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z}). \end{aligned}$$

But we know that the image of any faithful normalized trace on $K_0(H(\alpha_2, \beta_2))$ (and in fact there is only one) is equal to $\mathbf{Z} + \alpha_2\mathbf{Z} + \beta_2\mathbf{Z}$. Hence

$$1/r(\mathbf{Z} + \alpha_2\mathbf{Z} + \beta_2\mathbf{Z}) = \mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z}.$$

Thus there are integers $a, b, c, d, e, f, g, h, i$ with

$$\frac{1}{r}\alpha_2 = a\alpha_1 + b\beta_1 + c,$$
$$(1) \quad \frac{1}{r}\beta_2 = d\alpha_1 + e\beta_1 + f,$$
$$\frac{1}{r} = g\alpha_1 + h\beta_1 + i.$$

A routine calculation similar to that given in [**24**] using the fact that

$$\mathbf{Z} + \alpha_2\mathbf{Z} + \beta_2\mathbf{Z} = r(\mathbf{Z} + \alpha_1\mathbf{Z} + \beta_1\mathbf{Z})$$

shows that

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z}).$$

As for class 1 Heisenberg C^*-algebras, in [**14**] it is shown that they are all strongly Morita equivalent to $C^*(H)$, the universal rotation algebra.

With these preliminary results in hand we are now in a position to determine necessary and sufficient conditions that two Heisenberg C^*-algebras be strongly Morita equivalent. Our aim is to associate to each strong Morita equivalence class a $GL(3, \mathbf{Z})$-orbit in the real projective plane. This point of view was first suggested to us by Marc Rieffel, from consideration of our Theorem 1.7.

First, consider the real projective plane $\mathbf{R}P^2$ as lines through the origin in 3-space. We construct a correspondence

$$\psi:\mathbf{R}P^2 \to \{H(\alpha, \beta)\,|\alpha, \beta \in \mathbf{R}\}$$

as follows. Divide $\mathbf{R}P^2$ into the disjoint union $E_1 \cup E_2 \cup E_3$ where

$$E_1 = \{\text{the line passing through } (1, 0, 0) = l_1\},$$

$$E_2 = \{\text{lines contained in the } xy \text{ plane } z = 0\} - E_1,$$

$$E_3 = \mathbf{R}P^2 - (E_1 \cup E_2).$$

Note that each element in E_2 passes through the line $y = 1$, $z = 0$, and each element in E_3 passes through the plane $z = 1$. Thus we can parametrize E_2 by $\{ (\alpha, 1, 0):\alpha \in \mathbf{R}\}$ and E_3 by the set of points $\{ (\alpha, \beta, 1), \alpha, \beta \in \mathbf{R}\}$. Let

$$\pi:\mathbf{R}P^2 \to \mathbf{R}^2$$

be given by

$$\pi(l_1) = (1, 0), \quad l_1 \in E_1,$$

$$\pi(l_2) = (\alpha_{l_2}, 1), \quad l_2 \in E_2, l_2 \text{ goes through } (\alpha_{l_2}, 1, 0),$$

$$\pi(l_3) = (\alpha_{l_3}, \beta_{l_3}), \quad l_3 \in E_3, l_3 \text{ goes through } (\alpha_{l_3}, \beta_{l_3}, 1).$$

We define

$$\psi:\mathbf{R}P^2 \to \{H(\alpha, \beta)\,|\alpha, \beta \in \mathbf{R}\}$$

by

$$\psi(l) = H(\pi(l)).$$

The map ψ is not one-to-one but it is onto. Now $GL(3, \mathbf{Z})$ acts on $\mathbf{R}^3$ sending lines through the origin to lines through the origin and we thus obtain an action of $GL(3, \mathbf{Z})$ on $\mathbf{R}P^2$. We want to associate the strong Morita equivalence classes of Heisenberg algebras to the orbit space $GL(3, \mathbf{Z})\backslash\mathbf{R}P^2$.

THEOREM 1.8. *Let l_1, $l_2 \in \mathbf{R}P^2$. Then $\psi(l_1)$ is strongly Morita equivalent to $\psi(l_2)$ if and only if there exists $M \in GL(3, \mathbf{Z})$ with $M(l_1) = l_2$.*

Proof. Suppose $M(l_1) = l_2$ for $M \in GL(3, \mathbf{Z})$. We claim that $\psi(l_1)$ is strongly Morita equivalent to $\psi(l_2)$.

To show that $\psi(l)$ is strongly Morita equivalent to $\psi(M(l))$, it is enough to check that $\psi(M(l))$ is strongly Morita equivalent to $\psi(l)$, $\forall l \in \mathbf{R}P^2$ and $\forall M \in S$, where S is a finite set of generators for $GL(3, \mathbf{Z})$. It is well known that the group $GL(3, \mathbf{Z})$ is generated by

$$S = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

(see [**10**], p. 34 for details). Thus one need only check that $\psi(M(l))$ is strongly Morita equivalent to $\psi(l)$ for $M \in S$. One does this by employing the results of this section and Theorem 1.1; we leave details to the reader.

Since S generates $GL(3, \mathbf{Z})$ it follows that $\psi(M(l))$ is strongly Morita equivalent to $\psi(l)$ $\forall l \in \mathbf{R}P^2$, $\forall M \in GL(3, \mathbf{Z})$.

We now suppose that $\psi(l_1)$ is strongly Morita equivalent to $\psi(l_2)$. Suppose $\psi(l_1)$ is a class 3 C^*-algebra. Then $\psi(l_2)$ is also class 3 so we can identify $l_1 = (\alpha_1, \beta_1, 1)$, $l_2 = (\alpha_2, \beta_2, 1)$, $l_1, l_2 \in E_3$, where $\alpha_1, \beta_1, 1$ are linearly independent and $\alpha_2, \beta_2, 1$ are linearly independent. By Theorem 3.4 of the previous section, there exist

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z})$$

and $j, k \in \mathbf{Z}$ with

$$\alpha_2 = \frac{a\alpha_1 + b\beta_1 + c}{g\alpha_1 + h\beta_1 + i} + j, \quad \beta_2 = \frac{d\alpha_1 + e\beta_1 + f}{g\alpha_1 + h\beta_1 + i} + k.$$

But then set

$$M = \begin{pmatrix} 1 & 0 & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL(3, \mathbf{Z});$$

we have $M(l_1) = l_2$. Suppose that $\psi(l_1)$, thus $\psi(l_2)$ is of class 2. There are 4 possibilities:

(1) $l_1, l_2 \in E_3$,

(2) $l_1 \in E_2, l_2 \in E_3$,

(3) $l_1 \in E_3, l_2 \in E_2$,

(4) $l_1, l_2 \in E_2$.

The cases 2, 3 are symmetric. Hence it is enough to examine cases 1, 2 and 4. We prove only case 1 and leave cases 2 and 4 as an exercise.

Case 1. Suppose that $l_1 = (\alpha_1, \beta_1, 1)$, $l_2 = (\alpha_2, \beta_2, 1)$. Then by Proposition 1.6, there exist irrational $\tilde{\alpha}_1 \in \mathbf{R}$, $p_1, q_1 \in \mathbf{Z}$ with

$$(p_1, q_1) = 1, \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in GL(2, \mathbf{Z}), \quad j_1, k_1 \in \mathbf{Z},$$

and irrational $\tilde{\alpha}_2 \in \mathbf{Z}$, $p_2, q_2 \in \mathbf{Z}$ with

$$(p_2, q_2) = 1, \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in GL(2, \mathbf{Z}), \quad \text{and} \quad j_2, k_2 \in \mathbf{Z},$$

such that

$$\left(\tilde{\alpha}_1, \frac{p_1}{q_1}\right) = (a_1\alpha_1 + b_1\beta_1 + j_1, c_1\alpha_1 + d_1\beta_1 + k_1),$$

$$\left(\tilde{\alpha}_2, \frac{p_2}{q_2}\right) = (a_2\alpha_2 + b_2\beta_2 + j_2, c_2\alpha_2 + d_2\beta_2 + k_2).$$

Set

$$M_1 = \begin{pmatrix} a_1 & b_1 & j_1 \\ c_1 & d_1 & k_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 & j_2 \\ c_2 & d_2 & k_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both M_1 and M_2 are in $GL(3, \mathbf{Z})$ and

$$M_1(l_1) = \left(\tilde{\alpha}_1, \frac{p_1}{q_1}, 1\right), \quad M_2(l_2) = \left(\tilde{\alpha}_2, \frac{p_2}{q_2}, 1\right).$$

Since $\psi(l_1)$ is strongly Morita equivalent to $\psi(l_2)$, and since $\psi(M_1(l_1))$ and $\psi(M_2(l_2))$ are *-isomorphic to $\psi(l_1)$ and $\psi(l_2)$, respectively, it follows that $H(\tilde{\alpha}_1, p_1/q_1)$ is strongly Morita equivalent to $H(\tilde{\alpha}_2, p_2/q_2)$. We can find $m_1, n_1, m_2, n_2 \in \mathbf{Z}$ with

$$\begin{pmatrix} m_1 & p_1 \\ n_1 & q_1 \end{pmatrix} \in SL(2, \mathbf{Z}) \quad \text{and} \quad \begin{pmatrix} m_2 & p_2 \\ n_2 & q_2 \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m_1 & p_1 \\ 0 & n_1 & q_1 \end{pmatrix} = N_1 \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_2 & p_2 \\ 0 & n_2 & q_2 \end{pmatrix} = N_2$$

are both in $GL(3, \mathbf{Z})$, and

$$N_1^{-1}\left(\tilde{\alpha}_1, \frac{p_1}{q_1}, 1\right) = (q_1\tilde{\alpha}_1, 0, 1),$$

$$N_2^{-1}\left(\tilde{\alpha}_2, \frac{p_2}{q_2}, 1\right) = (q_2\tilde{\alpha}_2, 0, 1).$$

It follows that $H(q_1\tilde{\alpha}_1, 0)$ is strongly Morita equivalent to $H(q_2\tilde{\alpha}_2, 0)$. By Lemma 1.4 there exists

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL(2, \mathbf{Z}),$$

with

$$\frac{xq_1\tilde{\alpha}_1 + y}{zq_1\tilde{\alpha}_1 + w} = q_2\tilde{\alpha}_2.$$

Then

$$\begin{pmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & w \end{pmatrix} = P \in GL(3, \mathbf{Z})$$

and $P(q_1\tilde{\alpha}_1, 0, 1) = (q_2\tilde{\alpha}_2, 0, 1)$. It follows that

$$M_2^{-1}N_2PN_1^{-1}M_1(\alpha_1, \beta_1, 1) = (\alpha_2, \beta_2, 1).$$

Since $M_2^{-1}N_2PN_1^{-1}M_1 \in GL(3, \mathbf{Z})$ we obtain the desired result for case 1.

Now suppose that $\psi(l_1)$ and $\psi(l_2)$ are of class 1. We shall show that l_1 and l_2 are in the $GL(3, \mathbf{Z})$ orbit of (1, 0, 0). If $\psi(l_1)$ is of class 1, then

$$l_1 \in \{ (1, 0, 0) \} \cup \{ (p/q, 1, 0):p/q \in \mathbf{Q}\} \\ \cup \{ (p/q, r/s, 1):p/q, r/s \in \mathbf{Q}\} = Q.$$

Now

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix} \sim \begin{matrix} (a/g, d/g, 1), & g \neq 0, & \\ (a/d, 1, 0), & g = 0, & d \neq 0, \\ (\pm 1, 0, 0), & d = g = 0. & \end{matrix}$$

As

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

varies over $GL(3, \mathbf{Z})$, it is easily checked that

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

varies over all of the set Q; we leave details to the reader.

Thus if $\psi(l_1)$ and $\psi(l_2)$ are of class 1, l_1 and l_2 are in the same $GL(3, \mathbf{Z})$ orbit, as we desired to show. This completes the proof that if $\psi(l_1)$ and $\psi(l_2)$ are strongly Morita equivalent, l_2 is in the $GL(3, \mathbf{Z})$ orbit of l_1. Thus we may identify the strong Morita equivalence classes of Heisenberg algebras with $GL(3, \mathbf{Z})\backslash \mathbf{R}P^2$.

Remark. 1.9. The results of Theorem 1.8 taken together with Theorem 1.1 are natural generalizations of corresponding statements for the rational and irrational rotation algebras which can be made using results in [**19**] and [**21**]. The isomorphism classes of rotation algebras can be identified with

$$\{1, -1\}\backslash S^1 = GL(1, \mathbf{Z})\backslash S^1,$$

and the strong Morita equivalence classes of rotation algebras can be identified with $GL(2, \mathbf{Z})\backslash \mathbf{R}P^1$.

2. Construction of the positive cone of $K_0(H(\alpha, \beta))$ and cancellation for Heisenberg C^*-algebras of classes 2 and 3. The constructive theorems of the last section enabled us to form strong Morita equivalence bimodules over a wide variety of Heisenberg C^*-algebras. As Rieffel noted in [**19**], strong Morita equivalence bimodules for unital C^*-algebras can be viewed as finitely generated projective bimodules over the C^*-algebras in question, so that if $A - X - B$ is a strong Morita equivalence bimodule between the unital C^*-algebras A and B, X is a finitely generated projective left A-module and $B \cong \mathrm{End}_A X$. Likewise, X is a (f.g.) projective right B-module and $A \cong \mathrm{End}_B X$.

In this section we use this idea to construct projective modules representing all the elements in the positive cone of the K_0-group of a Heisenberg C^*-algebra. In all three classes, we shall see that in order for a non-zero element of $K_0(H(\alpha, \beta))$ to be in the positive cone it is necessary and sufficient that its image under τ_* be positive, where τ is a trace on $H(\alpha, \beta)$. We also show that the endomorphism rings of the projective modules constructed are themselves matrix algebras over Heisenberg C^*-algebras. For class 2 and 3 algebras this will allow us to prove analogues of the cancellation theorem of Rieffel for irrational rotation algebras [**21**].

We discuss first the construction of (f.g.) projective modules for Heisenberg C^*-algebras of classes 2 and 3. Theorem 1.2 and Lemma 1.4 of the last section will allow us to calculate all of these for the class 2 and 3 cases. Finding projective modules for class 3 Heisenberg C^*-algebras is fairly straightforward since they are parametrized entirely by their trace, and most of the necessary constructive work was done in the last section. Suppose $H(\alpha, \beta)$ is of class 3. Then if $A, B, C \in \mathbf{Z}$ are such that $p = A\alpha + B\beta + C > 0$, Theorem 1.7 allows us to construct a projective module of trace p as follows: Let k be the greatest (positive) common divisor of A, B, C and write

$$A\alpha + B\beta + C = k(g\alpha + h\beta + i).$$

In fact we can find integers i, D, G, H with $(i, D) = 1$, $(G, H) = 1$, $g = DG$, $h = DH$, $i = i$. As we did in the proof of Theorem 1.7, we can construct a strong Morita equivalence between $H(\alpha, \beta)$ and $H(\alpha', \beta')$ where

$$\alpha' = \frac{(nG\alpha + nH\beta + m)}{(DG\alpha + DH\beta + i)}, \quad \beta' = \frac{(h'\alpha + g'\beta)}{(DG\alpha + DH\beta + i)}$$

(we keep the same notation as in the first half of the proof of Theorem 1.7), where the matrix corresponding to this equivalence is given by

$$\begin{pmatrix} nG & nH & m \\ h' & g' & 0 \\ DG & DH & i \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1988-037-8 Published online by Cambridge University Press

We form the equivalence bimodule

$$H(\alpha', \beta') - X - H(\alpha, \beta).$$

From the proof of Theorem 1.7 we see that

$$n(\mathrm{Ind}_X(\tau)) * (K_0(H(\alpha', \beta'))) = r(\mathbf{Z}\alpha + \mathbf{Z}\beta + \mathbf{Z})$$

where

$$r = 1/(DG\alpha + DH\beta + i).$$

Thus the projection with trace 1 in $K_0(H(\alpha', \beta'))$ corresponds to the element in $K_0(H(\alpha, \beta))$ of trace $DG\alpha + DH\beta + i = g\alpha + h\beta + i$ in $K_0(H(\alpha, \beta))$. By Rieffel's results in Section 2 of **[19]**, the injection of the identity projection $H(\alpha', \beta')$ into some matrix algebra over $H(\alpha, \beta)$ gives a projection with trace $g\alpha + h\beta + i$. To complete the proof, we note that the projective module corresponding to

$$A\alpha + B\beta + C = kg\alpha + kh\beta + ki$$

is clearly given by the equivalence bimodule

$$M_k(H(\alpha', \beta')) - \bigoplus_{i=1}^{k} X_i - H(\alpha, \beta).$$

With this construction in mind we can use Rieffel's and Blackadar's results on cancellation to show

PROPOSITION 2.1. *Any Heisenberg C^*-algebra of class 3 has cancellation.*

Proof. $K_0(H(\alpha, \beta))$ for $H(\alpha, \beta)$ of class 3 is totally ordered, has arbitrarily small positive elements, and the endomorphism rings for the corresponding projective modules are all of the form $M_k(H(\alpha', \beta'))$ which have bounded Bass stable rank. Thus Theorem $A1$ of **[2]** may be applied to conclude that $H(\alpha, \beta)$ has cancellation.

We now turn to the construction of projective modules and their endomorphism rings for Heisenberg C^*-algebras of class 2. The study of the K-theory of these C^*-algebras involves both rational rotation algebras and irrational rotation algebras and for that reason is more involved than that of the class 3 and class 1 C^*-algebras which involve only irrational rotation algebras or rational rotation algebras respectively.

In order to parametrize projective modules it is convenient for us to standardize the generators for the K_0-groups under examination. We first consider the class 2 C^*-algebra $H_\alpha = H(\alpha, 0)$, which is generated by unitaries U, V and W satisfying

$$UV = e^{2\pi i\alpha}VU, \; WV = VW, \; UW = VWU,$$

for irrational α. Now H_α can be expanded as a crossed product of A_α by the automorphism $\theta: V \to V$, $U \to V^*U$, or also as the crossed product of $C(T^2)$ by

$$\alpha: V \to e^{2\pi i\alpha}V, \quad W \to VW.$$

In either of these formulations, we can see from the Pimsner-Voiculescu exact sequence that $K_0(H_\alpha)$ is generated by $[e_{|\alpha|}]_{K_0(A_\alpha)}$, $[\mathrm{Id}]_{K_0}$ and a third element constructed from a "twist" involving the element W (here $e_{|\alpha|}$ represents a projection in some $M_n(A_\alpha)$ of trace $|\alpha|$). We choose as our final generator for $K_0(H_\alpha)$ the class $[\mathrm{Id}] - [e(1, 1)]$, where $e(1, 1)$ is a projection in $M_2(C(T^2))$ of trace one and twist -1,

$$e(1, 1) = \begin{pmatrix} W^* & 0 \\ 0 & W^* \end{pmatrix} M_1^* + M_0 + M_1 \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

where

$$M_0 = \begin{pmatrix} \cos^2 \pi t & \cos \pi t \sin \pi t \chi_{[0,1/2]}(t) \\ \cos \pi t \sin \pi t \chi_{[0,1/2]}(t) & \sin^2 \pi t \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0 & -\cos \pi t \sin \pi t \chi_{[1/2,1]}(t) \\ 0 & 0 \end{pmatrix}.$$

Hence by using the result of Pimsner and Voiculescu (given in the Appendix to **[15]**) on the image of "Rieffel" projections under the boundary maps in their exact sequence for crossed projects by **Z**, one calculates that in the exact sequence

$$K_1(H_\alpha) \to K_0(A_\alpha) \to K_0(A_\alpha) \to K_0(H_\alpha)$$

$$\overset{\delta}{\to} K_1(A_\alpha) \to K_1(A_\alpha) \to K_0(A_\alpha \times_\theta \mathbf{Z})$$

we have

$$\delta([e(1, 1)]_{K_0(H_\alpha)}) = [V]_{K_1(A_\alpha)}.$$

Following Rieffel's terminology given in **[21]**, we say $e(1, 1)$ has trace 1 and twist -1, and we get

$$t_1 = [\mathrm{Id}] - [e(1, 1)] \in K_0(H_\alpha).$$

Notation 2.2. In the expression of $K_0(H_\alpha)$ as $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ we denote

$$[i(e_{\mathrm{Id}})]_{K_0(H_\alpha)}$$

by (0, 1, 0),

$$\mathrm{sgn}(\alpha)[i(e_{|\alpha|})]_{K_0(H_\alpha)}$$

by (1, 0, 0), and t_1 by (0, 0, 1). (Here $i: A_\alpha \to H_\alpha$ is the injection and

https://doi.org/10.4153/CJM-1988-037-8 Published online by Cambridge University Press

$$\operatorname{sgn}(\alpha) = \begin{Bmatrix} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{Bmatrix}.$$

Let us briefly examine another projection in H_α arising from an automorphism of H_α which will be of much use to us:

PROPOSITION 2.3. *Let $\mathcal{A}$ be the automorphism of H_α which sends $U \to WU$, $V \to V$, and $W \to W$ and let $\mathcal{A}_*$ be the corresponding isomorphism of $K_0(H_\alpha)$ onto itself. Then $\mathcal{A}_*$ can be denoted by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

with respect to the standard generators.

Proof. The proof is an easy calculation using the result of Pimsner and Voiculescu on Rieffel projections in the appendix to [**15**] cited above.

In order to construct representatives of the entire positive cone of $K_0(H_\alpha)$, we need to consider an isomorphism between $K_0(H_\alpha)$ and $K_0(H_{1/\alpha})$ which is determined by a strong Morita equivalence bimodule similar to those given in Lemma 1.4.

LEMMA 2.4. *Let irrational $\alpha > 0$ and let $A_\alpha - \overline{X} - A_{1/\alpha}$ be the strong Morita equivalence bimodule given in* [**19**]. *Then there exists a linear automorphism $Q:\overline{X} \to \overline{X}$ with the properties of Theorem* 1.2 *such that*

$$H_\alpha \cong A_\alpha \times \mathbf{Z} - \overline{X} \times {}_Q\mathbf{Z} - A_{1/\alpha} \times \mathbf{Z} \cong H_{1/\alpha}$$

is a strong Morita equivalence bimodule whose matrix

$$M_{H_\alpha}^{H_{1/\alpha}}: K_0(H_\alpha) \to K_0(H_{1/\alpha})$$

is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to the standard generators.

Proof. Since either α or $1/\alpha > 1$ it is enough to prove the lemma for the case $\alpha > 1$.

Let $X = C_C(\mathbf{R})$. Then as shown in [**19**], X has the structure of a pre-$A_{1/\alpha}$, A_α-bimodule if we set

$$V_{1/\alpha}^m U_{1/\alpha}^n f(t) = e^{(2\pi i(t/\alpha))m} f(t+n), \qquad m, n \in \mathbf{Z},$$

$$f(t) V_\alpha^m U_\alpha^n = f(t - n\alpha) e^{-2\pi i m n \alpha} e^{2\pi i m t}, \qquad m, n \in \mathbf{Z},$$

and define the inner products by

$$\langle F, F\rangle_{A_\alpha} = \sum_{m\in\mathbf{Z}} \left(\sum_{n\in\mathbf{Z}} \overline{F(t+n)}F(t+m\alpha+n)\right) U_\alpha^m,$$

$$\langle F, F\rangle_{A_{1/\alpha}} = \sum_{m\in\mathbf{Z}} \left(\sum_{n\in\mathbf{Z}} F(t+n\alpha)\overline{F(t+n\alpha+m)}\right) U_{1/\alpha}^m$$

(here $U_{1/\alpha}$, $V_{1/\alpha}$, U_α, V_α represent the generators of $A_{1/\alpha}$ and A_α respectively). Since $F \in C_C(\mathbf{R})$ these infinite sums are, in fact, finite.

Employing the methods of Lemma 1.3, we examine the linear automorphism $Q:X \to X$ defined by

$$(QF)(t) = \exp(-2\pi i g(t))F(t) \quad \text{where } g(t) = 1/2(t^2/\alpha - t).$$

By Theorem 1.2, Q determines actions γ and β on H_α and $H_{1/\alpha}$ respectively which are conjugate to the actions $\tilde{\gamma}$, $\tilde{\beta}$ defined by

$$\tilde{\gamma}(U_\alpha) = V_\alpha^* U_\alpha, \quad \tilde{\gamma}(V_\alpha) = V_\alpha,$$

$$\tilde{\beta}(U_{1/\alpha}) = V_{1/\alpha}U_{1/\alpha}, \quad \tilde{\beta}(V_{1/\alpha}) = V_{1/\alpha}.$$

The relations

$$\langle QF_1, QF_2\rangle_{A_{1/\alpha}} = \beta(\langle F_1, F_2\rangle_{A_{1/\alpha}}) \quad \text{and}$$

$$\langle QF_1, QF_2\rangle_{A_\alpha} = \gamma(\langle F_1, F_2\rangle_{A_\alpha})$$

are easily checked so that, as in Lemma 1.3, Q determines a strong Morita equivalence bimodule between $A_{1/\alpha} \times_{\tilde{\beta}} \mathbf{Z}$ and $A_\alpha \times_{\tilde{\gamma}} \mathbf{Z}$ which we denote by $\bar{X} \times_Q \mathbf{Z}$. It is clear that we can regard $A_\alpha \times_{\tilde{\gamma}} \mathbf{Z}$ as H_α with the action of $\tilde{\gamma}$ defined by Ad W_α, and $A_{1/\alpha} \times_{\tilde{\beta}} \mathbf{Z}$ can be regarded as $H_{1/\alpha}$ with the action of $\tilde{\beta}$ defined by Ad $W_{1/\alpha}^*$.

We now wish to calculate

$$M_{H_\alpha}^{H_{1/\alpha}}(\bar{X} \times_Q \mathbf{Z}).$$

It is evident that the first two columns of this matrix must be

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the coupling constant (cf. [**14**, Def. 2.1])

$$C_{H_\alpha}^{H_{1/\alpha}}(\bar{X} \times_Q \mathbf{Z}) = C_{A_\alpha}^{A_{1/\alpha}}(\bar{X}) = \alpha$$

(implying that the top two entries in the right-hand column of $M_{H_\alpha}^{H_{1/\alpha}}$ are zero), and it follows that the bottom-most diagonal entry must be ± 1 in order that the matrix be in $GL(3, \mathbf{Z})$.

Let k be the greatest positive integer less than α (by hypothesis, $\alpha > 1$). Following an argument analogous to that on p. 427 of [**19**], we can find a function $F \in X$ and $0 < \epsilon < \alpha - k$ with F supported on $[0, 1 + \epsilon]$ and such that

$$\langle F, F\rangle_{A_\alpha} = \mathrm{Id}_{A_\alpha}.$$

Thus, as shown in [19], $\langle F, F\rangle_{A_{1/\alpha}}$ is a projection in $A_{1/\alpha}$ of the form

(1) $\langle F, F\rangle_{A_{1/\alpha}} = U_{1/\alpha}^* \overline{m_1(t)} + m_0(t) + m_1(t)U_{1/\alpha}$

where $m_0(t) = F(t)\overline{F(t)}$ and $m_1(t) = F(t)\overline{F(t+1)}$ (evaluated on $[0, \alpha]$).

The map $M_{H_\alpha}^{H_{1/\alpha}}$ of $K_0(H_\alpha)$ to $K_0(H_{1/\alpha})$ is determined by the map

$$[p]_{K_0(H_\alpha)} \to [\langle F, Fp\rangle_{H_{1/\alpha}}]K_0(H_{1/\alpha})$$

for p a projection in H_α with the obvious extension to projections in $M_n(H_\alpha)$. If $p \in A_\alpha$ this formula becomes especially easy, since

$$\langle F, Fp\rangle_{H_{1/\alpha}} = \langle F, Fp\rangle_{A_{1/\alpha}}.$$

Let $p_{\alpha-k}$ be a projection in $A_\alpha \subset H_\alpha$ of trace $0 < \alpha - k < 1$, which we denote in terms of our standard generators for $K_0(H_\alpha)$ by $(1, -k, 0)$. Recall that we can take $p_{\alpha-k}$ of the form

$$U_\alpha^* h_1(t) + h_0(t) + h_1(t)U_\alpha$$

where the graphs of $h_0(t)$ and $h_1(t)$ are as on p. 621 of [6]. In the following, for any $F \in X$ we let $\widetilde{F}$ denote the element of $\overline{X} \times {}_Q\mathbf{Z}$ defined by

$$\widetilde{F}(t, m) = \begin{cases} F(t), & m = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We now examine $\langle \widetilde{F}, \widetilde{F}\mathscr{A}(p_{\alpha-k})\rangle_{H_{1/\alpha}}$, where $\mathscr{A}$ is the *-automorphism given in Proposition 2.3 so that

$$\mathscr{A}(p_{\alpha-k}) = U_\alpha^* W_\alpha^* h_1(t) + h_0(t) + h_1(t)W_\alpha U_\alpha.$$

Proposition 2.3 shows that

$$[\mathscr{A}(p_{\alpha-k})]_{K_0(H_\alpha)} = (1, -k, 1).$$

We wish to show that

$$[\langle \widetilde{F}, \widetilde{F}\mathscr{A}(p_{\alpha-k})\rangle_{H_{1/\alpha}}]_{K_0(H_{1/\alpha})} = (-k, 1, -1).$$

This will imply by linearity that

$$M_{H_\alpha}^{H_{1/\alpha}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since the coupling constant between H_α and $H_{1/\alpha}$ defined by $\overline{X} \times {}_Q\mathbf{Z}$ is α,

$$\langle \widetilde{F}, \widetilde{F}\mathscr{A}(p_{\alpha-k})\rangle_{H_{1/\alpha}}$$

is a projection of trace $1 - k/\alpha$, and one can calculate that

$$\langle \widetilde{F}, \widetilde{F}\mathscr{A}(p_{\alpha-k}) \rangle_{H_{1/\alpha}} = W_{1/\alpha}^* U_{1/\alpha}^{-k} \overline{f_1(t)} + f_0(t) + f_1(t) U_{1/\alpha}^k W_{1/\alpha},$$

where $f_0(t)$ and $f_1(t)$ satisfy Rieffel equations similar to those of Theorem 1.1 (1), (2), (3) in [**19**]. By using the previously mentioned result of Pimsner and Voiculescu given in the Appendix to [**15**], calculations show that

$$\begin{aligned} &\delta([M_{H_\alpha}^{H_{1/\alpha}}([\mathscr{A}(p_{\alpha-k})]_{K_0(H_\alpha)})]_{K_0(H_{1/\alpha})}) \\ &= \delta([\langle \widetilde{F}, \widetilde{F}\mathscr{A}(p_{\alpha-k}) \rangle_{H_{1/\alpha}}]) \\ &= \delta([W_{1/\alpha}^* U_{1/\alpha}^{-k} \overline{f_1(t)} + f_0(t) + f_1(t) U_{1/\alpha}^k W_{1/\alpha}]) \\ &= [\exp 2\pi i f_0 \Delta]_{K_1(A_{1/\alpha})} \\ &= [V_{1/\alpha}]_{K_1(A_{1/\alpha})} \end{aligned}$$

where we are considering the Pimsner-Voiculescu exact sequence

$$K_0(A_{1/\alpha}) \xrightarrow{\widetilde{\beta}_* - \mathrm{Id}} K_0(A_{1/\alpha}) \xrightarrow{i} K_0(H_{1/\alpha})$$

$$\xrightarrow{\delta} K_1(A_{1/\alpha}) \xrightarrow{\widetilde{\beta}_* - \mathrm{Id}} K_1(A_{1/\alpha})$$

with

$$H_{1/\alpha} = A_{1/\alpha} \times_{\widetilde{\beta}} \mathbf{Z}$$

and Δ the left support projection of $f_1(t)U_{1/\alpha}^k$ in the enveloping von Neumann algebra of A_α. But any element of $K_0(H_{1/\alpha})$ is completely determined by its trace in $\mathbf{R}$ and image under δ. Thus we have shown that

$$[M_{H_\alpha}^{H_{1/\alpha}}((p_{\alpha-k}))]_{K_0(H_{1/\alpha})} = (-k, 1, -1)_{K_0(H_{1/\alpha})},$$

in terms of the standard generators for $K_0(H_{1/\alpha})$, which implies

$$M_{H_\alpha}^{H_{1/\alpha}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

as we desired to show.

Remark 2.5. If $\alpha < 0$, then the matrix mapping

$$K_0(A_\alpha) \to K_0(A_{1/\alpha})$$

given by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represents the Morita equivalence $A_{1/\alpha} - X - A_\alpha$. It is clear that $H_{-\alpha}$ is *-isomorphic to H_α via the correspondence

$$U_\alpha \to U_\alpha, \quad V_{-\alpha} \to \lambda V_\alpha^*, \quad W_{-\alpha} \to W_\alpha^*.$$

Thus

$$H_\alpha \cong H_{-\alpha} - \bar{X} \times \mathbf{Z} - H_{-1/\alpha} \cong H_{1/\alpha}.$$

It is not hard to see in this case that the matrix $M_{H_\alpha}^{H_{1/\alpha}}$ is given by

https://doi.org/10.4153/CJM-1988-037-8 Published online by Cambridge University Press

$$M_{H_\alpha}^{H_{1/\alpha}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The next lemma is much easier to prove than the preceding one, and we leave the proof to the reader:

LEMMA 2.4. *Let α be an irrational number, and let n be any integer such that $\alpha + n > 0$. Then the matrix*

$$M_{H_\alpha}^{H_{\alpha+n}}:K_0(H_\alpha) \to K_0(H_{\alpha+n})$$

obtained via the identification of H_α with $H_{\alpha+n}$ is expressed in terms of the standard generators by

$$M_{H_\alpha}^{H_{\alpha+n}} = \begin{pmatrix} 1 & 0 & 0 \\ -n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 2.7. If $\alpha + n < 0$ then the matrix

$$M_{H_\alpha}^{H_{\alpha+n}}:K_0(H_\alpha) \to K_0(H_{\alpha+n})$$

will be given by

$$\begin{pmatrix} -1 & 0 & 0 \\ n & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are finally able to prove

PROPOSITION 2.8. *Let α be an irrational real number and let*

$$\beta = (a\alpha + b)/(q\alpha + p)$$

where

$$\begin{pmatrix} a & b \\ q & p \end{pmatrix} \in GL(2, \mathbf{Z}).$$

Then there is a strong Morita equivalence bimodule $H_\alpha - Z - H_\beta$ whose matrix

$$M_{H_\alpha}^{H_\beta}:K_0(H_\alpha) \to K_0(H_\beta)$$

is given by

$$\det\begin{pmatrix} a & b \\ q & p \end{pmatrix}\begin{pmatrix} \gamma p & -q\gamma & 0 \\ -b\gamma & a\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\gamma = \operatorname{sgn}(q\alpha + p)$.

Proof. As mentioned in the proof of Theorem 4 of **[19]** the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (or alternatively $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$) generate $GL(2, \mathbf{Z})$.

Lemmas 2.4 and 2.6 together with the subsequent remarks show that one can arrive at

$$\beta = (a\alpha + b)/(q\alpha + p)$$

by a finite chain of equivalences

$$H_{\beta=\alpha_n} - X_{n-1} - H_{\alpha_{n-1}} - X_{n-2} \dots - H_{\alpha_2} - X_1 - H_{\alpha=\alpha_1}$$

where the matrix $M_{H_{\alpha_i}}^{H_{\alpha_{i+1}}}(X_i)$ is given by an element

$$\begin{pmatrix} M^{-1} & & 0 \\ & & 0 \\ 0 & 0 & \det M \end{pmatrix}$$

for

$$M \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

One obtains the desired matrix by taking the product of the $n - 1$ matrices involved.

We now are prepared for the following result, which identifies the positive cone of the K_0 group for class 2 Heisenberg C^*-algebras as being those elements with positive trace:

LEMMA 2.9. *Let H_α be the C^*-algebra generated by three unitary elements. U, V, and W with the relations*

$$UV = e^{2\pi i\alpha}VU, \quad VW = WV, \quad \text{and } UW = VWU,$$

where α is irrational. Let (a, b, c) be an element of $K_0(H_\alpha)$ as represented in the standard generators of Notation 2.2. Then there exists a non-zero projection p in $M_n(H_\alpha)$ for some positive integer n with

$$[p]_{K_0(H_\alpha)} = (a, b, c)$$

if, and only if, $a\alpha + b > 0$.

Proof. One direction of the lemma is obvious, for if p is a non-trivial projection in $M_n(H_\alpha)$ with $[p] = (a, b, c)$, then

$$\tau^*([p]) = \tau(p) = a\alpha + b$$

which therefore must be greater than zero.

As for sufficiency, let d be the greatest (positive) common divisor of a, b, and c, and write $(a, b, c) = d(l, m, n)$, where l, m, n have no common factor. Let f be the greatest (positive) common divisor of l and m, and write

$$(a, b, c) = d(fg, fh, n) \quad \text{where } (g, h) = 1.$$

We note that $g\alpha + h > 0$ since

$$c\alpha + d = dfg\alpha + dfh > 0,$$

and that $(f, n) = 1$; hence there exist r, s with $rf - sn = 1$.

We now form the following chain of strong Morita equivalences:

$$H_\alpha - V_3 - H_\beta - V_2 - H(\beta/f, s/f) - V_1 - M_d(H(\beta/f, s/f))$$

where V_1 is given by $\oplus_{i=1}^d H(\beta/f, s/f)_i$,

$$\langle \bar{x}, \bar{y}\rangle_{M_d(H(\beta/f,s/f))ij} = x_i^* y_j, \quad \langle \bar{x}, \bar{y}\rangle_{H(\beta/f,s/f)} = \sum_{i=1}^d x_i y_i^*,$$

V_2 is given as in Example 2.8 of [**14**], and V_3 is given by Proposition 2.8, where

$$\beta = (x\alpha + y)/(g\alpha + h) \quad \text{for} \begin{pmatrix} x & y \\ g & h \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Then

$$H_\alpha - V_3 \underset{H_\beta}{\otimes} V_2 \underset{H(\beta/f,s/f)}{\otimes} V_1 - M_d(H(\beta/f, s/f)$$

is a strong Morita equivalence bimodule, so that if we denote by V the bimodule

$$V_3 \underset{H_\beta}{\otimes} V_2 \underset{H(\beta/f,s/f)}{\otimes} V_1,$$

V is a finitely generated projective H_α-module. The projection in $M_k(H_\alpha)$ corresponding to V is given by injecting the C^*-algebra $M_d(H(\beta/f, s/f))$ into a full corner of $M_k(H_\alpha)$ for some $k \in \mathbf{N}$ and then calculating the image of $\mathrm{Id}_{M_d(H(\beta/f,s/f))}$ in $M_k(H_\alpha)$, say p. Then $[p]_{K_0(H_\alpha)}$ is given in terms of the standard generators for $K(H_\alpha)$ by

$$\begin{aligned}
&M^{H_\alpha}_{M_d(H(\beta/f,s/f))}(V)([\mathrm{Id}]_{K_0(M_d(H(\beta/f,s/f)))}) \\
&= M^{H_\alpha}_{H(\beta/f,s/f)}(V_3 \otimes V_2)(d[\mathrm{Id}]_{K_0(H(\beta/f,s/f))}) \\
&= M^{H_\alpha}_{H_\beta}(V_3)((0, df, dn)^t_{K_0(H_\beta)}) \\
&= \begin{pmatrix} x & g & 0 \\ y & h & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ df \\ dn \end{pmatrix} \\
&= (dfg, dfh, dn)^t \quad \text{(by Proposition 2.8)} \\
&= (a, b,c)^t.
\end{aligned}$$

Thus, if $G_1, G_2, \ldots, G_k$ are elements of V such that

$$\sum \langle G_i, G_i\rangle_{M_d(H(\beta/f,s/f))} = \mathrm{Id}_{M_d(H(\beta/f,s/f))},$$

then $p \in M_k(H_\alpha)$ defined by

$$(p_{ij}) = (\langle G_i, G_j \rangle_{H_\alpha})$$

is a projection in $M_k(H_\alpha)$ with

$$[p]_{K_0(H_\alpha)} = (a, b, c),$$

as we desired to show.

Lemma 2.9 constructs examples of all finitely generated projective H_α-modules up to stable equivalence in $K_0(H_\alpha)$. We now can apply Theorem *A*1 of **[2]**; which is actually a Corollary of Theorem 2.2 of **[21]**, to conclude that

THEOREM 2.10. *Every Heisenberg C^*-algebra of class* 2 *or* 3 *has the cancellation property.*

Proof. The result has already been proven for class 3 in Proposition 2.1, so we need only concentrate on the class 2 case. Every $H(\alpha', \beta')$ of class 2 is strongly Morita equivalent to H_α for some irrational number α, so it suffices to show that H_α has cancellation. We note that H_α is simple and unital, contains arbitrarily small positive elements, and that the Bass stable ranks of the endomorphism rings of the projective modules constructed in Lemma 2.9 are always $\leqq 3$ (since they are of the form $M_n(H(\beta/q, p/q))$ which always has Bass stable rank $\leqq 3$). Hence Theorem *A*1 of **[2]** may be applied to conclude that H_α has cancellation.

Remark 2.11. Having proved cancellation, it is clear that corollaries analogous to Corollaries 2.3, 2.5 and 2.6 of **[21]** can be proved for the $H(\alpha, \beta)$ of classes 2 and 3, and hence for their matrix algebras also.

To complete this section we calculate representatives for the positive cone of the K_0-groups of class 1 Heisenberg C^*-algebras. Recall that any class 1 Heisenberg C^*-algebra is *-isomorphic to $H(1/n, 0)$ for some $n \in \mathbf{N}$. As we mentioned earlier, all such C^*-algebras are therefore strongly Morita equivalent to $C^*(H)$, the rotation algebra. Thus to examine the structure of the K_0-group of class 1 Heisenberg C^*-algebras it suffices to examine $K_0(C^*(H))$ which we do now. Some of the material which follows is also found in **[1]**, but we include it here as our approach is somewhat different.

It is clear from the Pimsner-Voiculescu exact sequence that

$$K_0(C^*(H)) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$$

and $K_0(C^*(H))$ is generated by three projections: the identity, a projection e_1 in $M_2(\langle U, V \rangle)$, and a projection $e_2 \in M_2(\langle V, W \rangle)$. Here, $\langle U, V \rangle$ and $\langle V, W \rangle$ are the C^*-subalgebras of $C^*(H)$ generated by V, U and V, W respectively. As in the beginning of this section, we choose

$$e_2 = U^*M_1^* + M_0 + M_1U,$$

$$e_1 = W^*M_1^* + M_0 + M_1W,$$

where $M_1, M_0 \in M_2(\langle V \rangle) = M_2(C(T))$.

Since $C^*(H)$ can be written as a crossed product in two different ways, $\langle V, U\rangle \times_{\theta_2} \mathbf{Z}$, or $\langle V, W\rangle \times_{\theta_1} \mathbf{Z}$, we have two Pimsner-Voiculescu exact sequences corresponding to the following decompositions, of which we examine three terms.

$$(1)\qquad \begin{array}{ccccc} K_0(\langle V\rangle) & \xrightarrow{\ i_1\ } & K_0(\langle V, U\rangle) & \xrightarrow{\ \delta_1\ } & K_1(\langle V\rangle) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\langle V, W\rangle) & \xrightarrow{\ \tilde{i}_1\ } & K_0(C^*(H)) & \xrightarrow{\ \tilde{\delta}_1\ } & K_1(\langle V, W\rangle) \end{array}$$

$$(2)\qquad \begin{array}{ccccc} K_0(\langle V\rangle) & \xrightarrow{\ i_2\ } & K_0(\langle V, W\rangle) & \xrightarrow{\ \delta_2\ } & K_1(\langle V\rangle) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\langle V, U\rangle) & \xrightarrow{\ \tilde{i}_2\ } & K_0(C^*(H)) & \xrightarrow{\ \tilde{\delta}_2\ } & K_1(\langle V, U\rangle) \end{array}$$

By naturality of the Pimsner-Voiculescu exact sequence these diagrams are commutative and we obtain

$$\tilde{\delta}_1(e_1) = [V]_{K_1(\langle V,W\rangle)}$$
$$\tilde{\delta}_2(e_2) = [V]_{K_1(\langle V,U\rangle)}$$

by using the argument given just before 2.2. It is clear that $K_0(C^*(H))$ is generated by the identity, e_1, and e_2. We now assert that an element of $K_0(C^*(H))$ is determined by its images under τ^*, $\tilde{\delta}_1$, and $\tilde{\delta}_2$ where τ is any normalized faithful trace. First we formulate standard generators for $K_0(C^*(H))$. Let $t_i \in K_0(C^*(H))$ be defined by

$$[\mathrm{Id}] - [e_i], \quad i \in \{1, 2\}.$$

Identify $K_0(C^*(H))$ with $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ by the correspondence

$$[\mathrm{Id}] \rightarrow (1, 0, 0),$$
$$[t_1] \rightarrow (0, 1, 0),$$
$$[t_2] \rightarrow (0, 0, 1).$$

Then we have the following proposition, whose verification we leave to the reader.

PROPOSITION 2.12. *Let* $(a, b, c) \in K_0(C^*(H))$ *with the generators as described above, for* $a, b, c \in \mathbf{Z}$. *If* τ *is any faithful normalized trace on* $C^*(H)$, *and* $\tilde{\delta}_1$, $\tilde{\delta}_2$ *are the maps in diagrams* (1) *and* (2) *above, then*

$$\tau_*((a, b, c)) = a,$$

$$\widetilde{\delta}_1((a, b, c)) = [V^{-b}]_{K_1(\langle U, V \rangle)},$$
$$\widetilde{\delta}_2(a, b, c) = [V^{-c}]_{K_1(\langle V, W \rangle)}.$$

We shall now calculate the non-trivial part of the positive cone of $K_0(C^*(H))$, i.e., those elements of $K_0(C^*(H))$ which can be represented as (non-trivial) projections in $M_n(C^*(H))$ for some $n \in \mathbf{N}$. It is clear that in order for $(a, b, c) \in K_0(C^*(H))$ to correspond to a projection it is necessary that

$$\tau(a, b, c) = a > 0.$$

We shall show that this condition is sufficient as well, by computing the effects of $K_0(C^*(H))$ of certain *-automorphisms of $C^*(H)$, similar to those constructed by Brenken for the irrational rotation algebra in [**3**], corresponding to elements $M \in GL(2, \mathbf{Z})$. Theorem 1.1 shows that there is a *-isomorphism between $H(\alpha, \beta)$ and $H(d\alpha + b\beta, c\alpha + a\beta)$ for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}),$$

which when $\alpha = \beta = 0$ is a *-automorphism of $C^*(H)$ onto itself. If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}),$$

then under the corresponding *-isomorphism,

$$W \to W^a U^c, \quad U \to W^c U^d, \quad \text{and } V \to V^{\det M}.$$

(If desired, we can perturb each A_M corresponding to $M \in SL(2, \mathbf{Z})$ by an inner automorphism and in fact obtain a group action of $SL(2, \mathbf{Z})$ on $C^*(H)$, but that is not essential here.) We wish to compute the corresponding maps on the K_0-group of $C^*(H)$. Let us first consider the automorphism corresponding to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ so that, under θ,

$$V_1 \to V, \quad W_1 \to W, \quad U_1 \to WU.$$

Then

$$\begin{aligned} \theta(e_2) &= \theta(U^*M_1^* + M_0 + M_1 U) = U^*W^*M_1^* + M_0 + M_1 WU \\ &= W^*U^*VM_1^* + M_0 + M_1 V^*UW. \end{aligned}$$

(Keep in mind that $M_1, M_0 \in M_2(C^*(H))$ so that we are actually considering the embeddings of U, V, W in $M_2(C^*(H))$ given by

$$U \to \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \quad V \to \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}, \quad W \to \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}.)$$

We may now apply the method of Pimsner and Voiculescu, given in the Appendix to [**15**], to calculate $\widetilde{\delta}_1(\theta(e_2))$ and $\widetilde{\delta}_2(\theta(e_2))$, for note that we may view $\theta(e_1)$ as a Rieffel projection either in $M_2(\langle V, W\rangle) \times \mathbf{Z}$ or in $M_2(\langle V, U\rangle) \times \mathbf{Z}$. Now

$$M_1 V^* U U^* V M_1^* = M_1 M_1^*,$$

and we compute Δ the left support function projection in $M_2(C(T))$ as being

$$\begin{pmatrix} \chi_{[1/2,1]}(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$M_0 \Delta_1 = \begin{pmatrix} \cos^2 \pi t \chi_{[1/2,1]}(t) & 0 \\ 0 & 0 \end{pmatrix}$$

so that

$$\widetilde{\delta}_1([\theta(e_1)]_{K_0(C^*(H))}) = [\exp(2\pi i M_0 \Delta_1)]_{K_1(\langle V,U\rangle)} = [V]_{K_1(\langle V,U\rangle)}.$$

Similarly,

$$\widetilde{\delta}_2([\theta(e_2)]_{K_0(C^*(H))}) = [V]_{K_1(\langle V,W\rangle)}.$$

It follows from Proposition 1.2 of [**14**] that

$$\theta_*([e_2]_{K_0(C^*(H))}) = (1, -1, -1)$$

in terms of the standard generators. Hence

$$\theta_*([\mathrm{Id}] - [e_2]) = (0, 1, 1).$$

We are now in a position to represent the automorphism θ_* as an element of $\mathrm{Aut}(\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z})$, therefore as an element of $GL(3, \mathbf{Z})$. With respect to our standard generators, our calculations have shown that we may express θ_* as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{(acting on column vectors on the left).}$$

A similar argument shows that examining the *-automorphism

$$\phi : C^*(H) \to C^*(H)$$

given by $\phi(V) = V$, $\phi(W) = WU$, $\phi(U) = U$, we obtain

$$\phi^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, the bottom right 2×2 corner of θ_*, is precisely the matrix in $SL(2, \mathbf{Z})$ associated with the automorphism θ. We now note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

and it is well known that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ generate $SL(2, \mathbf{Z})$. Since we already know the matrices in $GL(3, \mathbf{Z})$ which correspond to θ_* and ϕ_*, and these generate

$$\begin{pmatrix} 1 & 0 \quad 0 \\ 0 & SL(2, \mathbf{Z}) \\ 0 & \end{pmatrix},$$

we can conclude with the following

LEMMA 2.13. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$$

and suppose that ψ *is the *-automorphism of* $C^*(H)$ *onto itself corresponding to* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *so that*

$$V \to V, \ W \to W^a U^c, \ U \to W^b U^d$$

under ψ. *Then* ψ *induces the map*

$$\psi_*: K_0(C^*(H)) \to K_0(C^*(H))$$

given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$

with respect to the standard generators.

Proof. Up to multiplication by an inner *-automorphism of $C^*(H)$, ψ can be expressed as a product of powers of θ and ϕ (this follows from known facts about the automorphism group of H). Then ψ_* can be expressed as a product of powers of θ_* and ϕ_*. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $SL(2, \mathbf{Z})$ and these inject into $GL(3, \mathbf{Z})$ via

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

the theorem follows from the fact that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

is a monomorphism of $SL(2, \mathbf{Z})$ into $GL(3, \mathbf{Z})$.

We are now able to construct every element in the positive cone of $K_0(C^*(H))$:

THEOREM 2.14. *Let* $(q, a, b) \in K_0(C^*(H))$ *be given with respect to the standard generators. If* $q > 0$, *there exists* $d \in \mathbf{N}$ *and a projection* $p \in M_d(C^*(H))$ *with*

$$[p]_{K_0(C^*(H))} = (q, a, b).$$

Proof. Write $(q, a, b) = k(q', a', b')$ where q', a' and b' have no common factor and $k > 0$. Let d be the g.c.d. of a' and b'. Note that $(q', d) = 1$ and that we can write $a' = dm$ and $b' = dn$ where $(m, n) = 1$. Find $r, s \in \mathbf{Z}$ with

$$\begin{pmatrix} m & r \\ n & s \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Then by Lemma 2.13 there exists an automorphism

$$\psi: C^*(H) \rightarrow C^*(H)$$

which induces the map

$$\psi_*: K_0(C^*(H)) \rightarrow K_0(C^*(H))$$

described by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & r \\ 0 & n & s \end{pmatrix}$$

in terms of the standard generators. Now find $n' \in \mathbf{N}$ with

$$pq' - dn' = 1 \quad \text{for some } p \in \mathbf{Z}.$$

Then by Example 2.8 of **[14]** there exists a strong Morita equivalence bimodule

$$C^*(H) - X(q', -d) \times \mathbf{Z} - H(n'/q', 0) \cong H(1/q', 0).$$

The projection in $K_0(C^*(H))$ corresponding to $X(q', -d) \times \mathbf{Z}$ is precisely $(q', d, 0)$ again by Example 2.8 of **[14]**. Form now the strong Morita equivalence bimodule

$$H(1/q', 0) - \bigoplus_{i=1}^{k} H(1/q', 0) - M_k(H(1/q', 0)),$$

and construct the tensor product bimodule

$$C^*(H) - (X(q', -d) \times \mathbf{Z}) \underset{H(1/q',0)}{\otimes} \left(\bigoplus_{i=1}^{k} H(1/q',0) \right) - M_k(H(1/q',0)).$$

Then it is evident that $(kq', kd, 0)$ is represented by a projection in some $M_n(C^*(H))$, and is the element in $K_0(C^*(H))$ corresponding to the projective module

$$(X(q', -d) \times \mathbf{Z}) \underset{H(1/q',1)}{\otimes} \left(\bigoplus_{i=1}^{k} H(1/q', 0) \right).$$

It follows that

$$\psi^*(kq', kd, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & r \\ 0 & n & s \end{pmatrix} \begin{pmatrix} kq' \\ kd \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ a \\ b \end{pmatrix}$$

so that (q, a, b) is in the positive cone of $K_0(C^*(H))$, as desired. It is clear from the construction that the endomorphism ring of the corresponding projective module for $C^*(H)$ is *-isomorphic to $M_k(H(1/q', 0))$.

Remark 2.15. Theorem 2.14 constructs representatives for every element in the positive cone of $C^*(H)$ and shows that for these representatives, the endomorphism rings are of the form $M_k(H(1/q, 0))$ for some $k \in \mathbf{N}$. Recall that all Heisenberg C^*-algebras of class 1 are strongly Morita equivalent to the universal rotation algebra, $C^*(H)$, studied in [**1**], whose primitive ideal space was computed by Howe in [**9**, p. 283]. The question of whether or not cancellation holds for their projective modules is an open one.

References

1. J. Anderson and W. Paschke, *The rotation algebra*, preprint.

2. B. Blackadar, *A stable cancellation theorem for simple C*-algebras*, Proc. London Math. Soc. (3) *47* (1983), 303-305.

3. B. A. Brenken, *Representations and automorphisms of the irrational rotation algebra*, Pacific J. Math. *111* (1984), 257-282.

4. L. G. Brown, P. Green and M. Rieffel, *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math. *71* (1977), 349-363.

5. F. Combes, *Crossed products and Morita equivalence*, Proc. London Math. Soc. *49* (1984), 289-306.

6. A. Connes, *A survey of foliations and operator algebras*, Proceedings of Symposia in Pure Math. *38*, Part 1, 521-628.

7. R. Curto, P. Muhly and D. Williams, *Cross products of strongly Morita equivalent C*-algebras*, Proc. Amer. Math. Soc. *90* (1984), 528-530.

8. G. Elliott, *On the K-theory of the C*-algebra generated by a projective representation of a torsion-free discrete abelian group*, Operator Algebras and Group Representations (Pitman, London, 1984), 157-184.

9. R. Howe, *On representations of discrete, finitely generated, torsion-free nilpotent groups*, Pacific J. Math. *73* (1977), 281-305.

10. C. C. Macduffee, *The theory of matrices* (Chelsea Publishing Co., New York, 1956).

11. G. Mackey, *Ergodic theory and virtual groups*, Math. Annalen *166* (1966), 187-207.

12. J. Packer, *K-theoretic invariants for C*-algebras associated to transformations and induced flows*, J. Functional Anal. *67* (1986), 25-59.

13. ——— *C*-algebras corresponding to projective representations of the Heisenberg group, I, II*, preprints.

14. ——— *C*-algebras corresponding to projective representations of the Heisenberg group* (revised), J. Operator Theory *18* (1987), 42-66.

15. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed-product C*-algebras*, J. Operator Theory *4* (1980), 93-118.

16. J. Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Math. *793* (Springer-Verlag, Berlin, 1980).

17. M. Rieffel, *Induced representations of C*-algebras*, Adv. Math. *13* (1974), 176-257.

18. ——— *Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner*, Studies in Analysis, Adv. Math. Suppl. Studies *4* (1979), 43-82.

19. ——— *C*-algebras associated with irrational rotations*, Pacific J. Math. *93* (1981), 415-429.

20. ——— *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. (3) *46* (1983), 301-333.

21. ——— *The cancellation theorem for projective modules over irrational rotation algebras*, Proc. London Math. Soc. (3) *47* (1983), 285-302.

22. ——— *"Vector bundles" over higher-dimensional non-commutative tori*, Operator Algebras and their Connections with Topology and Ergodic Theory, Lecture Notes in Math. *1132* (Springer-Verlag, Berlin-Heidelberg, 1985), 456-467.

23. G. Zeller-Meier, *Products croisés d'une C*-algebre par un groupe d'automorphismes*, J. Math. Pures Appl. *47* (1968), 101-239.

National University of Singapore,
Singapore, Republic of Singapore

https://doi.org/10.4153/CJM-1988-037-8 Published online by Cambridge University Press