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107.18 A two-variable approach to some standard optimisation problems

Introduction

Problems of the following type are staples of introductory courses on differentiation.

A solid right circular cylinder has fixed volume. Show that its total surface area is minimised when the height is twice the radius.

With standard notation for the attributes of the cylinder, the usual method of solution is to use $\pi r^2 h = V_0$ (fixed) to eliminate h in the expression for total surface area, $S = 2\pi r^2 + 2\pi r \cdot \frac{V_0}{\pi r^2}$. The condition $\frac{dS}{dr} = 0$ leads to $r = \sqrt[3]{\frac{V_0}{2\pi}}$ and routine algebra then identifies $h = \sqrt[3]{\frac{4V_0}{\pi}} = 2r$. In such an approach, the pleasant minimising ratio for $\frac{h}{r}$ appears serendipitously and almost as an afterthought. In addition, the fact that the dual problem (of maximising the volume of a solid cylinder of fixed surface area) is solved by the same value of $\frac{h}{r}$ is obscured by the algebra. This point was raised by Prithwijit De and Des MacHale in their Note, [1]; re-reading this stimulated the reflections that follow. Observe that we shall not formally check whether our optimising values correspond to maxima or minima: in any specific situation (such as the above problem), consideration of extreme cases (e.g. $r \to 0$, $r \to \infty$) usually makes this clear.

An alternative approach

We will see here that a multivariable approach yields a rich insight into this type of problem and adds to the repertoire of methods of solution. Suppose then that $V(r, h) = V_0$ is fixed. This equation implicitly defines $h = h(r)$ so, by the chain rule and writing $V_r = \frac{\partial V}{\partial r}$, etc., we have $V_r + h'(r) V_h = 0$.

Similarly, the stationary values of $S(r, h) = S\big(r, h(r)\big)$ occur when $S_r + h'(r) S_h = 0$.

Eliminating $h'(r)$ from these two equations gives what we shall refer to as the Key Equation: $V_r S_h = V_h S_r$.

We shall see that the Key Equation often leads to a very efficient way of identifying the condition for optimisation. Moreover, by symmetry, precisely the same equation arises if S is fixed and V optimised. In the Appendix we show that, at the optimising value of r, $\dfrac{V''(r)}{S''(r)} = -\dfrac{V_r}{S_r}$. Since this is negative (provided $V_r, S_r > 0$), maximum values of V pair with minimum values of S and vice versa.

Illustrative examples

1. For our introductory example, $V = \pi r^2 h$ and $S = 2\pi r^2 + 2\pi rh$.
The Key Equation $V_r S_h = V_h S_r$ gives

$$2\pi rh.2\pi r = \pi r^2 (4\pi r + 2\pi h),$$

so that $4h = 4r + 2h$ and $h = 2r$.

Examples 2, 3, 4, 5 may be found in many sources, such as the Sixth Form textbook, [2].

2. Find the shape of the right circular cone with given volume and minimum curved surface area.

Here, $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$.
The Key Equation gives

$$\frac{2}{3}\pi rh \cdot \frac{\pi rh}{\sqrt{r^2 + h^2}} = \frac{1}{3}\pi r^2 \left(\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right),$$

so that $2h^2 = r^2 + h^2 + r^2$ and $\frac{h}{r} = \sqrt{2}$.

3. Find the shape of the right circular cylinder inscribed in a given sphere of radius a which:

(a) maximises its curved surface area;
(b) maximises its total surface area.

The same theory works for these types of constraint problems: here the constraint is $r^2 + \frac{1}{4}h^2 = a^2$.

For (a), optimising $C = 2\pi rh$ using the Key Equation gives

$$2\pi r \cdot 2r = 2\pi h \cdot \frac{2h}{4},$$

so that $\frac{h}{r} = 2$.

For (b), optimising $S = 2\pi r^2 + 2\pi rh$ using the Key Equation gives

$$(4\pi r + 2\pi h)\frac{2h}{4} = 2\pi r.2r$$

or $h^2 + 2rh - 4r^2 = 0$ whence $\frac{h}{r} = \sqrt{5} - 1$, so that $\frac{2r}{h} = \varphi$, the golden ratio. (A one-variable approach to (a) and (b) features in [3].)

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4. Find the shape of the right circular cone of greatest volume that can be inscribed in a given sphere of radius a.

Here, the constraint is $r^2 + (h - a)^2 = a^2$ or $r^2 + h^2 - 2ah = 0$. Using the Key Equation to optimise $V = \frac{1}{3}\pi r^2 h$ gives

$$\tfrac{2}{3}\pi rh\,(2h - 2a) = \tfrac{1}{3}\pi r^2.2r$$

or $2h^2 - 2ah = r^2$.

This time, we need the help of the constraint equation to get $2h^2 - \left(r^2 + h^2\right) = r^2$ from which $\frac{h}{r} = \sqrt{2}$.

5. Find the shape of the right circular cone of minimum volume that circumscribes a sphere of given radius a.

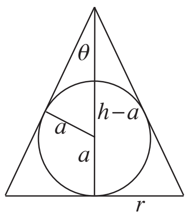

FIGURE 1

Here, from Figure 1, $\sin\theta = \dfrac{a}{h - a} = \dfrac{r}{\sqrt{r^2 + h^2}}$, leading to the constraint equation

$$a^2\left(r^2 + h^2\right) = r^2\,(h - a)^2 \qquad \text{or} \qquad 2ar^2 + a^2h - r^2h = 0.$$

Using the Key Equation to optimise $V = \frac{1}{3}\pi r^2 h$ gives

$$\tfrac{2}{3}\pi rh\left(a^2 - r^2\right) = \tfrac{1}{3}\pi r^2\left(4ar - 2rh\right)$$

so that $2ha^2 - 2hr^2 = 4ar^2 - 2hr^2$ or $2ar^2 = ha^2$. Substituting this into the constraint equation gives $ha^2 + a^2h - r^2h = 0$ so that $r^2 = 2a^2$ and $h = 4a$, giving $\frac{h}{r} = 2\sqrt{2}$.

6. Find the shape of the right square-based pyramid which maximises the volume when the total surface area is fixed.

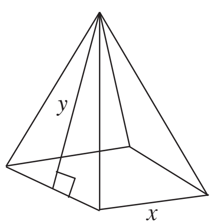

FIGURE 2

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Here, with notation as in Figure 2, $S = x^2 + 2xy$ and $V = \frac{1}{3}x^2\sqrt{y^2 - \frac{x^2}{4}}$.

The Key Equation gives

$$(2x + 2y).\frac{1}{3}x^2\frac{y}{\sqrt{y^2 - \frac{1}{4}x^2}} = 2x.\frac{1}{3}\left[2x\sqrt{y^2 - \tfrac{1}{4}x^2} - \frac{x^2.\left(\frac{1}{4}x\right)}{\sqrt{y^2 - \frac{1}{4}x^2}}\right]$$

so that $y(x + y) = 2y^2 - \frac{1}{2}x^2 - \frac{1}{4}x^2$ or $3x^2 + 4xy - 4y^2 = 0$.

Thus $(3x - 2y)(x + 2y) = 0$ and $\frac{y}{x} = \frac{3}{2}$.

7. Finally, we provide an alternative solution to the buoy problem considered in [1]. The challenge is to find the shape of the cone which maximises the volume of a buoy of fixed surface area, where the buoy consists of the cone attached to a hemisphere as in Figure 3.

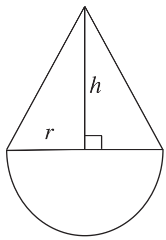

FIGURE 3

Here, $S = 2\pi r^2 + \pi r\sqrt{r^2 + h^2}$ and $V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$.

The Key Equation gives

$$\left(2\pi r^2 + \tfrac{2}{3}\pi rh\right).\frac{\pi rh}{\sqrt{r^2 + h^2}} = \tfrac{1}{3}\pi r^2\left[4\pi r + \pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right]$$

so that

$$(6r + 2h)h = \sqrt{r^2 + h^2}\left[4r + \sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}}\right]$$

and $h^2 + 6rh - 2r^2 = 4r\sqrt{r^2 + h^2}$.

Squaring produces $h^4 + 12rh^3 + 16r^2h^2 - 24r^3h - 12r^4 = 0$, a quartic in $\frac{h}{r}$ with four real roots, only one of which is positive; $\frac{h}{r} \approx 1.126$, which agrees with the answer in [1].

Appendix: Derivation of $\frac{V''(r)}{S''(r)} = -\frac{V_r}{S_r}$.

For the case $S(r, h(r)) = S_0$ fixed, $h'(r) = -\frac{S_r}{S_h}$, from which

$$h''(r) = \frac{-\left[S_h\left(S_{rr} + h'(r)S_{rh}\right) - S_r\left(S_{rh} + h'(r)S_{hh}\right)\right]}{S_h^2}$$

$$= \frac{-\left[S_h^2S_{rr} - 2S_rS_hS_{rh} + S_r^2S_{hh}\right]}{S_h^3}, \text{ on substituting for } h'(r).$$

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From $V'(r) = V_r + h'(r)V_h$, we have

$$V''(r) = V_{rr} + h'(r)V_{rh} + h'(r)\left[V_{rh} + h'(r)V_{hh}\right] + h''(r)V_h$$
$$= \frac{S_h^2(S_hV_{rr} - V_hS_{rr}) + 2S_rS_h(V_hS_{rh} - S_hV_{rh}) + S_r^2(S_hV_{rr} - V_rS_{hh})}{S_h^3},$$

on substituting for $h'(r)$ and $h''(r)$.

In the same way, for the case $V\left(r, \tilde{h}(r)\right) = V_0$, fixed, we obtain

$$S''(r) = \frac{-\left[V_h^2(S_hV_{rr} - V_hS_{rr}) + 2V_rV_h(V_rS_{rh} - S_hV_{rh}) + V_r^2(S_hV_{rr} - V_rS_{hh})\right]}{V_h^3}.$$

These are general calculations. If we evaluate at the optimising value of r for which the Key Equation $\frac{S_r}{S_h} = \frac{V_r}{V_h}$ holds, we obtain $\frac{V''(r)}{S''(r)} = -\frac{V_h}{S_h} = -\frac{V_r}{S_r}$, as stated in the paragraph before Example 1.

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107.19 A quadratic harmonic approximation

Introduction

Some eight hundred years ago the French archbishop Nicholas Oresme developed his beautiful proof that the n-th *harmonic number*:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

satisfies the following growth inequality:

$$H_{2^k} > 1 + \frac{k}{2},$$

and thereby presented the first example in the history of mathematics, and the first seen by countless generations of calculus students, of an infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges although its n-th term decreases to zero.

Unfortunately H_n has no (known) simple closed formula representation and so its further study demanded that mathematicians find suitable