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The Ranks of the Homotopy Groups of a Finite Dimensional Complex

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Abstract. Let X be an n-dimensional, finite, simply connected CW complex and set

$$\alpha_X = \limsup_i \frac{\log \operatorname{rank} \pi_i(X)}{i}.$$

When $0 < \alpha_X < \infty$, we give upper and lower bounds for $\sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(X)$ for k sufficiently large. We also show for any r that α_X can be estimated from the integers $\operatorname{rk} \pi_i(X)$, $i \le nr$ with an error bound depending explicitly on r.

1 Introduction

Recall that any finitely generated abelian group, G, has the form $G \cong \mathbb{Z}^k \oplus T$, where T is a finite group; k is called the *rank* of G, $\operatorname{rk} G$. Evidently $\operatorname{rk} G = \dim G \otimes_{\mathbb{Z}} \mathbb{Q}$, and so the definition may be extended to all abelian groups.

Definition The *rank* of an arbitrary abelian group, G, is defined by

$$\operatorname{rk} G = \dim G \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, since for any pointed topological space X the groups $\pi_i(X)$, $i \ge 2$, are abelian, the sequences $(\operatorname{rk} \pi_i(X))_{i\ge 2}$ are well defined.

It is a classical result that if $(k_i)_{i\ge 2}$ is an arbitrary sequence with each k_i a non-negative integer or ∞, then there are simply connected CW complexes X with

$$\operatorname{rk} \pi_i(X) = k_i, \quad i \ge 2.$$

In this paper we shall be concerned with the following question.

Question *What are the restrictions on the sequences* $(\operatorname{rk} \pi_i(X))_{i\ge 2}$ *imposed by the condition that X be a finite dimensional connected CW complex ?*

First note that the class of all pointed topological spaces, X, may be divided into the three distinct groups characterized by the following conditions:

(i) $\sum_{i\ge 2} \operatorname{rk} \pi_i(X) < \infty$;
(ii) for $i \ge 2$ each $\operatorname{rk} \pi_i(X) < \infty$, but $\sum_{i\ge 2} \operatorname{rk} \pi_i(X) = \infty$;

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(iii) for some $i \geq 2$, $\operatorname{rk} \pi_i(X) = \infty$.

Definition An n-dimensional, connected, finite CW complex, X, is called *rationally elliptic* (resp. *rationally hyperbolic*, π*-rank infinite*) if X belongs to group (i) (resp. group (ii), group (iii)) above.

Now a classical spectral sequence argument applied to Postnikov decompositions for the universal cover, $\widetilde{X}$, establishes the following equivalences:

$$\operatorname{rk} \pi_i(X) < \infty \text{ for } 2 \leq i \leq k \iff \dim H_{\leq k}(\widetilde{X}; \mathbb{Q}) < \infty. \tag{1.1}$$

Therefore, if X is rationally elliptic (resp. rationally hyperbolic), then $\widetilde{X}$ is rationally elliptic (resp. rationally hyperbolic) in the sense of [7].

Now consider the question above. In the elliptic case it is completely resolved in [9], where the authors establish a simple algorithm that decides whether any finite sequence $k_1, \dots, k_r$ of non-negative integers appears as the sequence $(\operatorname{rk} \pi_i(X))_{i \geq 2}$ for a rationally elliptic finite dimensional CW complex. For the rationally hyperbolic and π-rank infinite cases, however, such a characterization seems out of reach, especially given the fact that when n is odd, the space $S^n \vee S^n$ and $S^n \vee S^1$ satisfy $\operatorname{rk} \pi_i(X) = 0$ unless $i \equiv 1 (\operatorname{mod}(n-1))$. Thus, instead, we consider the sequence

$$\mu_k(X) = \max_{k+2 \leq i \leq k+n} \operatorname{rk} \pi_i(X).$$

Our principal result deals with the hyperbolic case, and we first need to recall the following definition.

Definition The *homotopy log index*, α_X, of a pointed topological space X is given by

$$\alpha_X = \limsup_k \frac{\log \operatorname{rk} \pi_k(X)}{k}.$$

This invariant, which provides one measure of the growth of the sequence $\operatorname{rk} \pi_k(X)$ is analogous to the classical Gelfand Kirillov dimension defined for a finitely generated graded algebra A by

$$\operatorname{GK} \dim A = \limsup_k \frac{\log \dim A^k}{\log k}.$$

This invariant was introduced by Gelfand and Kirillov for enveloping algebras of Lie algebras in [10] and [13]. When $\dim A^k \simeq k^n$, $\operatorname{GK} \dim A = n$. In our case, when $\operatorname{rk} \pi_k(X) \simeq a^k$, $\alpha_X = \log a$.

Associated with X are the Hilbert series

$$\pi(z) = \sum_{i \geq 2} \operatorname{rk} \pi_i(X) z^i \quad \text{and} \quad \Omega(z) = \sum_i \dim H_i(\Omega X; \mathbb{Q}) z^i.$$

When X is a simply connected, rationally hyperbolic, finite CW complex, then by a result of Babenko, [3], $\pi(z)$ and $\Omega(z)$ have the same radius of convergence ρ, and by definition, $\alpha_X = -\log\rho$. We prove in [7] that in this case $0 < \alpha_X < \infty$. In this paper we will give more precise estimations of α_X.

Now if X is a rationally hyperbolic, connected, n-dimensional CW complex, we have that $\dim H(\widetilde{X};\mathbb{Q}) < \infty$, and so we may set $h = \max_i \dim H_i(\widetilde{X};\mathbb{Q})$. To state our main theorem we introduce the notation

$$\beta(n,h) = 40\,(2n\log 2n + \log(h+1) + 1)\log nh,$$
$$\gamma(n,h) = n\log(h+1) + 2n\log 2n.$$

Then our first main theorem reads as follows.

Theorem 1.1 *Suppose X is an n-dimensional, connected, rationally hyperbolic CW complex. Then $0 < \alpha_X < \infty$, and for some K, and for every $k \geq K$,*

$$e^{(\alpha_X - \frac{\beta(n,h)}{\log k})k} \leq \max_{k+2\leq i\leq k+n} \operatorname{rk}\pi_i(X) \leq e^{(\alpha_X + \frac{\gamma(n,h)}{k})k}.$$

Moreover, for every $k \geq 2$,

$$\operatorname{rk}\pi_k(X) \leq \max\Big\{1, \frac{(2n)^n}{e^{\alpha_X}}\Big\}\, e^{\alpha_X k}.$$

This leaves the π-rank infinite case, and here we have a complete answer.

Theorem 1.2 *Suppose X is an n-dimensional connected CW complex. If X is π-rank infinite, then for all $k \geq 0$,*

$$\max_{k+2\leq i\leq k+n} \operatorname{rk}\pi_i(X) = \infty.$$

Remarks

(i) The principal result of [7] is equivalent to the assertion that (for X as in Theorem 1.1) if k is sufficiently large, then $\max_{k+2\leq i\leq k+n}\operatorname{rk}\pi_i(X) = e^{(\alpha_X+\varepsilon_k)k}$ with $\varepsilon_k \to 0$ as $k \to \infty$. Theorem 1.1 improves this result with precise estimates for ε_k depending only on n, h, and k.

(ii) While the result of [7] generalizes to spaces of finite Lusternik–Schnirelmann category, Theorem 1.1 does not, as we shall see in Theorem 1.7.

When combined with the results of [12] and [9], Theorems 1.1 and 1.2 have the following immediate corollaries.

Corollary 1.3 *Let X be an n-dimensional connected CW complex. Then*

(i) *X is rationally elliptic $\Longleftrightarrow$ $\operatorname{rk}\pi_i(X) = 0$, $i \geq 2n$;*

(ii) *X is rationally hyperbolic $\Longleftrightarrow$ $1 \leq \max_{k+2\leq i\leq k+n}\operatorname{rk}\pi_i(X) < \infty$ for all $k \geq 0$;*

(iii) *X is π-rank infinite $\Longleftrightarrow$ $\max_{k+2\leq i\leq k+n}\operatorname{rk}\pi_i(X) = \infty$ for all $k \geq 0$.*

Corollary 1.4 *Let X be an n-dimensional connected CW complex. Then*

(i) *X is rationally elliptic $\Longleftrightarrow$ $\alpha_X = -\infty$;*

(ii) *X is rationally hyperbolic* $\Longleftrightarrow 0 < \alpha_X < \infty$*;*
(iii) *X is π-rank infinite* $\Longleftrightarrow \alpha_X = \infty$.

Corollary 1.5 *Let X be an n-dimensional connected CW complex. Then X is rationally elliptic (resp. rationally hyperbolic, π-rank infinite) if and only if*

$$\max_{2n \leq i \leq 3n-2} \operatorname{rk} \pi_i(X) = 0$$

(resp. $\in (0, \infty)$*, resp.* $= \infty$*).*

Remark The asymptotic formula of Theorem 1.1 provides a good estimate of the homotopy log index α_X in terms of $\max_{k+2 \leq i \leq k+n} \operatorname{rk} \pi_i(X)$, provided $k \geq K$ for sufficiently large K. Unfortunately we are not able to give any estimate for K, and, indeed, nothing we know gives any suggestion that this might be possible. Thus Theorem 1.1 does not provide a "computational" tool for the estimation of α_X.

By contrast it is possible to directly estimate α_X from the integers $\operatorname{rk} \pi_i(X)$, $r \leq i < 2r$, or equivalently from the integers $\dim H_i(\Omega X; \mathbb{Q})$, $r \leq i < 2r$, with an error bound depending explicitly on r. Thus, our third main result reads as follows.

Theorem 1.6 *Let X be a rationally hyperbolic n-dimensional CW complex and set $h = \max_i \dim H^i(\widetilde{X}; \mathbb{Q})$. Then for $\log r > 2^{n+1} n^{2n+5} \log nh$,*

$$\max_{i \geq r} \frac{\log \operatorname{rk} \pi_i(X)}{i} - \frac{n \log 2n}{r} \leq \alpha_X \leq \max_{r \leq i < 2r} \frac{\log \operatorname{rk} \pi_i(X)}{i} + \frac{\beta(n,h)}{10 \log r}.$$

The main part of Theorem 1.1 asserts that for the "universal sequence" $\delta_k = 1/\log k$, given any n-dimensional rationally hyperbolic CW complex X there is a constant $c = c(n,h)$ such that for k sufficiently large

$$\max_{k+2 \leq i \leq k+n} \frac{\log \operatorname{rk} \pi_i(X)}{k} \geq \alpha_X - c\delta_k.$$

This is the assertion that does not generalize to rationally hyperbolic spaces of finite Lusternik Schnirelmann category. Our final main theorem reads as follows.

Theorem 1.7 *Let $\delta_k \to 0$ be any sequence of non-negative numbers and let $\alpha \in (0, \infty)$ be any number. Then there is a simply connected rationally hyperbolic wedge of spheres X such that $\alpha_X = \alpha$, and for any $c > 0$ and any integer $d > 0$ there are infinitely many k for which*

$$\max_{k \leq i \leq k+d} \frac{\log \operatorname{rk} \pi_i(X)}{k} < \alpha_X - c\delta_k.$$

The main theorem of [7] and Theorem 1.1 are steps to a better knowledge of the sequence $\operatorname{rk} \pi_n(X)$ for a connected finite CW complex, X. The results are inspired by what we know concerning some special families of spaces, like wedge of spheres (Theorem 1.7) or cofibers of maps between suspensions as illustrated by the following result of P. Lambrechts for cofibers of maps between suspensions.

Theorem ([15]) *Let X be a finite simply connected CW complex of dimension n that is the cofiber of a map between suspensions.*

(i) *There are constants A and B such that for k large enough,*

$$\frac{A}{k} e^{\alpha_X k} \leq \sum_{k+2 \leq i \leq k+n} \operatorname{rk} \pi_i(X) \leq \frac{B}{k} e^{\alpha_X k}.$$

(ii) *There is a polynomial $P(z)$ such that $\Omega(z)P(z)$ is the expansion at the origin of an analytic function without zero in a disc of radius $> \rho$.*

By comparison, Theorem 1.1 asserts that if X is a rationally hyperbolic n-dimensional CW complex, then there are constants A, B, and r depending only on n and the Betti numbers of $\widetilde{X}$ such that for k sufficiently large,

$$\frac{A}{k^r} e^{\alpha_X k} \leq \sum_{k+2 \leq i \leq k+n} \operatorname{rk} \pi_i(X) \leq B e^{\alpha_X k}.$$

We do not know if our estimates for A, B, and r can be significantly improved, although it seems clear that they are not sharp, nor we do know if B can be replaced by B'/k^ℓ with $\ell > 0$.

A second open question is whether the Hilbert series $\Omega(z)$ has a singularity at $z = \rho$ and if so to discover its nature. The special case that $H_*(\Omega X; \mathbb{Q})$ is a finitely generated algebra is settled by the following result of D. Anick.

Theorem ([2]) *Let H be a finitely generated graded connected algebra over a field $\Bbbk$ and $H(z)$ its Hilbert series with radius r. Then* $\liminf_{z \to r^-} (r - z)H(z) > 0$.

Finally, as indicated above, it is open whether the constant K in Theorem 1.1 can be estimated in terms of n and the Betti numbers of $\widetilde{X}$.

The proofs of Theorems 1.1, 1.2, 1.6, and 1.7 proceed by a careful analysis of the homotopy Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ with Lie bracket given by the Samelson product. The starting point for such an analysis is the minimal Sullivan model $(\wedge V, d)$ of a simply connected rationally hyperbolic space, X, which has the key property that $\dim V^i = \operatorname{rk} \pi_i(X)$ and $H(\wedge V, d) \cong H^*(X; \mathbb{Q})$. The proof of Theorem 1.1, which occupies almost the full paper, follows the same general strategy as [7]. In fact the analysis here is more delicate, the estimates more difficult, and considerable additional tools not needed in [7] are now required. In Section 5 we translate the algebra into proofs for Theorems 1.1 and 1.6, and give a direct proof of Theorems 1.2 and 1.7.

More precisely, in Section 2 we develop the relations between the growth of a Lie algebra L, its universal enveloping algebra UL and the indecomposable elements of some sub Lie algebras. Section 3 is a refinement of the exponential growth result obtained in [5]. Let n be the dimension of the space X. We prove in particular that for any integer N there is some $q < n^{n+5}N$ with $\operatorname{rk} \pi_{q+2}(X) > N$. This section contains all the material required for the proof of Theorem 1.6. Section 4 deals with the asymptotic formula and using partial results from [7] leads to the proof of Theorem 1.1.

2 Growth of Graded Lie Algebras

We work over an arbitrary ground field $\Bbbk$ of characteristic 0. If $W = \{W_i\}$ is any graded vector space, we adopt notation such as $W_{<k} = \{W_i\}_{i<k}$, $W_{[j,k]} = \{W_i\}_{j\le i\le k}$, and $W_+ = \{W_i\}_{i>0}$. We denote by $W(z) = \sum \dim W_i z^i$ the Hilbert series of W. Note that we will also use a graded vector space with a superscript notation $V = \{V^i\}$ with similar notations. Additionally we define the log index of W by

$$\log \operatorname{index} W = \limsup_k \frac{\log \dim W_k}{k}.$$

A graded Lie algebra L is a graded vector space equipped with a Lie bracket $[\,\cdot\,,\,\cdot\,]\colon L\otimes L \to L$, satisfying

$$[x,y] + (-1)^{\deg x\cdot \deg y}[y,x] = 0 \text{ and } [x,[y,z]] = [[x,y],z] + (-1)^{\deg x\cdot\deg y}[y,[x,z]].$$

We will consider graded Lie algebras L that are connected, $L = \{L_i\}_{i\ge 0}$, and of finite type (each L_i is finite dimensional). Graded Lie algebras satisfying those conditions are called cft graded Lie algebras. The rational homotopy Lie algebra, $L_X = \pi_*(\Omega X)\otimes \mathbb{Q}$ of a simply connected CW complex of finite type is a cft graded Lie algebra, and $\dim(L_X)_i = \operatorname{rank}\pi_{i+1}(X)$.

Throughout this article we work entirely in graded categories, so that, for example, subspace, sub Lie algebra, and subalgebra mean, respectively, graded subspace, graded sub Lie algebra, and graded subalgebra.

Lemma 2.1 *For any integer $s \ge 1$ the coefficients in the power series*

$$\sum_{k=0}^{\infty} a_k x^k = \frac{1-x}{1-x-x^s}$$

satisfy $a_k \le (s+1)^{k/s}$, $k \ge 1$. Moreover, the radius of convergence of this power series ρ_s satisfies

$$\rho_s > \Big(\frac{1}{s}\Big)^{1/s}, \quad \textit{if } s \textit{ is sufficiently large.}$$

Remark The reader will notice that the second assertion does not follow from the first, but the necessary improvement in the estimate for the a_k is very small, technical to obtain, and not necessary for this paper.

Proof Set $x_s = \big(\frac{1}{s}\big)^{1/s}$. To show that $\rho_s > \big(\frac{1}{s}\big)^{1/s}$ for large s we need only show that $1 - x_s - x_s^s > 0$, s large.

Set $u_s = \frac{1}{s}$. Then $1 - x_s - x_s^s = 1 - u_s^{1/s} - u_s$ and we have only to show that $1 - u^u - u > 0$ for $u \in (0,\varepsilon)$. An easy calculus argument shows that $1 - u^u - u$ is increasing in some interval $(0,\varepsilon)$ and that $\lim_{u\to 0}(1 - u^u - u) = 0$.

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It remains to show that for all $k, s \geq 1$, $a_k \leq (s+1)^{k/s}$. Note that

$$\frac{1-x}{1-x-x^s} = \frac{1}{1-x^s/(1-x)} = 1 + \sum_{\ell=1}^{\infty} x^{s\ell}\left(\frac{1}{1-x}\right)^{\ell}$$
$$= 1 + \sum_{\ell=1}^{\infty} x^{s\ell} \sum_{j=0}^{\infty} \binom{\ell+j-1}{\ell-1} x^j \quad NB. \binom{r}{0} = 1.$$

In particular, $a_0 = 1$ and $a_k = 0$, $1 \leq k < s$.

Thus it is sufficient to prove the inequality for $k \geq s$. Thus fix $k \geq s$ and let $q \geq 1$ and $i \in [0, s-1]$ be the unique integers such that $k = qs + i$. Then

$$a_k = \sum_{\ell=1}^{q} \binom{\ell + (q-\ell)s + i - 1}{\ell - 1}.$$

Write $m = \ell - 1$, so that

$$a_k = \sum_{m=0}^{q-1} \binom{m + (q-m-1)s + i}{m}.$$

But

$$m + (q-m-1)s + i \leq m + (q-m-1)s + s - 1 = (q-m)s + (m-1).$$

Thus,

$$\binom{m + (q-m-1)s + i}{m} \leq \frac{[(q-m)s + (m-1)] \cdots [(q-m)s + r] \cdots [(q-m)s]}{m!}$$
$$\leq s^m \frac{(q-1)(q-2) \cdots (q-m+r) \cdots (q-m)}{m!}$$
$$= s^m \binom{q-1}{m}.$$

Hence for $k \geq s$,

$$a_k \leq \sum_{m=0}^{q-1} s^m \binom{q-1}{m} = (1+s)^{q-1} \leq (1+s)^{k/s}. \qquad \blacksquare$$

Remark When $s = 1$,

$$\frac{1-x}{1-x-x^s} = \frac{1-x}{1-2x} = (1-x)\sum_{0}^{\infty} (2x)^k.$$

Thus in this case

$$a_k = 2^k - 2^{k-1} = (1+s)^k - (1+s)^{k-1}.$$

Lemma 2.2 *Suppose that a graded Lie algebra F satisfies*

$$F_k = 0,\ k < s \quad \textit{and} \quad \dim(F/[F,F])_k \le e^{\beta k}, k \ge s,$$

for some integer $s \ge 1$ and some $\beta \ge 0$. Then

$$\dim(UF)_k \le e^{(\beta + \frac{1}{s}\ \log(s+1))k}, \quad k \ge 0.$$

Proof Let W be a graded vector space satisfying $W_k = 0$, $k < s$, and $\dim W_k$ is the integral part of $e^{\beta k}$, $k \ge s$. The tensor algebra TW is the universal enveloping algebra of the free graded Lie algebra E generated by $W : TW = UE$. The respective Hilbert series satisfy ($\ll$ denotes coefficient-wise inequality $\le$)

$$W(z) \ll (e^\beta z)^s \sum_{k=0}^{\infty} (e^\beta z)^k = \frac{(e^\beta z)^s}{1 - e^\beta z}$$

and

$$\begin{aligned} UE(z) &= \frac{1}{1 - W(z)} = \sum_{k=0}^{\infty} \big[W(z)\big]^k \ll \sum_{k=0}^{\infty} \Big(\frac{(e^\beta z)^s}{1 - e^\beta z}\Big)^k \\ &= \frac{1}{1 - \frac{(e^\beta z)^s}{1 - e^\beta z}} = \frac{1 - e^\beta z}{1 - e^\beta z - (e^\beta z)^s}. \end{aligned}$$

Write $\frac{1-x}{1-x-x^s} = \sum_{k=0}^{\infty} a_k x^k$. Then

$$UE(z) \ll \sum_{k=0}^{\infty} a_k\, e^{\beta k} z^k.$$

Since $W_k = 0$, $k < s$, it follows that $(UE)_k = 0$, $1 \le k < s$ and so $a_k = 0$, $1 \le k < s$. On the other hand, Lemma 2.1 gives $a_k \le (s+1)^{k/s}$, $k \ge s$. Thus,

$$\dim(UE)_k \ll e^{\frac{k}{s}\log(s+1) + \beta k} = e^{[\beta + \frac{\log(s+1)}{s}]k}, \quad k \ge s.$$

By construction of W, there is a surjective linear map $W \to F/[F,F]$ that extends to a surjective map of graded Lie algebras $E \to F$ and to a surjective map of universal enveloping algebras $UE \to UF$. The lemma follows. ■

Lemma 2.3 *Suppose $E \subset L$ and $Y \subset L$ are a sub Lie algebra and a graded subspace of a cft graded Lie algebra L, respectively. Suppose further that for some integer $s \ge 1$ and some $\beta > 0$ they satisfy*

$$\dim(UE)_k \le e^{\beta k}, k \ge 0, \quad Y_k = 0, k < s, \quad \textit{and} \quad \dim Y_k \le e^{\beta k}, k \ge s.$$

Then the sub Lie algebra G generated by E and Y satisfies

$$\dim(UG)_k \le e^{(\beta + 3\frac{\log(s+1)}{s})k}, \quad k \ge 0.$$

Moreover, if $Y = Y_s$, then

$$\dim(UG)_k \le \begin{cases} e^{\beta k}, & k < s \\ e^{(\beta + \frac{2\log(s+1)}{s})k}, & k \ge s. \end{cases}$$

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Proof Denote the adjoint action of UL in L by "$\circ$" and note that for $j \geq s$,

$$\dim(UE \circ Y)_j \leq \sum_i \dim(UE)_i \dim Y_{j-i} \leq je^{\beta j} \leq e^{(\beta + \frac{\log(s+1)}{s})j},$$

because for $j \geq s \geq 1$, we have $\frac{\log j}{j} < \frac{\log(s+1)}{s}$.

It follows from Lemma 2.2 that the sub Lie algebra F generated by $UE \circ Y$ satisfies

$$\dim(UF)_k \leq e^{(\beta + \frac{2}{s}\log(s+1))k}, \quad k \geq 0.$$

Since $[E, F] \subset F$, it follows that $G = E + F$. Thus multiplication in UG gives a surjective map $UE \otimes UF \to UG$, whence for $k \geq s \geq 1$,

$$\dim(UG)_k \leq \sum_i \dim(UE)_i \dim(UF)_{k-i} \leq k\, e^{\beta k} e^{[\frac{2}{s}\log(s+1)]k} \leq e^{(\beta + \frac{3}{s}\log(s+1))k}.$$

Now, if $k < s$, $\dim UG)_k = \dim(UE)_k \leq e^{\beta k} \leq e^{(\beta + \frac{3}{s}\log(s+1))k}$.

Finally, if $Y = Y_s$, then for $k \geq s$, $(UE \circ Y)_k = (UE)_{k-s} \circ Y_s$ and so $\dim(UE \circ Y)_k \leq e^{\beta k}$. Since $(UE \circ Y)_k = 0$ for $k < s$ we obtain from Lemma 2.2 that $\dim(UF)_k \leq e^{(\beta + \frac{\log(s+1)}{s})k}$, $k \geq s$; clearly $(UF)_k = 0$ for $1 \leq k < s$. The same argument as above now gives the final assertion. ∎

Lemma 2.4 *Let L be a cft graded Lie algebra and assume given a positive integer i_0, an infinite sequence of integers $0 < r_0 < r_1 < \cdots$, and an infinite sequence of real numbers $(\lambda_i)_{i \geq i_0}$, such that*

$$\dim L_{r_i} \geq e^{\lambda_i r_i}, \quad i \geq 0, \quad \text{and} \quad \lambda_{i+1} \geq \lambda_i + 2\frac{\log(r_i + 1)}{r_i} + \frac{\log 3}{r_{i+1}}, \quad i \geq 0.$$

Then L contains a sub Lie algebra E, generated by subspaces in degrees r_i, $i \geq i_0$, such that

$$\frac{1}{2}e^{\lambda_i r_i} \leq \dim\big(E/[E, E]\big)_{r_i} \leq e^{\lambda_i r_i}, \quad i \geq 0.$$

Proof We construct inductively a sequence of subspaces $W_{r_i} \subset L_{r_i}$, $i \geq 0$ such that

$$\frac{1}{2}e^{\lambda_i r_i} \leq \dim W_{r_i} \leq e^{\lambda_i r_i}, \quad i \geq 0 \tag{2.1}$$

and such that the sub Lie algebra $E(i)$ generated by $W_{r_0}, \dots, W_{r_i}$ satisfies

$$\dim[UE(i)]_k \leq e^{(\lambda_i + 2\frac{\log(r_i+1)}{r_i})k}, \quad k \geq 0 \tag{2.2}$$

and

$$E(i) = [E(i), E(i)] \oplus \left(\oplus_{j=0}^{i} W_{r_j}\right), \quad i \geq 0. \tag{2.3}$$

It is then immediate from (2.1) and (2.3) that $E = \cup_i E(i)$ satisfies the conditions of the lemma. It remains to choose the W_{r_i}.

Suppose $W_{r_0}, \ldots, W_{r_\ell}$ are constructed. Then by (2.2) and the hypothesis on the λ_i,

$$\dim [UE(\ell)]_{r_{\ell+1}} \le \frac{1}{3} e^{\lambda_{\ell+1} r_{\ell+1}}.$$

Thus, since $\dim L_{r_{\ell+1}} \ge e^{\lambda_{\ell+1} r_{\ell+1}}$, we may choose $W_{r_{\ell+1}} \subset L_{r_{\ell+1}}$ so that (2.1) holds for $i = \ell+1$ and so that $W_{r_{\ell+1}} \cap E(\ell)_{r_{\ell+1}} = 0$. Then (2.3) is immediate for $i = \ell+1$. Since $\lambda_\ell + 2\frac{\log(r_\ell+1)}{r_\ell} \le \lambda_{\ell+1}$, we may apply the final assertion of Lemma 2.3 with $Y = W_{r_{\ell+1}}$, $\beta = \lambda_{\ell+1}$ and $s = r_{\ell+1}$ to obtain (2.2) for $i = \ell + 1$. ■

3 Approximating Log Index L

In this section we recall from [6] some basic definitions and properties for Sullivan algebras. Thus, a simply connected minimal Sullivan algebra $(\wedge V, d)$ is a graded commutative differential algebra in which $V = \{V^i\}_{i \ge 2}$ is a graded vector space, $\wedge V$ is the free graded commutative algebra generated by V, and $\operatorname{Im} d \subset (\wedge V)^+ \cdot (\wedge V)^+$.

For any graded vector space V, the algebra $\wedge V$ is the direct sum of the subspaces $\wedge^k V = V \wedge \cdots \wedge V$ (k factors). In the case of a simply connected minimal Sullivan algebra $(\wedge V, d)$ the surjection $\wedge V \to \wedge V / \wedge^{>m} V$ is a morphism of graded differential algebras, which extends to a quasi-isomorphism $(\wedge V \otimes \wedge W, d) \xrightarrow{\simeq} (\wedge V / \wedge^{>m} V, d)$ from a second simply connected minimal Sullivan algebra. The least m (or ∞) for which the identity morphism of $(\wedge V, d)$ extends to a morphism $(\wedge V \otimes \wedge W, d) \to (\wedge V, d)$ is called the *category* of $(\wedge V, d)$.

A graded vector space V has finite type if each $\dim V^i < \infty$, and a simply connected minimal Sullivan algebra $(\wedge V, d)$ has finite type if each $\dim V^i < \infty$. With each such Sullivan algebra is associated its *homotopy Lie algebra*, L, defined by $L_k = (V^{k+1})^\#$ and with Lie bracket dual to the component $d_1 \colon V \to \wedge^2 V$ of the differential d. (Here # denotes vector space dual.) For details *cf.* [6, Proposition 13.16].

The starting point for the main results of this paper is a growth theorem established in [5], which in turn depends on the following result.

Proposition 3.1 ([6, Theorem 29.5]) *Suppose* $\varphi \colon (\wedge V, d) \to (\wedge W, d)$ *is a surjective morphism between simply connected minimal Sullivan algebras of finite type. Then* $\operatorname{cat}(\wedge V, d) \ge \operatorname{cat}(\wedge W, d)$.

We now introduce the following hypotheses and notation for a simply connected minimal Sullivan algebra $(\wedge V, d)$ and its homotopy Lie algebra L:

$$(H)\begin{cases} \text{(i)} & \dim V^i < \infty \text{ for all } i \text{ and } \dim V = \infty,\\ \text{(ii)} & \dim H(\wedge V, d) < \infty, \text{ and } n = \max\{i \mid H^i(\wedge V, d) \ne 0\},\\ & \text{and } h = \max_i \dim H^i(\wedge V, d).\\ \text{(iii)} & \operatorname{cat}(\wedge V, d) = m.\end{cases}$$

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Remark Recall ([6, Corollary to Proposition 29.3]) that

$$m+1 \leq \frac{n}{2}+1 \leq n. \tag{3.1}$$

Moreover, since the only simply connected minimal Sullivan algebra with $n=2$ and $h=1$ has the form $\wedge(v,w)$ with $dw=v^2$, it follows that

$$nh \geq 3. \tag{3.2}$$

It is the interplay between the hypotheses of (H) and the Lie structure of L that provides the ingredients for the proof of our main theorems. In particular, it is shown in [5, pp. 188–189],that (H) implies that the integers $\dim V^k$ are unbounded. Here we establish a more precise statement.

Proposition 3.2 *Suppose $(\wedge V, d)$ is a simply connected minimal Sullivan algebra satisfying (H). Then for some q, $\dim V^{q+2} > [2(m+1)]^{m+1}$. Moreover for any integer $N>0$ there is some $q < 6(m+1)^{m+4}n^2N$ for which $\dim V^{q+2} > N$.*

Corollary *If $n \geq 4$, $\dim V^{q+2} > N$ for some $q < n^{n+5}N$.*

Proof Recall from (3.1) that $m+1 \leq \frac{n}{2}+1$. If $n \geq 4$, then $(n+2)^{\frac{n}{2}+3} \leq n^{n+3}$ and $6(m+1)^{m+4}n^2N < n^{n+5}N$. ∎

Proof of Proposition 3.2 The first assertion is [5, Lemma 4.3]. Now let $r=\ell n$. Since $[r+1,2r] = \cup_{i=\ell}^{2\ell-1}[in+1, i(n+1)]$, we deduce from [12, Theorem C(i)] that $\dim V^{[r+1,2r]} \geq \ell$. Choose $\ell = 3n(m+1)^{m+3}N$, and set $s = \dim V^{[r+1,2r]}$. Then trivially, $s \geq 2m$.

Now, divide $\wedge V$ by the ideal $V^{\leq r}\cdot \wedge V$. This yields a quotient minimal Sullivan algebra $(\wedge V^{>r}, \overline{d})$. There is a unique linear map of degree 1, $\theta\colon V^{>2r} \to \wedge^{\geq 2}V^{[r+1,2r]}$ such that

$$\overline{d}-\theta\colon V^{>2r} \to V^{>2r}\cdot \wedge V^{>r}.$$

Write

$$\theta = \sum_{i\geq 2}\theta_i, \quad \text{where } \theta_i\colon V^{>2r} \to \wedge^i V^{[r+1,2r]}.$$

Since $\mathrm{cat}(\wedge V^{>r},\overline{d}) \leq m$, by Proposition 3.1, and since $\overline{d}(V^{[r+1,2r]})=0$, it follows that

$$\wedge^{m+1}V^{[r+1,2r]} \subset \mathrm{Im}\,\overline{d}.$$

This implies that

$$\wedge^{m+1}V^{[r+1,2r]} = \sum_{i=2}^{m+1}\theta_i(V^{>2r})\wedge^{m+1-i}V^{[r+1,2r]}. \tag{3.3}$$

Now we make the following observations:

- $\wedge^i V^{[r+1,2r]}$ is concentrated in degrees $k \in [(r+1)i, 2ri]$ and so

$$\theta_i(V^{>2r}) = \theta_i(V^{[(r+1)i-1,2ri-1]});$$

- $\dim \wedge^{m+1} V^{[r+1,2r]} \geq \binom{s}{m+1} \geq \frac{s^{m+1}}{2^{m+1}(m+1)!}$ (because $s \geq 2m$);
- $\dim \wedge^{m+1-i} V^{[r+1,2r]} < s^{m+1-i}$.

On the other hand, setting

$$\lambda = \max\{\dim V^j \mid 2r+1 \leq j \leq 2r(m+1)-1\},$$

we find that

$$\begin{aligned}
&\dim \wedge^{m+1} V^{[r+1,2r]} \\
&\quad \leq \sum_{i=2}^{m+1} \dim V^{[(r+1)i-1,2ri-1]} \dim \wedge^{m+1-i} V^{[r+1,2r]} < \lambda \sum_{i=2}^{m+1} (ri-i+1) s^{m+1-i} \\
&\quad \leq \lambda(m+1)\big((r-1)(m+1)+1\big) s^{m-1}.
\end{aligned}$$

In other words, since $(m+1)! < e(\frac{m+1}{2})^{m+1}$,

$$\lambda > \frac{s^2}{2^{m+1}(m+1)!\,\ell\, n(m+1)^2} > \frac{s^2}{3\ell(m+1)^{m+3} n}.$$

Since $s = \dim V^{[r+1,2r]} \geq \ell$, it follows that $\lambda > N$. But $\lambda = \dim V^{q+2}$, some $q+2 \in [2r+1, 2r(m+1)-1]$ and so $q < 2\ell n(m+1) = 6(m+1)^{m+4} n^2 N$. ∎

Definition The *critical degree* for a simply connected minimal Sullivan algebra satisfying (H) is the least integer σ such that $\dim V^{\sigma+2} > [2(m+1)]^{m+1}$.

Remark Since $m+1 \leq n$, it follows from the corollary to Proposition 3.2 that if $n \geq 4$, then the critical degree satisfies $\sigma < 2^n n^{2n+5}$. The same is true when $n = 2$ or 3. In those cases, $(\wedge V, d)$ is the cochain algebra on a free graded Lie algebra, L, generated in degrees ≤ 2, and with at least two generators. Then $m = 1$, and $(2(m+1))^{m+1} = 16$. If L has at least two generators x, y in degree 1, then the elements $(\mathrm{ad}\, x)^i (\mathrm{ad}\, y)^{k-i}[x,y]$ are linearly independent and so $\dim V^{19} = \dim L_{18} \geq 17$. It follows that $\sigma \leq 17 < 2^n n^{2n+5}$. The case when $\dim L_1 \leq 1$ is similar.

Lemma 3.3 *Let L be the homotopy Lie algebra and let σ be the critical degree of a simply connected minimal Sullivan algebra, $(\wedge V, d)$ satisfying (H).*

(i) *If $\dim L_q > [2(m+1)]^{m+1}$ (in particular if $q = \sigma+1$), then q extends to an infinite sequence $q = q_0 < q_1 < \cdots$ such that for each $i \geq 1$, $q_i + 1 = \ell_i(q_{i-1}+1) - 1$ with $2 \leq \ell_i \leq m+1$, and*

$$\dim V^{q_i+1} \geq \left[\frac{1}{2(m+1)}\right]^{m+1} \left(\dim V^{q_{i-1}+1}\right)^{\ell_i}.$$

(ii) *Given a sequence (q_i) as in* (i), *then* $\dim V^{q_i+1} > [2(m+1)]^{m+1}$ *for all $i \geq 0$, and for all $i > j \geq 0$,*

$$\frac{\log \dim V^{q_i+1}}{q_i+1} \geq \frac{\log \dim V^{q_j+1}}{q_j+1} - \frac{(m+1)\log 2(m+1)}{q_j+1}.$$

(iii)
$$\log \operatorname{index} V \geq \sup_q \Big(\frac{\log \dim V^{q+1}}{q+1} - \frac{(m+1)\log 2(m+1)}{q+1} \Big).$$

Proof The proof is essentially contained in [5, pp. 188–189]. To make this paper more self contained we reproduce here the main lines of the proof. Denote $N = \dim V^{q+1}$ and $a = (1/2(m+1))^{m+1}$. By hypothesis $N > [2(m+1)]^{m+1}$, and so $Na > 1$. We have $N/2 \geq (m+1)\frac{N}{2(m+1)}$, and since $N \geq 2m$, we have for $1 \leq i \leq m$,

$$\frac{N-i}{i+1} \geq \frac{N-m}{m+1} \geq \frac{N}{2(m+1)}.$$

It follows from (3.3) that $\wedge^{m+1} V^{q+1} = \sum_{i=2}^{m+1} \theta_i(V^{i(q+1)-1}) \wedge^{m+1-i} V^{q+1}$, and hence

$$(m+1)\Big(\frac{N}{2(m+1)}\Big)^{m+1} \leq \binom{N}{m+1} \leq \dim \wedge^{m+1} V^{q+1} \leq \sum_{i=2}^{m+1} \dim V^{i(q+1)-1} \cdot N^{m+1-i}.$$

Therefore, since $Na > 1$, for some $j \in [2, m+1]$,

$$\dim V^{j(q+1)-1} \geq N^j \cdot a \geq N^2 a > N > (2(m+1))^{m+1}.$$

We set $\ell_1 = j$, and we iterate the procedure to construct the sequence $(q_i)_{i\geq 0}$. Thus we have a sequence of integers $q = q_0 < q_1 < \cdots$ with $q_i + 1 = \ell_i(q_{i-1}+1) - 1$, where ℓ_i is an integer in $[2, m+1]$ and

$$\dim V^{q_i+1} \geq \Big[\frac{1}{2(m+1)}\Big]^{m+1} \big(\dim V^{q_{i-1}+1}\big)^{\ell_i}.$$

This proves (i). It follows that $\dim V^{q_i+1} \geq a^{1+\ell_i+\ell_{i-1}\ell_i+\cdots+\ell_2\cdots\ell_i} \cdot N^{\ell_1\ell_2\cdots\ell_i}$. Since each $\ell_i \geq 2$, $(1+\ell_i+\cdots+\ell_2\cdots\ell_i)/(\ell_1\ell_2\cdots\ell_i) \leq \frac{1}{2^i} + \cdots + \frac{1}{2} < 1$, and so

$$\dim V^{q_i+1} \geq (aN)^{\ell_1\ell_2\cdots\ell_i} \geq \bigg(\Big[\frac{1}{2(m+1)}\Big]^{m+1} \dim V^{q+1} \bigg)^{\frac{q_i+1}{q+1}}.$$

We deduce part (ii) of the lemma, namely that

$$\frac{\log \dim V^{q_i+1}}{q_i+1} \geq \frac{\log \dim V^{q+1} - (m+1)\log 2(m+1)}{q+1}.$$

Finally, (iii) is a direct consequence of (ii). ■

Lemma 3.4 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra, $(\wedge V, d)$, satisfying (H).*

(i) *For any $j, k \geq 0$,*

$$\log \operatorname{index} L = \limsup_{\ell \geq k} \frac{\log \sum_{i=\ell}^{\ell+j} \dim L_i}{\ell}.$$

(ii) $\dim L_\ell \le (nh)^\ell$ *for all* $\ell \ge 1$.
(iii) *If* $\dim L_i \le c$ *for* $i < k$, *then* $\dim L_k \le h + m(k+2)^m\, c^{m+1}$.
(iv) $0 < \log \operatorname{index} L \le \log(h+1)$.

Proof Part (i) is obvious.

(ii) Let S be a space with minimal model $(\wedge V, d)$. Then $\pi_*(\Omega S) \otimes \mathbb{Q} \cong L$, and a classical result of Milnor–Moore [16] states that $UL \cong H_*(\Omega S; \mathbb{Q})$. An even earlier result of Adams–Hilton [1] states that $H_*(\Omega S; \mathbb{Q}) \cong H(TY, D)$, where (TY, D) is a differential graded tensor algebra on Y and $Y = \{Y_i\}_{i \ge 1}$ with $\dim Y_i = \dim H^{i+1}(\wedge V, d)$.

Let $\{y_{ij}\}$ be a basis of Y_i and let $X = X_1$ be a vector space concentrated in degree 1 with basis $\{x_{ij}\}$ in 1-1 correspondence with $\{y_{ij}\}$. Then the elements $w_{ij} = x_{ij}^i$ generate a sub algebra of TX isomorphic with TY. Since $\dim X = \dim Y = \sum_{i=2}^n \dim H^i(\wedge V, d) \le (n-1)h$, we obtain

$$\dim (TY)_\ell \le \dim (TX)_\ell \le \Big[(n-1)h\Big]^\ell,$$

and (ii) follows.

(iii) It follows from the minimal model translation of a theorem of Ginsburg [6, Theorem 29.14] that the kernel of the linear map

$$q\colon V^{k+1} \xrightarrow{d} \wedge V \to \wedge V / \wedge^{>m+1} V$$

is isomorphic with a subspace of $H^{k+1}(\wedge V, d)$. Thus,

$$\dim L_k = \dim V^{k+1} \le h + \sum_{\ell=2}^{m+1} \dim(\wedge^\ell V)^{k+2}.$$

Moreover,

$$\dim(\wedge^\ell V)^{k+2} \le \sum_{\substack{2 \le k_1 \le \cdots \le k_\ell \\ k_1 + \cdots + k_\ell = k+2}} \dim V^{k_1} \cdots \dim V^{k_\ell}.$$

Denote by $\rho_\ell(k+2)$ the number of partitions of $k+2$ of length ℓ and recall that $\rho_\ell(k+2) \le (k+2)^{\ell-1}$. (In fact, this is obvious for $\ell = 1$ and follows by a simple induction argument in general.) Since, by hypothesis, $\dim V^i \le c$, $i \le k$, we obtain $\dim(\wedge^\ell V)^{k+2} \le (k+2)^{\ell-1} c^\ell$, whence

$$\dim L_k \le h + m(k+2)^m c^{m+1}.$$

(iv) Recall that $W(z) = \sum_i \dim W_i\, z^i$ denotes the Hilbert series of a graded vector space $W = \{W_i\}_{i \ge 1}$ and recall that $\ll$ denotes coefficient-wise inequality between power series. Then, with the notation in the proof of (ii),

$$L(z) \ll UL(z) \ll TY(z) \ll \frac{1}{1 - \sum_{i=1}^{n-1} hz^i} = \frac{1-z}{1-(h+1)z + hz^n}.$$

Denote by r_L the radius of convergence of $L(z)$. These inequalities show that $r_L > \frac{1}{h+1}$. But

$$\log \text{index}\, L = \limsup_{\ell} \frac{\log \dim L_\ell}{\ell} = -\log r_L < \log(h+1).$$

Finally, it is immediate from Proposition 3.2 and Lemma 3.3 that $0 < \log \text{index}\, L$. ■

Lemma 3.5 *Suppose given integers $1 \le s < r$, a constant $b > 0$, an integer $c \ge 0$ and a graded vector space W concentrated in degrees $1 \le i < r$. If*

$$\dim W^i \le c, \quad 1 \le i \le s \quad \textit{and} \quad \dim W^i \le e^{bi}, \quad i > s,$$

then

$$\dim(\wedge W)^k \le (k+1)^{sc} e^{(b+\frac{3}{\sqrt{k}})k}, \quad k \ge 1.$$

Proof If s_i is the number of monomials of degree i in sc variables, then

$$\dim(\wedge W^{[1,s]})^i \le s_i.$$

But $s_i = \binom{i+sc-1}{i} = \binom{i+sc-1}{sc-1}$ is also the number of monomials of degree $sc-1$ in $i+1$ variables, and so $s_i \le (i+1)^{sc-1}$.

On the other hand,

$$\begin{aligned}\dim(\wedge W^{[s+1,r-1]})^j &\le \dim\big(\otimes_{\ell=s+1}^{r-1} T(W^\ell)\big)^j \\ &= \sum_{\sum \ell k_\ell = j} \dim\big(\otimes^{k_{s+1}} W^{s+1}\big) \cdots \dim\big(\otimes^{k_{r-1}} W^{r-1}\big) \le \rho(j) e^{bj},\end{aligned}$$

where $\rho(j)$ is the number of partitions of j, *i.e.*, the number of solutions of $\sum_{i=1}^n i p_i = j$ with $p_i \ge 0$ ([17, §15]). Thus,

$$\dim(\wedge W)^k \le \sum_{i+j=k} (i+1)^{sc-1} \rho(j) e^{bj} \le (k+1)^{sc} \rho(k) e^{bk}.$$

But by [17, Theorem 15.7],

$$\rho(k) < \frac{\pi}{\sqrt{6(k-1)}} e^{\pi\sqrt{2k/3}}.$$

It follows (from this for $k \ge 3$ and by inspection when $k = 1$ or 2) that $\rho(k) \le e^{3\sqrt{k}}$, $k \ge 1$. Thus,

$$\dim(\wedge W)^k \le (k+1)^{sc} e^{(b+3/\sqrt{k})k}, \quad k \ge 1.$$

■

Lemma 3.6 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra $(\wedge V, d)$ satisfying (H). Suppose for constants $c \ge 0$ and $b > 0$, and for integers $0 < s < r$, that*

$$\dim V^{i+1} \le c, \quad i \le s \quad \textit{and} \quad \dim V^{i+1} \le e^{bi}, \quad s < i < r.$$

Then

$$\dim L_k = \dim V^{k+1} \le (n+1)h(k+2)^{sc} e^{(b+\frac{3}{\sqrt{k+1}})(k+1)}, \quad r \le k \le 2r-1.$$

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Proof Define a graded vector space W by $W^i = V^{i+1}$, $1 \leq i \leq r-1$ and $W^i = 0$, $i \notin [1, r-1]$. There is then ([6, p. 181]) a quasi-isomorphism

$$(\wedge V \otimes \wedge W, D) \xrightarrow{\simeq} (\wedge V^{\geq r+1}, \overline{d})$$

in which $(\wedge V^{\geq r+1}, \overline{d})$ is the quotient Sullivan algebra obtained by dividing $\wedge V$ by the ideal generated by $V^{[2,r]}$ and $(\wedge V, d) \to (\wedge V \otimes \wedge W, D)$ is a relative Sullivan algebra for the projection $(\wedge V, d) \to (\wedge V^{\geq r+1}, \overline{d})$. Since $(\wedge V^{\geq r+1}, \overline{d})$ is minimal, it follows that $V^{[r+1,2r]}$ embeds in $H(\wedge V^{\geq r+1}, \overline{d})$.

On the other hand, filtering by the degree in $\wedge V$ gives a spectral sequence converging from $H(\wedge V, d) \otimes H(\wedge W, \overline{D})$ to $H(\wedge V^{\geq r+1}, \overline{d})$. Thus we may apply Lemma 3.5 to obtain for $k \in [r, 2r-1]$ that

$$\begin{aligned} \dim L_k = \dim V^{k+1} \leq \dim H^{k+1}(\wedge V \otimes \wedge W, D) &\leq \sum_{i+j=k+1} \dim H^i(\wedge V, d) \dim(\wedge W)^j \\ &\leq (n+1)h \max_{j \leq k+1} \dim(\wedge W)^j \leq (n+1)h(k+2)^{sc} e^{(b+\frac{3}{\sqrt{k+1}})(k+1)}. \end{aligned}$$ ■

Proposition 3.7 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra $(\wedge V, d)$ satisfying (H). Suppose for an integer $s \geq 0$ and for some $c \geq 0$ that $\dim L_i \leq c$ for $i \leq s$. Then for any integer $r > s$ and $r \geq 3$,*

$$\sup_{s<i} \frac{\log \dim L_i}{i} \leq \max_{s<i<r} \frac{\log \dim L_i}{i} + \frac{2\log n(n+1)h^2}{r} + \frac{5sc\log r}{r} + \frac{15}{\sqrt{r}}.$$

Proof Consider s as fixed and set

$$\mu_r = \max_{s<i<r} \frac{\log \dim L_i}{i} \quad \text{and} \quad \mu_\infty = \sup_{s<i} \frac{\log \dim L_i}{i}.$$

Apply Lemma 3.6 with $b = \mu_r$ and $r \geq 2$ to obtain, for $k \in [r, 2r-1]$, that

$$\begin{aligned} \frac{\log \dim L_k}{k} &\leq \frac{\log(n+1)h}{k} + sc\frac{\log(k+2)}{k} + \mu_r + \frac{\mu_r}{k} + 3\frac{\sqrt{k+1}}{k} \\ &\leq \frac{\log(n+1)h}{k} + sc\frac{\log 2k}{k} + \mu_r + \frac{\mu_r}{k} + 3\sqrt{\frac{2}{k}} \\ &\leq \mu_r + \frac{\log(n+1)h}{r} + sc\frac{\log 2r}{r} + \frac{\mu_r}{r} + 3\sqrt{\frac{2}{r}}. \end{aligned} \tag{3.4}$$

Now by Lemma 3.4(ii) we have $\dim L_\ell \leq (nh)^\ell$, $\ell \geq 1$, and it follows that $\mu_r \leq \log nh$, whence

$$\frac{\log(n+1)h}{r} + \frac{\mu_r}{r} \leq \frac{\log n(n+1)h^2}{r}.$$

Thus, because the previous inequalities (3.4) hold for $k \in [r, 2r-1]$, $r \geq 2$, we may conclude that either $\mu_{2r} - \mu_r = 0$ or else

$$\mu_{2r} - \mu_r = \max_{r \leq k \leq 2r-1} \frac{(\log \dim L_k)}{k} - \mu_r.$$

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Thus in either case,

$$\begin{aligned}\mu_{2r}-\mu_r &\le \frac{\log(n+1)h}{r}+\frac{sc\log 2r}{r}+\frac{\mu_r}{r}+3\sqrt{\frac{2}{r}}\\ &\le \frac{\log n(n+1)h^2}{r}+\frac{sc\log 2r}{r}+3\sqrt{\frac{2}{r}}\end{aligned}$$

Now replace r by $2^i r$ $(0\le i<\infty)$ in this inequality and sum over i to obtain, for $r\ge 3$, that

$$\mu_\infty-\mu_r\le$$
$$\Big(\sum_{i=0}^{\infty}\frac{1}{2^i}\Big)\frac{\log n(n+1)h^2}{r}+\Big(\sum_{i=0}^{\infty}\frac{1+\log 2^{i+1}}{2^i}\Big)\frac{sc\log r}{r}+\Big(3\sum_{i=0}^{\infty}\sqrt{\frac{2}{2^i}}\Big)\frac{1}{\sqrt{r}}.$$

Standard sums and a little calculation give

$$\sum_{i=0}^{\infty}\frac{1}{2^i}=2,\qquad \sum_{i=0}^{\infty}\frac{1+\log 2^{i+1}}{2^i}\le 5,\qquad\text{and}\qquad 3\sum_{i=0}^{\infty}\sqrt{\frac{2}{2^i}}\le 15.$$

Substitution in the previous inequality then gives the assertion in the proposition. ∎

Corollary *Let σ be the critical degree for $(\wedge V,d)$. Then for $r>\max(\sigma,9)$,*

$$\begin{aligned}\log \operatorname{index} L &\le \max_{\sigma<i<r}\frac{\log\dim L_i}{i}+\frac{2\log n(n+1)h^2}{r}+5\,2^{2n}n^{3n+5}\,\frac{\log r}{r}+\frac{15}{\sqrt{r}}\\ &\le \max_{\sigma<i<r}\frac{\log\dim L_i}{i}+\frac{1}{\sqrt{r}}\left(2\log n(n+1)h^2+5\,2^{2n}n^{3n+5}+15\right).\end{aligned}$$

Proof Since $\dim L_i\le[2(m+1)]^{m+1}$ for $i\le\sigma$, we may substitute σ for s and $[2(m+1)]^{m+1}$ for c in Proposition 3.7. Now since $m+1\le n$,

$$sc\le(2^n n^{2n+5})\big[2(m+1)\big]^{m+1}\le 2^n n^{2n+5}2^n n^n=2^{2n}n^{3n+5}.$$

This gives the first inequality, because $\log\operatorname{index}L\le\sup_{\sigma<i}(\log\dim L_i)/i$. Finally, because $r>9$, it follows that $1/\sqrt{r}<\log r/\sqrt{r}<1$. This gives the second inequality. ∎

Lemma 3.8 *Let L be the homotopy Lie algebra and let σ be the critical degree of a simply connected minimal Sullivan algebra satisfying* (H). *Suppose $0<s<r$ are integers such that $r>\sigma$. Then*

$$\max_{s<i<(m+1)r}\frac{\log\dim L_i}{i+1}\le\max_{r\le i<(m+1)r}\frac{\log\dim L_i}{i+1}+\frac{(m+1)\log 2(m+1)}{s}.$$

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Proof Let $i_0 \in (s, (m+1)r)$ be an integer for which

$$\frac{\log \dim L_{i_0} - (m+1)\log 2(m+1)}{i_0+1} = \max_{s<i<(m+1)r} \frac{\log \dim L_i - (m+1)\log 2(m+1)}{i+1}.$$

Since $r > \sigma$, it follows from Lemma 3.3 that $\dim L_{i_0} > [2(m+1)]^{m+1}$. Thus, by Lemma 3.3, i_0 extends to an infinite sequence $i_0 < i_1 < \cdots$ for which, in particular,

$$\frac{\log \dim L_{i_\nu}}{i_\nu+1} \ge \frac{\log \dim L_{i_0} - (m+1)\log 2(m+1)}{i_0+1}, \quad \text{all } \nu \ge 0.$$

Since $i_\nu \le (m+1)(i_{\nu-1}+1) - 2$, it follows that some i_k satisfies $r \le i_k < (m+1)r$. Thus it follows from Lemma 3.3 that

$$\begin{aligned}
\max_{r\le i<(m+1)r} \frac{\log \dim L_i}{i+1} \ge \frac{\log \dim L_{i_k}}{i_k+1} &\ge \frac{\log \dim L_{i_0} - (m+1)\log 2(m+1)}{i_0+1} \\
&= \max_{s<i<(m+1)r} \frac{\log \dim L_i - (m+1)\log 2(m+1)}{i+1} \\
&\ge \max_{s<i<(m+1)r} \frac{\log \dim L_i}{i+1} - \frac{(m+1)\log 2(m+1)}{s}.
\end{aligned}$$

This establishes the lemma. ∎

Theorem 3.9 *Let L be the homotopy Lie algebra and let σ be the critical degree of a simply connected minimal Sullivan algebra $(\wedge V, d)$ satisfying* (H). *Suppose $k \ge 2$ and r are fixed integers for which $\frac{\log kr}{2\log nh} > \max(20, k, \sigma)$. Then*

$$\begin{aligned}
&\max_{i\ge r}\Big(\frac{\log \dim L_i}{i+1}\Big) - \frac{n\,\log 2n}{r} \\
&\quad \le \log \operatorname{index} L \\
&\quad \le \max_{r\le i<kr}\Big(\frac{\log \dim L_i}{i+1}\Big) + \frac{4\,(2n\log 2n + \log(h+1)+1)\log nh}{\log r}.
\end{aligned}$$

Corollary *The conclusion of Theorem 3.9 holds for $k = 2$ if $\log r \ge 2^{n+1} n^{2n+5} \log nh$.*

Proof As observed in the remark following the proof of Proposition 3.2, $\sigma < 2^n n^{2n+5}$. ∎

Proof of Theorem 3.9 Recall from (3.1) that $m+1 \le n$. Thus for $i \ge r$, Lemma 3.3(iii) gives

$$\frac{\log \dim L_i}{i+1} - \frac{n\log 2n}{r} \le \frac{\log \dim L_i - (m+1)\log 2(m+1)}{i+1} \le \log \operatorname{index} L.$$

To prove the second inequality, let s be the largest integer satisfying $s < \frac{\log kr}{2\log nh}$. Then $s \ge k$ and $(nh)^{2s} < kr$. Now note that

$$\log \operatorname{index} L = \limsup_i \frac{\log \dim L_i}{i} \le \sup_{s<i} \frac{\log \dim L_i}{i}.$$

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By Lemma 3.4(ii) we have $\dim L_\ell \le (nh)^\ell$, $\ell \ge 1$. Thus we may use Proposition 3.7 with $c = (nh)^s$ to obtain

$$\sup_{s<i} \frac{\log \dim L_i}{i} \le \max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i} + \frac{2\log n(n+1)h^2}{(nh)^{2s}} + \frac{5s(nh)^s \log(nh)^{2s}}{(nh)^{2s}} + \frac{15}{(nh)^s}.$$

Since $s \ge 20$ and $\frac{\log x}{x} \le \frac{1}{e}$ for $x > 0$, it follows that $2\log n(n+1)h^2 \le (nh)^s$. Thus

$$\frac{2\log n(n+1)h^2}{(nh)^{2s}} + \frac{5s(nh)^s \log(nh)^{2s}}{(nh)^{2s}} + \frac{15}{(nh)^s} \le \frac{16 + 10s^2 \log nh}{(nh)^s}.$$

Since $s \ge 20$ and (by (3.2)) $nh \ge 3$, we have $16 < s^2 \log nh$ and

$$\frac{11s^2 \log nh}{(nh)^s} \le \frac{11s^2}{e(nh)^{s-1}} \le \frac{11s^2}{e^s} < \frac{1}{s}.$$

Altogether then, we obtain

$$\log \operatorname{index} L \le \sup_{s<i} \frac{\log \dim L_i}{i} \le \max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i} + \frac{1}{s}.$$

Next, for $i > s$, Lemma 3.3(iii) gives

$$\frac{\log \dim L_i}{i+1} \le \log \operatorname{index} L + \frac{(m+1)\log 2(m+1)}{s},$$

whence

$$\begin{aligned}\max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i} &\le \frac{s+1}{s} \max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i+1} \\ &\le \max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i+1} + \frac{1}{s}\Big[\log \operatorname{index} L + \frac{(m+1)\log 2(m+1)}{s}\Big].\end{aligned}$$

Since $\frac{1}{s^2} < \frac{1}{s}$, while $\frac{1}{s}\log \operatorname{index} L \le \frac{1}{s}\log(h+1)$ by Lemma 3.4(iv), the inequalities above reduce to

$$\log \operatorname{index} L \le \max_{s<i<(nh)^{2s}} \frac{\log \dim L_i}{i+1} + \frac{1}{s}\big[\log(h+1) + (m+1)\log 2(m+1) + 1\big].$$

Finally, since $s < (\log kr)/(2\log nh)$ it follows that $(nh)^{2s} < kr$. Since (*cf.* (3.2)) $nh \ge 3$, we have $e^{2s} < kr$. A little calculation shows that $x^2 < e^{2x}$ for $x \ge 0$, and so, since $s \ge k$ we have $sk \le s^2 < kr$; *i.e.*, $s < r$. On the other hand, we have chosen r so that $(\log kr)/(2\log(nh)) > \sigma$ and since s is the largest integer for which $(\log kr)/(2\log nh) > s$, we have $\sigma \le s < r$. Now we may use Lemma 3.8 to obtain

$$\begin{aligned}\max_{s<i<(nh)^{2s}} \frac{\log\dim L_i}{i+1} &\le \max_{s<i<kr} \frac{\log\dim L_i}{i+1}\\ &\le \max_{r\le i<kr} \frac{\log\dim L_i}{i+1} + \frac{(m+1)\log 2(m+1)}{s}.\end{aligned}$$

Indeed, if $\max_{s<i<kr}(\log\dim L_i)/i+1 = (\log\dim L_{i_0})/(i_0+1)$ with $r \le i_0 < kr$ then the inequality is trivially true. Otherwise,

$$\max_{s<i<kr} \frac{\log\dim L_i}{i+1} = \frac{\log\dim L_{i_0}}{i_0+1}$$

with $s < i_0 < r$. Then

$$\max_{s<i<kr} \frac{\log\dim L_i}{i+1} = \max_{s<i<(m+1)r} \frac{\log\dim L_i}{i+1},$$

and we apply Lemma 3.8. But because s is the greatest integer less than $\frac{\log kr}{2\log nh}$, it follows that

$$s \ge \frac{\log kr - 2\log nh}{2\log nh}.$$

On the other hand, by the choice of r, $2\log nh < \frac{\log kr}{k}$. Thus, because $k \ge 2$,

$$s \ge \frac{k-1}{k}\,\frac{\log kr}{2\log nh} \ge \frac{\log r}{4\log nh}.$$

Thus since $k \le s < r$, substitution in the inequality above yields the second inequality of the theorem. ∎

4 The Asymptotic Formula

In this section we again consider the homotopy Lie algebra L of a simply connected minimal Sullivan algebra $(\wedge V, d)$ that satisfies (H). In particular, we introduce the following additional notation:

$$\begin{aligned}\alpha_L &= \log \operatorname{index} L,\\ \lambda(n,h) &= 4\big(2n\log 2n + \log(h+1) + 1\big)\log nh.\end{aligned}$$

We have from Lemma 3.4(iv) that $0 < \alpha_L < \infty$.

The *depth* of L is the least integer, k, (or ∞) such that $\operatorname{Ext}^k_{UL}(\mathbb{Q}, UL) \ne 0$, and we recall from [6, Theorem 35.13] that depth $L \le m$, where, as set out in (H), $m = \operatorname{cat}(\wedge V, d)$.

Lemma 4.1 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra $(\wedge V, d)$ satisfying (H). There is then an infinite sequence $r_0 < r_1 < \cdots$ and a sub Lie algebra $E \subset L$ such that*

(i) *for $i \geq 0$, $2r_i \leq r_{i+1} < (m+1)r_i$;*
(ii) *E is generated in degrees r_i, $i \geq 0$, and for $i \geq 0$*

$$\frac{1}{2}\, e^{(\alpha_L - \frac{2\lambda(n,h)}{\log r_i})r_i} \leq \dim(E/[E,E])_{r_i} \leq e^{(\alpha_L - \frac{2\lambda(n,k)}{\log r_i})r_i}.$$

Proof We first use Theorem 3.9 to construct infinite sequences (r_i) satisfying (i) and such that

$$\frac{\log \dim L_{r_i}}{r_i} \geq \alpha_L - \frac{2\lambda(n,h)}{\log r_i}, \quad i \geq 0. \tag{4.1}$$

Indeed, let $r \geq n$ be any integer such that

$$\frac{(\log(m+1)r)}{(2\log nh)} > \max(20, m+1, \sigma),$$

where σ is the critical degree of $(\wedge V, d)$. Then choose $r_0 \in [r, (m+1)r)$ to maximize $(\log \dim L_j)/(j+1)$ for $r \leq j < (m+1)r$. Then Theorem 3.9 with $k = m+1$ yields

$$\frac{\log \dim L_{r_0}}{r_0} \geq \frac{\log \dim L_{r_0}}{r_0 + 1} \geq \alpha_L - \frac{\lambda(n,h)}{\log r} \geq \alpha_L - \frac{2\lambda(n,h)}{\log r_0}.$$

(The last inequality follows because $m \leq n/2$, and so $r_0 < (n/2+1)r$, and hence $\log r_0 < \log(n/2+1) + \log r < 2\log r$.)

Next suppose by induction that the r_j, $j \leq i$, have been constructed. Since $r_i > r_0$ we may apply Theorem 3.9 exactly as above to find $r_{i+1} \in [2r_i, (m+1)r_i)$ and such that (4.1) holds.

We complete the proof by applying Lemma 2.4, and for this we shall suppose that the initial r above was chosen so that

$$\frac{2(\log 2)\lambda(n,h)}{\log x \log 2x} \geq \frac{3\log 2x}{x}$$

for $x \geq r$. To obtain this lemma from Lemma 2.4 it is sufficient, in view of (4.1), to show that for $i \geq 0$,

$$2\lambda(n,h)\left[\frac{1}{\log r_i} - \frac{1}{\log r_{i+1}}\right] \geq 2\frac{\log(r_i+1)}{r_i} + \frac{\log 3}{r_{i+1}}.$$

Since $2r_i \leq r_{i+1} < (m+1)r_i$, it is enough to show that

$$\frac{2\log 2\lambda(n,h)}{\log r_i\ \log 2r_i} \geq \frac{3\log 2r_i}{r_i},$$

and this is exactly our additional hypothesis above on r. ■

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We will use the Hochschild–Serre spectral sequence in the next proposition. We refer the reader to [7] for the notation and properties of the spectral sequence, noting that since char $\Bbbk = 0$, $\wedge U = \Gamma U$, the free divided powers algebra on U, for any graded vector space $U = \{U_i\}_{i\geq 1}$. We denote by sU the graded vector space defined by $(sU)_i = U_{i-1}$. Then for each sub Lie algebra $E \subset L$, we write $I = [E, E]$, and observe that the proof of (X_4) in [7, §4] implies that

$$\mathrm{Ext}^*_{U(E/I)}\big(\mathrm{Tor}^{UI}_p(\Bbbk, \Gamma^q s(L/E)), U(E/I)\big) \neq 0 \tag{4.2}$$

for some $p + q = s \leq m$.

In fact, the Hochschild–Serre spectral sequence converging from

$$\mathrm{Ext}^i_{UE}\big(\wedge^q s(L/E), UL\big)$$

to $\mathrm{Ext}^{i+q}_{UL}(\Bbbk, UL)$ gives $\mathrm{Ext}^i_{UE}(\wedge^q s(L/E), UL) \neq 0$ for some $i + q \leq m$, because depth $L \leq m$. Since UL is UE-free, this implies that $\mathrm{Ext}^i_{UE}(\wedge^q s(L/E), UE) \neq 0$). Now the proof of [8, Lemma 4.2] applies verbatim to give $\mathrm{Ext}^i_{UE}(\wedge^q s(L/E), UE/I) \neq 0$.

On the other hand there is a Hochschild–Serre spectral sequence converging from $\mathrm{Ext}^{i-p}_{U(E/I)}(\mathrm{Tor}^{UI}_p(\Bbbk, \wedge^q s(L/E)), U(E/I))$ to $\mathrm{Ext}^i_{UE}(\wedge^q s(L/E), U(E/I))$. Formula (4.2) follows.

Finally [7, Proposition 1, §2] asserts that if F is an abelian Lie algebra, M is an F-module, and $\mathrm{Ext}_{UF}(M, UF) \neq 0$, then for some $x \in M$ and some r the map $U(F_{\geq r}) \to M$, $a \mapsto a \cdot x$ is injective. This, together with (4.2), shows that for any sub Lie algebra E of L, for some integer λ and some $\omega \in \mathrm{Tor}^{UI}_p(\Bbbk, \wedge^q s(L/E))$, the natural action of $U(E/I)$ satisfies

$$U\big((E/I)_{\geq\lambda}\big) \longrightarrow U\big((E/I)_{\geq\lambda}\big) \cdot \omega \text{ is injective.}$$

Proposition 4.2 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra satisfying (H). There is then an integer R such that for all $r \geq R$,*

$$\max_{r<i<r+\sqrt{r}-1} \frac{\log \dim L_i}{i} \geq \log \operatorname{index} L - \frac{9\lambda(n,h)}{\log r}.$$

Proof Let (r_i) and $E \subset L$ satisfy the conditions of Lemma 4.1, and write $[E, E] = I$. There is thus some integer λ and some $\omega \in \mathrm{Tor}^{UI}_p(\Bbbk, \wedge^q s(L/E))$ for which $p + q = s \leq m$ and

$$U((E/I)_{\geq\lambda}) \longrightarrow U((E/I)_{\geq\lambda}) \cdot \omega \text{ is injective.} \tag{4.3}$$

Write $W = sI \oplus sL/E$ and represent ω by a cycle $z \in \wedge^s(W_{\leq N})$, some N. Then choose $i_0 \geq 0$ so that $r_{i_0} > N$.

We will impose a number of conditions on R. To begin, we require that

$$\frac{\sqrt{R}}{(m+2)^2} > \lambda \quad \text{and} \quad \sqrt{R} > r_{i_0} + N.$$

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Now let $r \geq R$ be any integer. Then $r_{i_0} + N < \sqrt{r}$, and so there is a greatest integer i such that $r_i + N < \sqrt{r}$. Clearly then $r_i < \sqrt{r} < r$. Let $k \geq 2$ be the least integer for which $r_i k \geq r$. Since $(k-1)r_i < r$ we have $kr_i + N = (k-1)r_i + r_i + N < r + \sqrt{r} - 1$; *i.e.*,

$$[kr_i, kr_i + N] \subset [r, r + \sqrt{r} - 1).$$

Next, let $\{u_j\} \subset E_{r_i}$ represent a basis of $(E/I)_{r_i}$ and denote the adjoint representation of E in ΓW by " $\circ$ ". Further, for $\ell \geq 1$, denote by $A^\ell \subset UE$ the linear span of the elements $u_{j_1} \cdots u_{j_\ell}$ with $j_1 < \cdots < j_\ell$. Note that $A^{\ell+\ell'} \subset A^\ell \cdot A^{\ell'}$. Since i is the largest integer for which $r_i + N < \sqrt{r}$, we have $(m+2)r_i = (m+1)r_i + r_i \geq r_{i+1} + r_{i_0} > r_{i+1} + N \geq \sqrt{r}$. Thus,

$$r_i > \frac{\sqrt{r}}{m+2} \geq \frac{\sqrt{R}}{m+2} > \lambda.$$

It therefore follows from (4.3) that for any $\ell \geq 1$, $A^\ell \to A^\ell \circ z$ is injective.

Now recall that $z \in \Gamma^s(W_{\leq N})$, where $s \leq m$. Set $K = \dim \Gamma^{\leq m}(W_{\leq N})$. As in the proof of (X_4) in [7, §4], we have

$$A^\ell \circ z \subset \sum_{t=1}^{s} \sum_{\substack{\ell_1 + \cdots + \ell_t = \ell \\ \ell_1 \leq \cdots \leq \ell_t}} (A^{\ell_1} \circ W_{\leq N}) \cdots (A^{\ell_t} \circ W_{\leq N}) \Gamma^{s-t}(W_{\leq N}).$$

When $\ell = ks$, we have $\ell_t \geq \frac{ks}{t} \geq k$ (because $s \geq t$). Since $A^{\ell+\ell'} \subset A^\ell A^{\ell'}$, and since $\ell_t \geq k$, it follows that $A^{\ell_t} \circ W_{\leq N} \subset A^{\ell_t - k} \circ A^k \circ W_{\leq N}$. Since $s \leq m$, $\dim \Gamma^{s-t}(W_{\leq N}) \leq \dim \Gamma^{\leq m}(W_{\leq N}) = K$. Thus, since $A^{ks} \to A^{ks} \circ z$ is injective, we obtain

$$\dim A^{ks} \leq \sum_{t=1}^{s} \sum_{\substack{\ell_1 + \cdots + \ell_t = ks \\ \ell_1 \leq \cdots \leq \ell_t}} (K \dim A^{\ell_1}) \cdots (K \dim A^{\ell_{t-1}}) \dim A^{\ell_t - k} \dim(A^k \circ W_{\leq N}) K.$$

Next, write $d_i = \log \operatorname{index} L - \frac{2\lambda(n,h)}{\log r_i}$ and set

$$d = \max_{r < j < r + \sqrt{r} - 1} \frac{\log \dim L_j}{r}.$$

Now recall that A^k is the linear span of the elements $u_{j_1} \cdots u_{j_k}$ with $j_1 < \cdots < j_k$. Thus the elements of A^k have degree kr_i. Since $[kr_i + 1, kr_i + N] \subset (r, r + \sqrt{r} - 1)$, it follows that

$$\dim A^k \circ W_{\leq N} \leq \sum_{j = kr_i + 1}^{kr_i + N} \dim W_j \leq N \cdot \max_{r+1 \leq j \leq r + \sqrt{r} - 2} \dim L_j \leq N e^{dr}.$$

On the other hand (Lemma 4.1) $\dim (E/I)_{r_i} \leq e^{d_i r_i}$, and so for any ℓ, $\dim A^\ell \leq e^{d_i r_i \ell}$. Denote by ρ the number of partitions of ks. Then, because $t \leq s \leq m$, the inequalities above yield

$$\dim A^{ks} \leq \rho K^m e^{d_i r_i (s-1)k} N e^{dr}. \tag{4.4}$$

If $x \geq l$ are respectively a real number and a strictly positive integer, we write

$$\binom{x}{\ell} = \frac{x(x-1)\cdots(x-\ell+1)}{\ell!}$$

and note that $\binom{x}{\ell} \geq (\frac{x}{\ell})^{\ell}$. We now require the following further condition on R: for any $y \geq R$,

$$\Big(\log \operatorname{index} L - \frac{2\lambda(n,h)}{\log y}\Big)\, y > \log\big(4m(m+2)^2 y\big)\,.$$

Then, with this condition satisfied, we establish the inequality

$$\dim A^{ks} \geq \binom{\frac{1}{2}\, e^{d_i r_i}}{ks} \geq \Big(\frac{e^{d_i r_i}}{2ks}\Big)^{ks}. \tag{4.5}$$

In fact, since (Lemma 4.1) we have $\dim(E/I)_{r_i} \geq \frac{1}{2}e^{d_i r_i}$, it is sufficient to show that $e^{d_i r_i} \geq 2ks$. But since $(k-1)r_i < r$ and $r_i < r$, it follows that $kr_i < 2r$. Moreover, as observed earlier in the proof, $r_i > \frac{\sqrt{r}}{m+2}$. It follows that $k < 2(m+2)\sqrt{r}$, and so $2ks \leq 2km < 4m(m+2)\sqrt{r} < 4m(m+2)^2 r_i$. Since $r_i \geq R$, our hypothesis above on R implies that $d_i r_i > \log(4m(m+2)^2 r_i) > \log(2ks)$, *i.e.*, $e^{d_i r_i} > 2ks$. Thus (4.5) is established.

Now recall, as observed in the proof of Lemma 3.5, that $\rho < e^{3\sqrt{ks}}$. Recall also that $r \leq r_i k$ and that $s \leq m$. With these observations, combine the inequalities (4.4) and (4.5) and take logs to obtain

$$dr \geq d_i r - km\log(2km) - 3\sqrt{km} - m\log K - \log N.$$

It follows that for some c depending only on m, K, and N that

$$dr \geq d_i r - ck\log k.$$

But $k < 2(m+2)\sqrt{r}$, and so

$$d - d_i \geq -\frac{2c(m+2)}{\sqrt{r}}\log\big(2(m+2)\sqrt{r}\big)\,.$$

We now impose one final requirement on R, namely that for $y \geq R$,

$$\frac{2c(m+2)}{\sqrt{y}}\,\log\big(2(m+2)\sqrt{y}\big) < \frac{\lambda(n,h)}{\log y}.$$

Then, because $r \geq R$, we have

$$d \geq d_i - \frac{\lambda(n,h)}{\log r} = \log \operatorname{index} L - \frac{2\lambda(n,h)}{\log r_i} - \frac{\lambda(n,h)}{\log r}.$$

Finally, as observed earlier, $r_i > \frac{\sqrt{r}}{m+2}$ and so $\log r_i > \frac{1}{2}\log r - \log(m+2)$. But $r \geq R > (m+2)^4 j_0^2$ and so $\log(m+2) < \frac{1}{4}\log r$. Thus $\log r_i > \frac{1}{4}\log r$ and

$$d \geq \log \operatorname{index} L - \frac{9\lambda(n,h)}{\log r}. \qquad \blacksquare$$

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Theorem 4.3 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra satisfying (H). Then, given $\varepsilon > 0$, there is an infinite sequence of even integers $q_0 < q_1 < \cdots$ and a constant $C_0 \geq 0$ such that*

$$q_{i+1} - q_i < \sqrt{q_i} + C_0, \quad i \geq 0,$$

and

$$\frac{\log \dim L_{q_i}}{q_i} \geq \log \operatorname{index} L - \frac{(9+\varepsilon)\lambda(n,h)}{\log q_i}, \quad i \geq 0.$$

Proof As in the proof in [7, §4] that $(X_4) \Rightarrow (X_3)$, there are finitely many elements $y_1, \dots, y_r$ of odd degree in L such that for ℓ odd and sufficiently large there is some $\mu(\ell) \in [1, r]$ for which $\dim L_{\ell + \deg y_{\mu(\ell)}} \geq \frac{1}{r} \dim L$.

Next, recall from Proposition 4.2 that if an integer q is sufficiently large, then for some $\ell \in (q, q + \sqrt{q} - 1)$,

$$\frac{\log \dim L_\ell}{\ell} \geq \log \operatorname{index} L - \frac{9\lambda(n,h)}{\log q}.$$

Define k by

$$k = \begin{cases} \ell & \text{if } \ell \text{ is even,} \\ \ell + \deg y_{\mu(\ell)} & \text{if } \ell \text{ is odd,} \end{cases}$$

and write $C_0 = \max_{1 \leq i \leq r} \deg y_i$. Then k is even and $q < k < q + \sqrt{q} + C_0$, whether ℓ is even or odd.

In the case ℓ is odd we have

$$\begin{aligned} \text{(4.6)} \qquad \frac{\log \dim L_k}{k} &\geq \frac{\log \dim L_\ell}{k} - \frac{\log r}{k} \\ &= \frac{\log \dim L_\ell}{\ell} - \frac{1}{k}\Big(\deg y_{\mu(\ell)} \frac{\log \dim L_\ell}{\ell} + \log r\Big). \end{aligned}$$

It follows from Lemma 3.4(iv) that the set $\{\frac{\log \dim L_\ell}{\ell} \mid \ell \geq 1\}$ has a finite upper bound, and so for sufficiently large q,

$$\text{(4.7)} \qquad \frac{1}{k}\Big(\deg y_{\mu(\ell)} \frac{\log \dim L_\ell}{\ell} + \log r\Big) < \frac{\varepsilon/2\lambda(n,h)}{\log k}$$

Moreover, whether ℓ is even or odd we have $k \leq 2q$, and so

$$\frac{\log k}{\log q} \leq \frac{\log q + \log 2}{\log q} = 1 + \delta,$$

where for q sufficiently large $\delta < \varepsilon/2$. Then

$$\begin{aligned} \text{(4.8)} \qquad \frac{\log \dim L_\ell}{\ell} &\geq \log \operatorname{index} L - \frac{9\lambda(n,h)}{\log q} \\ &\geq \log \operatorname{index} L - \frac{(9 + \frac{\varepsilon}{2})\,\lambda(n,h)}{\log k}. \end{aligned}$$

Formula (4.8), combined with (4.6) and (4.7) when ℓ is odd, establishes the inequality

$$\frac{\log \dim L_k}{k} \geq \log \operatorname{index} L - \frac{(9+\varepsilon)\lambda(n,h)}{\log k},$$

always provided $q \geq Q$ for some fixed Q. For such q set $k = q_0$ and then iterate the construction to obtain the sequence (q_i). ∎

Proposition 4.4 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra satisfying (H). Then, given $\varepsilon > 0$, there is a constant $C > 0$ such that for $k \geq C$,*

$$\max_{k+2 \leq i \leq k+n} \frac{\log \dim L_i}{i} \geq \log \operatorname{index} L - \frac{(9+3\varepsilon)\lambda(n,h)}{\log(k+1)}.$$

Proof The principal step in the proof is to show that for some $d \geq n$,

$$\max_{\ell \leq i \leq \ell+d} \frac{\log \dim L_i}{i} \geq \log \operatorname{index} L - \frac{(9+2\varepsilon)\lambda(n,h)}{\log \ell}, \tag{4.9}$$

if ℓ is sufficiently large. We establish this by showing that the hypothesis that (4.9) fails leads to a contradiction, and we do so following the general idea of the proof of [7, §3, Thm. 2].

To begin, recall that we set $\alpha_L = \log \operatorname{index} L$, and remark that, by Lemma 3.4(i), since $\dim L = \infty$, we have $\alpha_L > 0$. Now simplify notation by writing

$$c = (9+2\varepsilon)\,\lambda(n,h).$$

Then if (4.5) is false, we may find an infinite sequence $0 < \ell_0 < \ell_1 < \cdots$ such that for $r \geq 0$,

$$\max_{\ell_r \leq i \leq \ell_r + r} \frac{\log \dim L_i}{i} < \alpha_L - \frac{c}{\log \ell_r}. \tag{4.10}$$

Moreover, by choosing a subsequence if necessary, we may also arrange that for $r \geq 0$,

$$\alpha_L - \frac{c}{\log \ell_r} > 0, \tag{4.11}$$

and for $r \geq 0$ and $x \geq \ell_{r+1}$,

$$\frac{c}{\log \ell_r} - \frac{6\log(\ell_r+1)}{\ell_r} > \frac{c}{\log x}. \tag{4.12}$$

For the last equation, remark that when $\ell_r > 5$, $\frac{c}{\log \ell_r} - \frac{6\log(\ell_r+1)}{\ell_r} > 0$, so we can find ℓ_{r+1} so that this quantity is bigger than $\frac{c}{\log \ell_{r+1}}$.

Recall next from Theorem 4.3 that there is a constant C_0 and an infinite sequence $2 \leq q_0 < q_1 < \cdots$ of even integers such that for $j \geq 0$,

$$\begin{cases} q_{j+1} < q_j + \sqrt{q_j} + C_0 \\ \dfrac{\log \dim L_{q_j}}{q_j} \geq \alpha_L - \dfrac{(9+\varepsilon)\,\lambda(n,h)}{\log q_j}. \end{cases} \tag{4.13}$$

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By beginning the sequence at a sufficiently large q_0 we may also arrange that for $j \geq 0$,

$$\alpha_L - \frac{(9+\varepsilon)\,\lambda(n,h)}{\log q_j} > 0 \quad \text{and} \quad q_j + \sqrt{q_j} + C_0 < 2q_j. \tag{4.14}$$

In particular, comparing (4.10) with (4.13) we find that for all $j, r \geq 0$,

$$q_j \notin [\ell_r, \ell_r + r]. \tag{4.15}$$

We now construct an infinite sequence (z_i) of elements of L and a strictly increasing infinite subsequence $(s_i) = (\ell_{r_i})$ of the sequence (ℓ_r) such that the following holds for $i \geq 0$:

(C1) $z_i \neq 0$, $\deg z_i$ is even, and $\deg z_i > \deg z_{i-1}$.

(C2) The sub Lie algebra, $E(i)$, generated by $z_0, \dots, z_i$ together with the subspaces $L_{[s_j, s_j + r_j]}$, $0 \leq j \leq i$, satisfies

$$\dim(UE(i))_j \leq e^{(\alpha_L - \frac{c}{\log s_{i+1}})j}, \quad j \geq 0.$$

(C3) The subspace $(UE(i-1))_+ \circ z_i$ is finite dimensional and concentrated in odd degrees, where "$\circ$" denotes the adjoint representation.

Remark In interpreting (C1)–(C3) for $i = 0$, we set $E(-1) = 0$, and $z_{-1} = 0 \in L_0$.

Now once the sequence $s_i = \ell_{r_i}$ and z_i are constructed, we will use the same ideas as in the proof of [7, Theorem 2, §3] to yield a contradiction and hence establish (4.9). Thus first we will recall how to obtain a contradiction from (4.9), and then we will construct these two sequences by induction on i.

Recall first that an L-module M is *weakly locally finite* if M is the increasing union of finite dimensional subspaces $M(1) \subset M(2) \subset \cdots$ such that $M(k)$ is preserved by $L_{\leq k}$. Then the *weak depth* of L, w-depth L, is the least k (or ∞) such that $\operatorname{Ext}^k_{UL}(M, UL) \neq 0$ for some weakly locally finite L-module, M.

Now set $E = \cup_r E(r)$. Since $(L/E)_i = 0$ for $i \in [s_j, s_j + r_j]$, L/E is a weakly locally finite L-module, and so, by [7, Lemma 5], w-depth $E \leq$ w-depth $L \leq m$.

The relation (C3) clearly implies that for any r, and any $i > 0$, $[E(r), z_{r+i}]$ is finite dimensional and concentrated in odd degrees.

Let w-depth $E = k$, and let $Z(r) \subset E(r)$ be the sub Lie algebra of elements that commute with each z_{r+i}, $0 < i \leq k$. Since $[E(r), z_{r+i}]$ is finite dimensional and concentrated in odd degrees it follows that $E(r)/Z(r)$ is also finite dimensional and concentrated in odd degrees. By [7, Lemma 8], the restriction morphism

$$\operatorname{Ext}_{UE(r)}(M, UE) \longrightarrow \operatorname{Ext}_{UZ(r)}(M, UE)$$

is injective for any E-module M.

On the other hand, since w-depth $E = k$, there is a weakly locally finite E-module M such that $\operatorname{Ext}^k_{UE}(M, UE) \neq 0$. Moreover, since $E = \cup_r E(r)$, for some r the restriction morphism $\operatorname{Ext}^k_{UE}(M, UE) \to \operatorname{Ext}^k_{UE(r)}(M, UE)$ is non-zero. Hence also the composite

$$\operatorname{Ext}^k_{UE}(M, UE) \longrightarrow \operatorname{Ext}^k_{UZ(r)}(M, UE)$$

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is non-zero.

Since each $[E(i), z_{i+1}]$ is concentrated in odd degrees, it follows that $[z_i, z_j] = 0$, $0 \le j < i < \infty$. Thus $Z = \oplus_{i=1}^{k} \Bbbk z_{r+i}$ is an abelian Lie algebra commuting with $Z(r)$. Because $Z(r) \to E$ factors as $Z(r) \to Z(r) + Z \to E$, the restriction map $\mathrm{Ext}^k_{UE}(M, UE) \to \mathrm{Ext}^k_{UZ(r)}(M, UE)$ factors through $\mathrm{Ext}^k_{U(Z(r)+Z)}(M, UE)$. Since UE is a free $U(Z(r) + Z)$-module, it follows that w-depth$(Z(r) + Z) \le k$. Since Z is an ideal in $Z(r) + Z$, there is a Hochschild-Serre spectral sequence converging from $\mathrm{Ext}^p_{U((Z(r)+Z)/Z)}(\Bbbk, \mathrm{Ext}^q_{U(Z)}(\,\cdot\,,\,\cdot\,))$ to $\mathrm{Ext}^{p+q}_{U(Z(r)+Z)}(\,\cdot\,,\,\cdot\,)$. It follows that w-depth $Z \le k$. But Z is finitely generated, so [7, Lemma 5] asserts that depth $Z \le k$. On the other hand, since Z is abelian and concentrated in even degrees, [6, Theorem 36.4] asserts that depth $Z = \dim Z = k + 1$, and we have the desired contradiction.

Now we proceed to the construction of the sequences s_i and z_i. To begin, we set $r_0 = 0$ so that $s_0 = \ell_0$, and we let z_0 be any non-zero element in some L_{q_j} with $q_j > \ell_0$, noting that (4.13) and (4.14) imply that this space is non-zero. Thus, because $E(-1) = 0$ and $z_{-1} = 0 \in L_0$, (C1) and (C3) are trivially satisfied for $i = 0$. Moreover, since $E(0)$ is generated by $L_{\ell_0} + \Bbbk z_0$ it follows from (4.10) and (4.11) that

$$\dim\Big(\frac{E(0)}{[E(0), E(0)]}\Big)_j \le e^{(\alpha_L - \frac{c}{\log \ell_0})j}, \quad j \ge 1.$$

Also, since $E(0)_j = 0$ for $j < \ell_0$, we have from Lemma 2.2 that

$$\dim UE(0)_j \le e^{(\alpha_L - \frac{c}{\log \ell_0} + \frac{\log(\ell_0+1)}{\ell_0})j}, \quad j \ge 1.$$

Now apply (4.12), noting that whatever the choice of s_1 we will have $s_1 \ge \ell_1$, to obtain that $E(0)$ satisfies (C2).

Next, suppose $s_0, \dots, s_p$ and $z_0, \dots, z_p$ have been selected so that (C1)–(C3) hold. Set

$$d(p) = 1 + \max(s_p + r_p, \deg z_p),$$

and choose q to satisfy the following three conditions:

(i) $q > d(p)$;

(ii) for $x \ge q$, $\frac{\log x}{\log(x+\sqrt{x}+C_0)} > \frac{(9+1.5\varepsilon)}{(9+2\varepsilon)}$;

(iii) for $x \ge q$, $\alpha_L \frac{2\sqrt{x}+2C_0}{x} + \frac{\log d(p)}{x} < \frac{.5\varepsilon}{q+2\varepsilon}\frac{c}{\log x}$.

Then choose r_{p+1} and $s_{p+1} = \ell_{r_{p+1}}$ so that for some $q_i > q$,

$$r_{p+1} > d(p) \quad \text{and} \quad s_{p+1} > q_i > q.$$

Because of (4.15), no q_j is in the interval $[s_{p+1}, s_{p+1} + r_{p+1}]$. It follows that for some (unique) $q_j > q$,

$$[s_{p+1}, s_{p+1} + r_{p+1}] \subset [q_j, q_{j+1}).$$

Now dualize the adjoint representation of $UE(p)$ in L to linear maps

$$\theta_i \colon L_{q_j} \to L_i \otimes (UE(p)_{i-q_j})^{\#},$$

where # denotes vector space dual. From (4.10) and (C2) we find that for $i \in [s_{p+1}, s_{p+1} + r_{p+1}]$,

$$\dim L_i \otimes (UE(p)_{i-q_j})^{\#} \leq e^{(\alpha_L - \frac{c}{\log s_{p+1}})i} e^{(\alpha_L - \frac{c}{\log s_{p+1}})i - q_j}.$$

From (4.13) we have $s_{p+1} + d(p) < s_{p+1} + r_{p+1} < q_{j+1} \leq q_j + \sqrt{q_j} + C_0$. Substitution in the inequality above yields

$$\dim \operatorname{Im} \theta_i \leq e^{(\alpha_L - \frac{c}{\log s_{p+1}})(q_j + 2\sqrt{q_j} + 2C_0)}.$$

for $s_{p+1} \leq i < s_{p+1} + d(p)$. Thus,

$$\frac{1}{q_j} \log\Bigl(\sum_{i=s_{p+1}}^{s_{p+1}+d(p)-1} \dim |IM\theta_i| \Bigr) \leq \alpha_L - \frac{c}{\log s_{p+1}} + \alpha_L \frac{2\sqrt{q_j} + 2C_0}{q_j} + \frac{\log d(p)}{q_j}.$$

But

$$\frac{c}{\log s_{p+1}} > \frac{c}{\log(q_j + \sqrt{q_j} + C_0)} = \frac{c}{\log q_j} \frac{\log q_j}{\log(q_j + \sqrt{q_j} + C_0)}.$$

Thus, since $q_j > q$, it follows from our conditions above on q that

$$\frac{1}{q_j} \log\Bigl(\sum_{i=s_{p+1}}^{s_{p+1}+d(p)-1} \dim \operatorname{Im} \theta_i \Bigr) \leq \alpha_L - \frac{(9+\varepsilon)\lambda(n,h)}{\log q_j} < \frac{\log \dim L_{q_j}}{q_j}.$$

This implies in turn that for some non-zero $z \in L_{q_j}$,

$$\theta_i z = 0, \quad s_{p+1} \leq i < s_{p+1} + d(p).$$

Equivalently, where, as usual, "$\circ$" denotes the adjoint representation,

$$UE(p)_i \circ z = 0 \quad \text{for } i \in \bigl[s_{p+1} - q_j, s_{p+1} - q_j + d(p)\bigr). \tag{4.16}$$

But since $d(p) > s_p + r_p$ and $d(p) > \deg z_p$, it follows from the definition of $E(p)$ that this sub Lie algebra is generated by elements of degree $< d(p)$. Thus from (4.16) we obtain

$$UE(p)_i \circ z = 0, \quad i \geq s_{p+1} - q_j;$$

i.e., $UE(p) \circ z$ is finite dimensional. Moreover, $z \in L_{q_j}$ is a non-zero element of even degree in $UE(p) \circ z$. Let z_{p+1} be a non-zero element of maximal even degree in $UE(p) \circ z$. Then $\deg z_{p+1} \geq \deg z = q_j > d(p) \geq \deg z_p$, and so (C1) holds for $i = p+1$. Obviously (C3) holds by the very choice of z_{p+1}. Finally, note that $E(p+1)$ is generated by $E(p)$ together with

$$W = L_{[s_{p+1}, s_{p+1}+r_{p+1}]} + \Bbbk z_{p+1}.$$

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In particular, it follows from (4.10) and (4.11) that for all $j \geq 1$, $\dim W_j \leq e^{(\alpha_L - \frac{c}{\log s_{p+1}})j}$.

Furthermore, since $\deg z_{p+1} \geq q_j$ and $s_{p+1} \geq q_j$, we have $W_i = 0$, $i < q_j$. Thus in view of (C2) for $i = p$ and Lemma 2.3(ii), we may conclude that for any $j \geq 1$,

$$\frac{\log \dim UE(p+1)_j}{j} \leq \alpha_L - \frac{c}{\log s_{p+1}} + \frac{3\log(q_j+1)}{q_j}.$$

Since $q_0 \geq 2$ and $\frac{\log(x+1)}{x}$ decreases as x increases for $x \geq 2$, we obtain from (4.14) that

$$\frac{3\log(q_j+1)}{q_j} = \frac{6\log(q_j+1)}{2q_j} \leq \frac{6\log(q_j+\sqrt{q_j}+C_0+1)}{q_j+\sqrt{q_j}+C_0} < \frac{6\log(s_{p+1}+1)}{s_{p+1}}.$$

Finally, apply (4.12) to obtain that for all $j \geq 0$ that

$$\frac{\log \dim UE(p+1)_j}{j} \leq \alpha_L - \Big(\frac{c}{s_{p+1}} - \frac{6\log(s_{p+1}+1)}{s_{p+1}}\Big) \leq \alpha_L - \frac{c}{\log x},$$

whenever $x \geq \ell_{r_{p+1}} + 1$. Since any choice of r_{p+2} will satisfy $r_{p+2} > r_{p+1} + 1$, we will have $s_{p+2} \geq \ell_{r_{p+1}+1}$, so (C2) follows for $i = p+1$. This completes the induction, and with it the proof of (4.9).

It remains to deduce the proposition from (4.9). Thus we suppose (4.9) holds for $\ell \geq C_1$, some $C_1 > 0$. Then, because $\dim L_k = \dim V^{k+1}$ (all $k \geq 1$) with $(\wedge V, d)$ a minimal Sullivan algebra satisfying $\dim H^i(\wedge V, d) \leq h$, all $i \geq 0$, and $H^i(\wedge V, d) = 0$, all $i > n$, we may apply [14, Corollary 7]. This asserts that for some $C_2 > 0$ and any integer $\ell \geq C_2$ there is an integer $i \in (\ell, \ell+n)$ for which

$$\dim L_i \geq \frac{1}{nh} \dim L_\ell.$$

It follows that if $k+n-d \geq C_2$, then for any $\ell \in [k+n-d, k+n]$ there is an integer $i \geq \ell$ such that $i \in [k+2, k+n]$ and

$$\dim L_i \geq \Big(\frac{1}{nh}\Big)^{d-n+2} \dim L_\ell.$$

Now suppose $k+n-d \geq C_2$ and $k+n-d \geq C_1$ and choose $\ell \in [k+n-d, k+n]$ so that

$$\frac{\log \dim L_\ell}{\ell} = \max_{k+n-d \leq j \leq k+n} \frac{\log \dim L_j}{j}.$$

Then because of (4.9), we will have for the $i \in [k+2, k+n]$ above that

$$\begin{aligned}\frac{\log \dim L_i}{\ell} &\geq \frac{\log \dim L_\ell}{\ell} - \frac{(d-n+2)\log nh}{\ell} \\ &\geq \alpha_L - \frac{(q+2\varepsilon)\,\lambda(n,h)}{\log(k+n-d)} - \frac{(d-n+2)\log nh}{\ell}.\end{aligned}$$

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Next, choose $C_3 > 0$ so that for $k \geq C_3$,

$$\begin{cases} \frac{\log(k+1)}{\log(k+n-d)} \leq \frac{9+2.5\varepsilon}{9+2\varepsilon}, \\ \frac{d\alpha_L+(d-n+2)\log nh}{k+2} \leq \frac{.5\varepsilon\,\lambda(n,h)}{\log(k+1)}. \end{cases} \tag{4.17}$$

Set $C = \max\{C_1, C_2, C_3\}$. Then for $k \geq C$ the $i \in [k+2, k+n]$ above satisfies

$$\begin{aligned} \frac{\log\dim L_i}{i} &\geq \Big(\alpha_L - \frac{(9+2\varepsilon)\,\lambda(n,h)}{\log(k+n-d)}\Big)\Big(1 - \frac{i-\ell}{i}\Big) - \frac{(d-n+2)\log nh}{i} \\ &\geq \alpha_L - \frac{d}{i}\,\alpha_L - \frac{(9+2\varepsilon)\,\lambda(n,h)}{\log(k+n-d)} - \frac{(d-n+2)\log nh}{i} \\ &= \alpha_L - \frac{(9+2\varepsilon)\,\lambda(n,h)}{\log(k+1)}\,\frac{\log(k+1)}{\log(k+n-d)} - \frac{(d-n+2)\log nh + d\alpha_L}{k+2}. \end{aligned}$$

In view of (4.17) this yields

$$\frac{\log\dim L_i}{i} \geq \alpha_L - \frac{(9+3\varepsilon)\,\lambda(n,h)}{\log(k+1)},$$

and since $i \in [k+2, k+n]$, the proof of Proposition 4.4 is complete. ∎

Next, recall the notation

$$\begin{aligned} \beta(n,h) &= 40(2n\log n + \log(h+1) + 1)\log nh, \\ \gamma(n,h) &= n\log(h+1) + 2n\log 2n \end{aligned}$$

from the introduction. In particular, $\beta(n,h) = 10\lambda(n,h)$.

Theorem 4.5 *Let L be the homotopy Lie algebra of a simply connected minimal Sullivan algebra that satisfies (H). Then the homotopy log index α_L satisfies $0 < \alpha_L < \infty$. Moreover, for some $K \geq 0$ and for all $k \geq K$,*

$$e^{(\alpha_L - \frac{\beta(n,h)}{\log(k+1)})(k+1)} \leq \max_{k+2\leq i\leq k+n} \dim L_i \leq e^{(\alpha_L + \frac{\gamma(n,h)}{k+1})(k+1)}.$$

Proof It follows from Lemmas 3.3(i)(ii), and 3.4(iv) that $0 < \alpha_L < \log(h+1)$. Moreover, the first inequality follows from Proposition 4.4, provided that $K \geq C$ and $3\varepsilon < 1$. On the other hand, the first inequality of Theorem 3.9 implies that for an appropriate K, and for all $k \geq K$,

$$\max_{k+2\leq i\leq k+n} \frac{\log\dim L_i}{k+n+1} \leq \max_{k+2\leq i\leq k+n} \frac{\log\dim L_i}{i+1} \leq \alpha_L + \frac{n\log 2n}{k+2}.$$

Therefore, for $i \in [k+2, k+n]$, we have $\dim L_i \leq e^{(\alpha_L + \frac{n\log 2n}{k+2})(k+n+1)}$. Since by Lemma 3.4(iv) $\alpha_L \leq \log(h+1)$, the second inequality of the theorem follows, provided that $k+2 \geq n$. ∎

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5 Topological Results

We begin by proving Theorem 1.2.

Lemma 5.1 *Let Y be a simply connected n-dimensional CW complex such that for some k, $\max_{k+2\le i\le k+n} \operatorname{rk}\pi_i(Y) < \infty$. Then $\operatorname{rk}\pi_{k+1}(Y) < \infty$.*

Proof We may assume that Y has a single zero cell and no 1-cells, and that the attaching maps for all the cells preserve base points. We shall assume $\operatorname{rk}\pi_{k+1}(Y) = \infty$ and deduce a contradiction. Indeed, given any subcomplex $i\colon W \subset Y$, consider based maps $f\colon S^{k+1} \to Y$ for which $f \vee i\colon S^{k+1} \vee W \to Y$ extends to a map $S^{k+1} \times W \to Y$, also denoted by f. This property depends only on the homotopy class $[f] \in \pi_{k+1}(Y)$, and the homotopy classes in $\pi_{k+1}(Y)$ with this property form a subgroup $G_{k+1}(W, Y)$ first introduced by Gottlieb in [11].

Now suppose $Z \supset W$ is a second subcomplex of Y obtained by the addition of a single cell: $Z = W \cup_g D^q$. Given a map $f\colon S^{k+1} \times W \to Y$, we let $\pi\colon D^{k+1} \to D^{k+1}/S^k = S^{k+1}$ and $\rho\colon S^k \times Z \to Z$ be the projections and form

$$f(\pi \times id) \cup \ell\rho\colon (D^{k+1} \times W) \cup (S^k \times Z) \longrightarrow Y,$$

where $\ell\colon Z \to Y$ is the inclusion. The obstruction $O(f)$ to extending this to a map $D^{k+1} \times Z \to Y$ is the obstruction to extending

$$\widetilde{f}\colon D^{k+1} \times S^{q-1} \cup (S^k \times D^q) \longrightarrow Y$$

to $D^{k+1} \times D^q$; *i.e.*, it is an element of $\pi_{k+q}(Y)$.

Lemma 5.2 *The correspondence $f \mapsto O(f)$ defines a linear map*

$$G_{k+1}(W, Y) \longrightarrow \pi_{k+q}(Y)$$

whose kernel is $G_{k+1}(Z, Y)$.

Proof Recall first the classical definition of the addition in $\pi_{k+q}(Y)$. Denote by H the hyperplane $x_1 = 0$ in $\mathbb{R}^{k+q+1}$. Then the pinch map $\nabla\colon S^{k+q} \to S^{k+q} \vee S^{k+q}$ is the quotient by the subspace $S^{k+q} \cap H$, and the sum of two elements h_1 and h_2 is the composition

$$S^{k+q} \xrightarrow{\nabla} S^{k+q} \vee S^{k+q} \xrightarrow{h_1 \vee h_2} Y.$$

The map ∇ decomposes as follows

$$\begin{aligned}(D^{k+1} \times S^{q-1}) \cup (S^k \times D^q) \to \big[(D_+^{k+1} \vee D_-^{k+1}) \times S^{q-i}\big] \cup \big[(S_+^k \vee S_-^k) \times D^q\big] = \\ \big[(D_+^{k+1} \times S^{q-1}) \cup (S_+^k \times D^q)\big] \vee \big[(D_-^{k+1} \times S^{q-1}) \cup (S_-^k \times D^q)\big],\end{aligned}$$

where D_+^{k+1} denotes the quotient of $\{(x_1, \dots, x_{k+1}) \in D^{k+1} \mid x_1 \ge 0\}$ by the subspace $D^{k+1} \cap H$. We use similar definitions for D_-^{k+1}, S_+^k and S_-^k. Clearly $S_-^k \cong S_+^k \cong S^k$ and $D_-^{k+1} \cong D_+^{k+1} \cong D^{k+1}$.

Now let $f, g \in G_{k+1}(W, Y)$. The extension of $f+g$ to $S^{k+1} \times W$ is the composition

$$(S^{k+1} \times W) \xrightarrow{\nabla \times id} (S^{k+1}_+ \vee S^{k+1}_-) \times W = (S^{k+1}_+ \times W) \cup_{(\{*\} \times W)} (S^{k+1}_- \times W) \xrightarrow{f \cup g} Y.$$

Since the projection $\rho\colon S^k \times D^q \to D^q$ is the composition

$$S^k \times D^q \overset{\nabla \times id}{\to} (S^k_+ \vee S^k_-) \times D^q = (S^k_+ \times D^q) \cup_{(\{*\} \times D^q)} (S^k_- \times D^q) \overset{\rho \cup \rho}{\to} D^q,$$

the maps $\widetilde{f+g}$ and $\widetilde{f} + \widetilde{g}$ coincide. ∎

Since $2 \leq q \leq n$, $\pi_{k+q}(Y) \otimes \mathbb{Q}$ is finite dimensional, it follows that the image of $G_{k+1}(Z, Y) \otimes \mathbb{Q}$ has finite codimension in $G_{k+1}(W, Y) \otimes \mathbb{Q}$.

A trivial induction now shows that for finite subcomplexes $W \subset Z$ of Y the image of $G_{k+1}(Z, Y) \otimes \mathbb{Q}$ has finite codimention in $G_{k+1}(W, Y) \otimes \mathbb{Q}$.

Next, given a based map $g\colon S^p \to Y$, we denote by $[g] \otimes 1$ the corresponding element of $\pi_p(Y) \otimes \mathbb{Q}$. Similarly if $\varphi\colon W \to Y$ is the inclusion of a subcomplex we denote by $\pi_p(\varphi) \otimes \mathbb{Q}$ the induced maps between the rationalized homotopy groups. We shall construct a sequence $pt = W(0) \subset W(1) \subset \cdots$ of finite subcomplexes of Y together with maps $f(i)\colon S^{k+1} \times W(i) \to W(i+1)$ such that

$$\left[f(i)_{| S^{k+1} \times pt} \right] \otimes 1 \notin \operatorname{Im} \pi_{k+1} \big(f(i)_{| pt \times W(i)} bigr) \otimes \mathbb{Q}.$$

Indeed, first note that $G_{k+1}(pt, Y) \otimes \mathbb{Q} = \pi_{k+1}(Y) \otimes \mathbb{Q}$ is supposed infinite dimensional, so we may choose for $f(0)$ any representative of a non-zero element of $\pi_{k+1}(Y) \otimes \mathbb{Q}$. Then if $W(0) \subset \cdots \subset W(\ell)$ and $f(0), \ldots, f(\ell)$ are constructed, we let $W(\ell+1)$ be any finite subcomplex of Y that contains $IM f(\ell)$. Then $\pi_{k+1}(W(\ell+1)) \otimes \mathbb{Q}$ is finite dimensional and so has finite dimensional image in $\pi_{k+1}(Y) \otimes \mathbb{Q}$. Since the image of $G_{k+1}(W(\ell+1), Y) \otimes \mathbb{Q}$ has finite codimension in the infinite dimensional space $G_{k+1}(pt, Y) = \pi_{k+1}(Y) \otimes \mathbb{Q}$, we may find $f(\ell+1)\colon S^{k+1} \times W(\ell+1) \to Y$ such that

$$\left[f(\ell+1)_{| S^{k+1} \times pt} \right] \otimes 1 \notin \operatorname{Im} \left[\pi_{k+1}(W(\ell+1)) \otimes \mathbb{Q}) \to \pi_{k+1}(Y) \otimes \mathbb{Q} \right].$$

In particular, for each $\ell \geq 1$ we have the maps

$$\varphi(\ell)\colon \underbrace{S^{k+1} \times \cdots \times S^{k+1}}_{\ell} \to \underbrace{S^{k+1} \times \cdots \times S^{k+1}}_{\ell-1} \times W(1) \to \underbrace{S^{k+1} \times \cdots \times S^{k+1}}_{\ell-2} \times W(2)$$
$$\to \cdots \to W(\ell+1)$$

with each $\pi_{k+1}(\varphi(\ell)) \otimes \mathbb{Q}$ injective.

Finally, let $F \to W(\ell+1) \to B$ be the Postnikov fibration in which $\pi_i(B) = 0$, $i \geq k+1$, and $\pi_i(F) = 0$, $i < k+1$. Then $\varphi(\ell)$ is homotopic to a map $\psi(\ell)\colon S^{k+1} \times \cdots \times S^{k+1} \to F$. The restriction of $\psi(\ell)$ to the $(k+1)$-spheres are linearly independent elements of $\pi_{k+1}(F) \otimes \mathbb{Q} = H_{k+1}(F; \mathbb{Q})$. Label these spheres $S^{k+1}_1, \ldots, S^{k+1}_\ell$. Then there are cohomology classes $\gamma_1, \ldots, \gamma_\ell \in H^{k+1}(F; \mathbb{Q})$ such that

$$\left\langle H^{k+1}(\psi(\ell))\gamma_i, [S^{k+1}_j] \right\rangle = \left\langle \gamma_i, [\psi(\ell)_{| S^{k+1}_j}] \right\rangle = \delta_{ij}.$$

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(Here $\langle \cdot\,,\,\cdot \rangle$ denotes the pairing between cohomology and homology.) It follows that

$$H^{\ell(k+1)}(\psi(\ell)) \prod_{i=1}^{\ell} \gamma_i = \prod_{i=1}^{\ell} H^{k+1}(\psi(\ell))\gamma_i \neq 0,$$

and thus $\gamma_1 \cdots \gamma_\ell \neq 0$.

But since $\pi_*(F)\otimes\mathbb{Q} \to \pi_*(W(\ell+1))\otimes\mathbb{Q}$ is injective, it follows from the Mapping Theorem [4] that the rational category, cat_0, satisfies $\mathrm{cat}_0 F \leq \mathrm{cat}_0(W(\ell+1)) \leq \dim W(\ell+1) \leq n$. In particular the product of $n+1$ classes in $H^{k+1}(F;\mathbb{Q})$ vanishes, which contradicts the above assertion for $\ell = n+1$. ■

Proof of Theorem 1.2 Let $\widetilde{X}$ be the universal cover of the n-dimensional CW complex X. If for some k, $\max_{k+2\leq i\leq k+n} \mathrm{rk}\,\pi_i(X) < \infty$, then Lemma 5.1 applied to $Y = \widetilde{X}$, also asserts that $\mathrm{rk}\,\pi_{k+1}(X) < \infty$. It follows that

$$\mathrm{rk}\,\pi_i(X) < \infty, \quad 2 \leq i \leq k+n.$$

Hence (*cf.* (1.1)) $H_i(\widetilde{X};\mathbb{Q})$ is finite dimensional for $i \leq k+n$, and so (since $\widetilde{X}$ is n-dimensional), $H_*(\widetilde{X};\mathbb{Q})$ is finite dimensional. This implies in turn that $\mathrm{rk}\,\pi_i(X)$ is finite for all i and thus that X is not π-rank infinite. ■

Proof of Theorems 1.1 and 1.6 Theorems 1.1 and 1.6 deal with the ranks of a rationally hyperbolic n-dimensional connected CW complex X. Thus in proving these theorems we may replace X by its universal cover; *i.e.,* we may restrict to the case where X is simply connected. Since $\mathrm{rk}\,\pi_i(X) < \infty$ for $i \geq 2$, the Betti numbers of X are finite. We can thus suppose that X is a finite complex.

In this case we let $(\wedge V, d)$ be the minimal Sullivan model of X ([6]). Then $H(\wedge V, d) \cong H^*(X;\mathbb{Q})$ and $V \cong \pi_*(X)\otimes\mathbb{Q}$. It follows that $(\wedge V, d)$ satisfies (H) with $n = \dim X$ and $h = \max_i \dim H_i(X;\mathbb{Q})$. Moreover the homotopy log index, α_X, satisfies

$$\alpha_X = \limsup_k \frac{\log\ dim V^k}{k} = \limsup_k \frac{\log \dim L_k}{k},$$

where $L = \{L_k\}$ is the homotopy Lie algebra of $(\wedge V, d)$. In particular, in the terminology of Sections 3 and 4, $\alpha_X = \log \mathrm{index}\, L = \alpha_L$.

With these translations the first assertion of Theorem 1.1 coincides with Theorem 4.5, while Theorem 1.6 coincides with the corollary to Theorem 3.9. It remains to prove the second assertion of Theorem 1.1. In fact, the right-hand inequality in the first assertion of Theorem 1.1 implies that for some fixed $c > 0$, $\mathrm{rk}\,\pi_i(X) \leq c e^{\alpha_X i}$ for all $i \geq 2$. Now suppose for some k that $\mathrm{rk}\,\pi_k(X) > \max\{1, \frac{(2n)^n}{e^{\alpha_X}}\} \cdot e^{\alpha_X k}$. Then $\dim V^k > (2n)^n \geq [2(m+1)]^{m+1}$, where $m = \mathrm{cat}(\wedge V, d)$. Thus, as in Lemma 3.3, k extends to an infinite sequence $k = k_0 < k_1 < \cdots$ such that for each $i \geq 1$, $k_i = \ell_i k_{i-1} - 1$, some $2 \leq \ell_i \leq m+1$, and also

$$\dim V^{k_i} \geq \Big[\frac{1}{2(m+1)}\Big]^{m+1} \big(\dim V^{k_{i-1}}\big)^{\ell_i}.$$

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Now write $\dim V^{k_i} = \lambda_i e^{\alpha_X k_i}$, $i \geq 0$. Then with $i = 1$ this inequality gives

$$\lambda_1 e^{\alpha_X(\ell_1 k_0 - 1)} \geq \left[\frac{1}{2(m+1)}\right]^{m+1} \left[\lambda_0 e^{\alpha_X k_0}\right]^{\ell_1}.$$

Thus

$$\lambda_1 \geq \left[\frac{1}{2(m+1)}\right]^{m+1} e^{\alpha_X} \lambda_0^{\ell_1}.$$

Since $n \geq m+1$ and since by hypothesis $\lambda_0 = \sigma(2n)^n e^{-\alpha_X}$ for some $\sigma > 1$, we may write $\lambda_1 = \sigma \lambda_0^{\ell_i - 1}$. But we also supposed that $\lambda_0 \geq 1$, and thus $\lambda_1 \geq \sigma\lambda_0$. In particular $\lambda_1 > \sigma \max\{1, (2n)^n e^{-\alpha_X}\}$. We may now iterate to obtain $\lambda_{i+1} > \sigma^i \max\{1, (2n)^n e^{-\alpha_X}\}$. Since $\sigma > 1$, this gives $\lambda_i \to \infty$ in contradiction with our earlier observation that $\lambda_i \leq c$ for all i. ■

Proof of Theorem 1.7 We are given a number $\alpha \in (0, \infty)$ and a sequence $\delta_k \to 0$ of nonnegative numbers, and we have to construct a rationally hyperbolic simply connected wedge of spheres X such that $\alpha_X = \alpha$ and such that for any $c, d > 0$ there is an infinite sequence k_ℓ for which

$$\max_{k_\ell \leq i \leq k_\ell + d} \frac{\operatorname{rk} \pi_i(X)}{k_\ell} < \alpha - c\delta_{k_\ell}.$$

First recall from [6] that if Y is any finite wedge of spheres of dimension ≥ 2, then the Poincaré series $\Omega Y(z) = \sum_{i=0}^{\infty} \dim H_i(\Omega Y; \mathbb{Q}) z^i$ is given by

$$\Omega Y(z) = \frac{1}{1 - f_Y(z)},$$

where $f_Y(z) = \sum_{j \geq 1} \dim H_{j+1}(Y; \mathbb{Z}) z^j$. Thus the log index α_Y is the unique number such that $1 - f_Y(e^{-\alpha_Y}) = 0$.

We construct X as the union of an increasing sequence of spaces $X(\ell)$ such that

$$X(1) = \bigvee_{j=1}^{r_1} S_j^{q_1+1} \quad \text{and} \quad X(\ell+1) = X(\ell) \vee \bigvee_{j=1}^{r_{\ell+1}} S^{q_{\ell+1}+1}.$$

Here (r_ℓ) and (q_ℓ) are sequences to be determined, and the $S_j^{q_\ell+1}$ are all copies of the $(q_\ell + 1)$-sphere.

To begin, set $r_1 = 2$ and choose q_1 so large that $1 - 2e^{-\alpha q_1} > 0$. Then $\alpha_{X(1)} < \alpha$. Next, suppose $X(\ell)$ has been constructed with $\alpha_{X(\ell)} < \alpha$ and, for simplicity, denote $X(\ell)$ by Y: $\alpha_Y < \alpha$. Suppose then that $Z = Y \vee \bigvee_{j=1}^{r} S_j^{q+1}$ for some q and r and define $f_Y(z)$ and $\Omega Y(z)$ as above. Then α_Z is determined by the equation

$$1 - f_Y(e^{-\alpha_Z}) - re^{-\alpha_Z q} = 0.$$

Next, choose an integer p_ℓ so that the following conditions hold:

(i) $\dfrac{\log \operatorname{rk} \pi_j(Y)}{j} < \alpha_Y + \dfrac{p_\ell}{p_\ell + \ell} \dfrac{\alpha - \alpha_Y}{3}$ for $j \geq p_\ell$;

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(ii) $\ell\delta_j < \dfrac{\alpha - \alpha_Y}{3}$ for $j \geq p_\ell$;

(iii) $\dfrac{\ell}{p_\ell}\alpha_Y < \dfrac{\alpha - \alpha_Y}{3}$.

Then for $j \in [p\ell, p_\ell + \ell]$ we have

$$\frac{\log \operatorname{rk} \pi_j(Y)}{p_\ell} < \alpha - \ell\delta_j. \tag{5.1}$$

Indeed,

$$\begin{aligned}\frac{\log \operatorname{rk} \pi_j(Y)}{p_\ell} &< \frac{p_\ell + \ell}{\ell} \frac{\log \operatorname{rk} \pi_j(Y)}{j} < \frac{p_\ell + \ell}{\ell}\alpha_Y + \frac{\alpha - \alpha_Y}{3} \\ &< \alpha_Y + \frac{\alpha - \alpha_Y}{3} + \frac{\alpha - \alpha_Y}{3} = \alpha - \frac{1}{3}(\alpha - \alpha_Y) \\ &< \alpha - \ell\delta_j.\end{aligned}$$

Now, given q, choose $r = r(q)$ to be the least integer satisfying $r > e^{\frac{1}{2}(\alpha+\alpha_Y)q}$. The log index of $\vee_{j=1}^r S_j^{q+1}$ is then greater than $\frac{1}{2}(\alpha + \alpha_Y)$ and so, trivially,

$$\alpha_Z > \frac{1}{2}(\alpha + \alpha_Y). \tag{5.2}$$

Moreover, we have, for sufficiently large q, that

$$\alpha_Z < \alpha. \tag{5.3}$$

Indeed, since $r - 1 \leq e^{\frac{1}{2}(\alpha+\alpha_Y)}$, it follows that

$$1 - f_Z(e^{-\alpha}) = 1 - f_Y(e^{-\alpha}) - (r-1)e^{-\alpha q} - e^{-\alpha q} \geq 1 - f_Y(e^{-\alpha}) - e^{-\frac{1}{2}(\alpha-\alpha_Y)q} - e^{-\alpha q}.$$

Thus for q sufficiently large, $1 - f_Z(e^{-\alpha}) > 0$, and (5.3) follows.

Now choose $q_{\ell+1}$ so that (5.3) holds for $q = q_{\ell+1}$ and also so that $q_{\ell+1} > p_\ell + \ell$. Then set $r_{\ell+1} = r(q_{\ell+1})$; thus $X(\ell+1) = Z$ and by (5.3), $\alpha_{X(\ell+1)} < \alpha$. We may thus iterate the construction to produce the infinite sequence $X(1) \subset X(2) \subset \cdots$ and we set $X = \cup_\ell X(\ell)$.

To complete the proof we observe first that (5.2) implies that

$$\alpha_{X(\ell+1)} > \alpha - \frac{1}{2}(\alpha - \alpha_{X(\ell)}), \quad \ell \geq 1.$$

Since $\alpha_X \geq \alpha_{X(\ell)}$ for all ℓ, it follows that $\alpha_X \geq \alpha$. Moreover, by construction there is some fixed λ such that $\dim H_{i+1}(X;\mathbb{Q}) \leq \lambda^i$ for all $i \geq 0$. As stated in [7, Theorem 4] this implies that $\alpha_X < \infty$ and, for some integer s,

$$\max_{k \leq i \leq k+s} \frac{\log \operatorname{rk} \pi_i(X)}{k} \to \alpha_X \quad \text{as } k \to \infty.$$

In particular this holds for the subsequence $k_\ell = p_\ell + \ell - s$:

$$\max_{k_\ell \le i \le k_\ell + s} \frac{\log \operatorname{rk} \pi_i(X)}{k_\ell} \to \alpha_X. \tag{5.4}$$

Since for $\ell > s$ we have $p_\ell < k_\ell \le k_\ell + s = p_\ell + \ell$, it follows from (5.1) that the limit in (5.4) is at most α; *i.e.,* $\alpha_X \le \alpha$. Thus $\alpha_X = \alpha$.

Finally let $c > 0$ and an integer $d > 0$ both be arbitrary, and choose any integer $\ell > \max(c, d)$. Then $\pi_i(X) = \pi_i(X(\ell))$ for $i \le q_{\ell+1}$, and so since $q_{\ell+1} > p_\ell + \ell$, we have from (5.1) that

$$\begin{aligned} \max_{p_\ell+\ell-d \le i \le p_\ell+\ell} \frac{\operatorname{rk} \pi_j(X)}{p_\ell + \ell - d} &\le \max_{p_\ell+\ell-d \le i \le p_\ell+\ell} \frac{\operatorname{rk} \pi_j(X(\ell))}{p_\ell} \\ &< \alpha_X - \ell\delta_{p_\ell+\ell-d} < \alpha_X - c\delta_{p_\ell+\ell-d}. \end{aligned}$$

Now the sequence $k_\ell = p_\ell + \ell - d$ goes to ∞ with ℓ, and we have

$$\max_{k_\ell \le i \le k_\ell + d} \frac{\operatorname{rk} \pi_j(X)}{k_\ell} < \alpha - c\delta_{k_\ell}. \qquad \blacksquare$$

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