A NOTE ON RIGHT INVARIANT INTEGRALS ON LOCALLY COMPACT SEMIGROUPS

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An integral on a locally compact Hausdorff semigroup S is a non-trivial, positive linear functional μ on the space $K(S)$ of real-valued continuous functions on S with compact support. If S has the property:

$$(P) \qquad As^{-1} = \{x | \, x \in S \text{ and } xs \in A\}$$

is compact whenever A is compact subset of S and $s \in S$, then the function f_a defined by $f_a(x) = f(xa)$ is in $K(S)$ whenever $f \in K(S)$ and $a \in S$. An integral on a locally compact semigroup S with the property (P) is said to be *right invariant* if $\mu(f_a) = \mu(f)$ for all $f \in K(S)$ and $a \in S$.

In [1] Michael has shown that if S is a separable, metric locally compact semigroup with the properties:

(A) for each pair of compact sets C, D of S, the set

$$\begin{aligned} CD^{-1} &= \{x | \, x \in S \text{ and } xy \in C \text{ for some } y \in D\} \\ &= \bigcup_{y \in D} Cy^{-1} \end{aligned}$$

is compact;

(B) for each pair of compact sets C, D of S, the set

$$\begin{aligned} D^{-1}C &= \{x | \, x \in S \text{ and } yx \in C \text{ for some } y \in D\} \\ &= \bigcup_{y \in D} y^{-1}C \end{aligned}$$

is compact;

(C) S contains exactly one minimal left ideal;

then S admits a right invariant integral.

In this note our aim is to prove the following:

THEOREM 1. *If there exists a right invariant integral μ on a locally compact semigroup S with the properties (P) and (B) then $\int d\mu$ is always infinite unless S is compact in which case $\int d\mu$ is always finite.*

PROOF. Suppose S is not compact and $\int d\mu < \infty$. We can select $f \in K(S)$

such that $\int f d\mu > 0$ and $f \geqq 0$. Without loss of generality we can assume that $f \leqq 1$. It suffices to determine a sequence $\{s_n\}$ in S such that the supports of the functions f_{s_n} are disjoint. Clearly then, it will follow that $\sum_{i=1}^{n} f_{s_i} \leqq 1$ whence $n \int f d\mu \leqq \int d\mu < \infty$ for each n. This leads to a contradiction. Let us denote the support of f by A. Then the supports of f_{s_n} will be contained in As_n^{-1}. Choose $s_1 \in S$ and assume that $s_1, s_2, \cdots, s_n$ have been constructed. Then in order that As_{n+1}^{-1} be disjoint from As_i^{-1} $(i = 1, 2, \cdots, n)$, we must have s_{n+1} not belonging to $\bigcup_{i=1}^{n} (As_i^{-1})^{-1}A$. Since S is not compact, such an s_{n+1} exists. This completes the proof.

For the next theorem, we need the following result of Rosen [3].

THEOREM 2. *There exists a right invariant integral on a compact semigroup S if and only if S contains exactly one minimal left ideal.*

Theorem 1 in conjunction with Theorem 2 yields

THEOREM 3. *In order that on a locally compact semigroup S with the properties (P) and (B) there exist a right invariant integral μ with $\int d\mu < \infty$, it is necessary and sufficient that S be compact and contain exactly one minimal left ideal.*

REMARK. We observe that conditions (P) and (B) of Theorem 1 cannot entirely be dropped. This we show by quoting an example, due to E. Granirer ([5], page 58), of a locally compact semigroup in which conditions (P) and (B) do not hold but there exists a right invariant integral with respect to which the semigroup has finite measure.

Let $Z_n = \{e_1, e_2, \cdots, e_n\}$ with the relations $e_i \cdot e_j = e_i$, and let A be a finite group. Then $A \times Z_n = \{(a, e_i) | \, a \in A \text{ and } e_i \in Z_n\}$ becomes a semigroup under the multiplication $(a, e_i) \cdot (b, e_j) = (ab, e_i)$. One easily sees that the set $A_i = \{(a, e_i) \,|\, a \in A\}$ is a finite group isomorphic to A. Let G_0 be an infinite group. In $G = G_0 \cup (A \times Z_n)$ we define a multiplication $*$ which renders G a semigroup structure as follows: if g', g'' are both in G_0 (or both in $A \times Z_n$) then $g' * g''$ means multiplication in G_0 (or in $A \times Z_n$). If $g' \in G_0$ and $g'' \in A \times Z_n$, define $g' * g'' = g'' * g' = g''$. It is shown in [5] that $A_1, A_2, \cdots, A_n$ are the only subsets of G, which simultaneously are finite groups, are right ideals and satisfy the right cancellation law.

Now G has exactly n finite disjoint groups which are right ideals with right cancellation. As in Granirer [4, lemma 3.1], one can prove that $I = \bigcup_{i=1}^{n} A_i$ is a finite left ideal in G. I is also a right ideal. Thus G contains a finite ideal I.

G endowed with the discrete topology becomes a locally compact semigroup and I a compact ideal in G. For $s \in I$,

$$Is^{-1} = \{x \in G \,|xs \in I\} = G \text{ and } s^{-1}I = \{x \in G \,|sx \in I\} = G.$$

Thus (P) and (B) do not hold in G. Define $\mu_i(f) = \sum_{a \in A_i} f(a)$ for $f \in K(G)$. One can easily see that μ_i is a right invariant integral on G for each i. Also the measure of G with respect to μ_i is N for each i, where N is the number of elements of A. Thus we have shown that G admits a right invariant integral with respect to which G has finite measure.

Incidentally, this gives an example of a locally compact semigroup admitting at least 'n' different invariant integrals.

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References

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