

# Modeling gravitational few-body problems with tsunami and okinami

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**Abstract.** In recent years, an increasing amount of attention is being paid to the gravitational few-body problem and its applications to astrophysical scenarios. Among the main reasons for this renewed interest there is large number of newly discovered exoplanets and the detection of gravitational waves. Here, we present two numerical codes to model three- and few-body systems, called TSUNAMI and OKINAMI. The TSUNAMI code is a direct few-body code with algorithmic regularization, tidal forces and post-Newtonian corrections. OKINAMI is a secular, double-averaged code for stable hierarchical triples. We describe the main methods implemented in our codes, and review our recent results and applications to gravitational-wave astronomy, planetary science and statistical escape theories.

**Keywords.** stars: kinematics and dynamics, methods: numerical, gravitational waves, gravitation

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## 1. Introduction

The gravitational three-body problem has a 300 year old history, dating back to Newton, Poincaré and many others. Rather than just being a didactic tool for the mathematical physicist, the three-body problem has numerous applications to modern astrophysical conundrums. Thanks to the recent advancement in observational astronomy, the three-body problem (and more generally, the few-body problem) is experiencing a renewed interest. Such interest is driven by the detection of gravitational waves in 2015, and the subsequent birth of gravitational-wave astronomy ([The LIGO Scientific Collaboration et al. 2021](#)). In fact, three-body interactions between compact objects have been proposed as one of the key formation mechanisms of gravitational-wave sources.

Another area of interest for three-body problems is exoplanet formation and evolution. This was made possible thanks to the rapid increase in exoplanet detections from transit surveys (K2, TESS, [Howell et al. 2014](#); [Ricker et al. 2015](#)), and the characterization of numerous exotic planetary systems (i.e. hot Jupiters, ultra-short period planets, compact resonant chains). The formation of such exotic systems can be explained with gravitational few-body interactions between planets or passing stars. In addition, the recent

reports on exomoon candidates have opened up questions on how extrasolar moons form and evolve.

Modeling few-body gravitational interactions is not an easy task. One issue arises from the nature of the gravitational force, which scales as  $\propto r^{-2}$ , where  $r$  is the separation between two particles. When two particles get very close,  $r \rightarrow 0$  and the acceleration increases dramatically. Using traditional integrators, like the Runge-Kutta or Hermite methods, as the acceleration increases, the timestep needs to be reduced accordingly, in order to time-resolve the trajectory of the particles with sufficient accuracy. This can possibly lead to the halt of the integration, or to the faster accumulation of integration errors due to the increased number of timesteps.

We have developed two codes, named TSUNAMI and OKINAMI that employ different techniques in order to accurately model few-body gravitational interactions. Here, we describe the main numerical methods that we implemented, along with their applications to astrophysical scenarios.

## 2. Overview of the codes

TSUNAMI and OKINAMI implement different methods and therefore have slightly different scopes. The main difference is that while TSUNAMI can simulate systems of hundreds of particles in arbitrary configurations, OKINAMI can model only hierarchical stable triples. Both codes can be interfaced through a dedicated Python library and come with several example scripts.

### 2.1. The TSUNAMI code

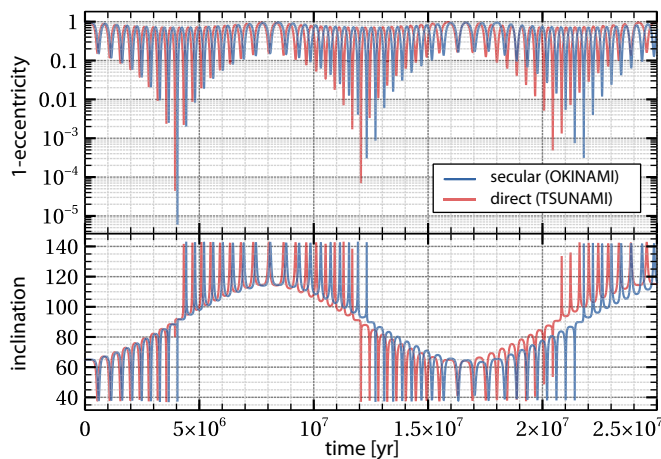
TSUNAMI is based on the following techniques: regularization of the equations of motion, chain coordinates to reduce round-off errors and Bulirsch–Stoer extrapolation. The first technique (regularization) takes care of the singularity of the gravitational potential for  $r \rightarrow 0$ . The second technique (chain coordinates) helps reducing the round-off errors in hierarchical systems, which arise with the center-of-mass coordinates, without the need to include numerically expensive techniques of compensated summation. The third method (Bulirsch–Stoer extrapolation) increases the accuracy of the integration and makes it adaptable over a wide dynamical range.

TSUNAMI solves the Newtonian equations of motion derived from a modified, extended Hamiltonian (Mikkola and Tanikawa 1999a,b). As a consequence, time is another variable that is integrated along positions and velocities of the particles. Along one timestep of fictitious time  $\Delta S$ , the physical time  $\Delta T$  is advanced by:

$$\Delta T = \frac{\Delta S}{\alpha U + \beta \Omega + \gamma} \quad (2.1)$$

where  $U$  is the potential energy,  $\Omega$  is a function of positions, and  $\alpha$ ,  $\beta$  and  $\gamma$  are arbitrary coefficients. Setting the values for  $(\alpha, \beta, \gamma)$  effectively changes the regularization algorithm.  $(\alpha, \beta, \gamma) = (1, 0, 0)$  corresponds to the logarithmic Hamiltonian algorithm,  $(\alpha, \beta, \gamma) = (0, 1, 0)$  is equivalent to the time-transformed leapfrog scheme, and for  $(\alpha, \beta, \gamma) = (0, 0, 1)$  the integration scheme reduces to the non-regularized leapfrog.

Positions and velocities are integrated in a chain coordinate system, rather than in the center-of-mass coordinates. This has the effect of reducing by 1 the number of equations to be integrated, and more importantly it reduces round-off errors when calculating distances between close particles far from the center of mass of the system (Mikkola and Aarseth 1993). These errors can quickly arise if the inter-particle separation is very small compared to the distance from the center of mass, due to the limits of floating-point arithmetic, which can happen, for example, in case of close binaries far



**Figure 1.** Comparison between TSUNAMI (red, direct integration) and OKINAMI (blue, secular average) in evolving a Jupiter-Sun-brown dwarf system. The top panel shows the eccentricity of the Jupiter’s orbit around the Sun-like star, while the bottom panel shows the mutual inclination between the Jupiter’s orbit and the brown dwarf’s orbit. As the Jupiter is perturbed by the outer brown dwarf, the Jupiter exhibits von Zeipel-Kozai-Lidov oscillations, with the typical flip of the orbit associated to the octupole-level secular interaction. This figure reproduces fig. 3 from (Naoz et al. 2013).

from a massive black hole. The chain of inter-particle vectors is formed so that all particles are included in the chain. The first segment of the chain is chosen to be the shortest inter-particle distance in the system. The next segment is included so that it connects the particle closest to one of the ends of the current chain. This process is repeated until all particles are included. As the system evolves, care is taken to update the chain so that any chained vector is always shorter than adjacent non-chained vectors. It is possible to directly transform the old coordinates into new chain coordinates without passing through the center-of-mass coordinates.

Finally, a simple leapfrog integration might not be accurate enough for some applications. Therefore, the accuracy of the integration can be improved with the Bulirsch-Stoer extrapolation. The idea behind Bulirsch-Stoer extrapolation is to consider the results of a numerical integration as being an analytic function of the stepsize  $h$ . The solution of a given time interval  $\Delta S$  is computed for smaller and smaller substeps  $h = \Delta S/N_{\text{steps}}$  and then it is extrapolated to  $h \rightarrow 0$ , using rational or polynomial functions.

The above integration scheme works well for Newtonian gravity. However, this is not enough to model some systems like binary black holes or planets, which require additional physics. TSUNAMI implements additional forces, like equilibrium tides (Hut 1981), dynamical tides (Samsing et al. 2018) and post-Newtonian corrections of order 1, 2 and 2.5, using the midpoint step described in Mikkola and Merritt (2008).

## 2.2. The OKINAMI code

Unlike TSUNAMI, OKINAMI is limited to stable hierarchical triples, that is, a binary whose center of mass forms another binary with a tertiary body. At its core, OKINAMI integrates the equations of motion derived from a three-body Hamiltonian, expanded at the octupole-level interaction and averaged over the mean anomalies of the inner and outer orbits. The double-average has the advantage of considerably speeding up the integration, because it avoids the integration of the “fast angles”. On the other hand, this has two consequences: the information about the individual positions of the

bodies along their orbit is lost, and the equations cannot describe the evolution of the system on timescales shorter than the inner and outer orbital periods. After the double average, what we obtain is a set of ordinary differential equations for the inner and outer eccentricities,  $(e_1, e_2)$ , arguments of pericenter  $(\omega_1, \omega_2)$ , longitudes of the ascending nodes  $(\Omega_1, \Omega_2)$  and orbital inclinations  $(i_1, i_2)$ . OKINAMI integrates these equations using an adaptive Runge-Kutta-Fehlberg of order 7. In addition, OKINAMI implements also equilibrium tides and post-Newtonian terms of orders 1, 2 and 2.5 for the inner orbit.

As an example, Figure 1 shows the evolution of inclination and eccentricity of the orbit of a Jupiter-sized planet around a Sun-like star, orbited by a distant brown dwarf. The integration with TSUNAMI takes about 5 minutes, while the one with OKINAMI takes less than a second.

### 3. Applications

#### 3.1. Exoplanets and exomoons

In Trani et al. (2020) we investigated the fate of exomoons around migrating hot Jupiters. We considered the scenario in which hot Jupiters experience high-eccentricity, tidally driven migration, due to the gravitational perturbation from a distant companion star. Physically, the system is a 4-body problem composed of two nested hierarchical triple systems: one triple is composed of the primary star, companion star and the Jupiter, the second triple is constituted by the moon, its host Jupiter and the main star.

This kind of system is an ideal test bench for TSUNAMI. Our code can accurately model the tidal forces required for the migration of the Jupiter and general relativity precession that can alter the long-term dynamics of the Jupiter and its moon. We found that exomoons are unlikely to survive the migration process of the host Jupiter. Massive moons can prevent the migration process entirely, by suppressing the eccentricity excitation induced by the secondary star. If the moon cannot shield the planet from perturbations, the Jupiter's orbit becomes increasingly eccentric, triggering the dynamical instability of the moon. Subsequently, most exomoons end up being ejected from the system or colliding with the primary star and the host planet. Only a few escaped exomoons can become stable planets after the Jupiter has migrated, or by tidally migrating themselves.

Even though close-in giants are ideal candidates for exomoon detections, our results suggest that it is unlikely for exomoons to be discovered around them, at least for planets migrated via high-eccentricity tidal circularization. Nonetheless, tidally disruptions or collisions of exomoons can still leave observational signatures, such as debris disks or chemically altered stellar atmospheres.

Besides exotic scenarios like exomoons and hot Jupiters, TSUNAMI is an excellent tool to assess the stability and long-term evolution of planetary systems (e.g. Livingston et al. 2019).

#### 3.2. Gravitational-wave radiation sources

The astrophysical origin of gravitational-wave events from coalescing black holes binaries is still debated, though many of the proposed formation scenarios involve some kind of few-body gravitational interaction. Specifically, chaotic three-body interactions between compact-object binaries and single black holes can happen frequently in dense stellar systems. These interactions can alter the spin-orbit orientation of black hole binaries, which can then be inferred from gravitational-wave observations.

In Trani et al. (2021b), we estimated the spin parameter distributions of merging black-hole binaries, comparing them with the currently available data. Here, we introduced a new formation scenario that combines elements from both the isolated and the dynamical formation scenarios. We ran an extensive set of highly-accurate simulations with

TSUNAMI, and we used the results to estimate the intrinsic merger rates of black-hole binaries in combination with a semi-analytic model.

Assuming low natal black-hole spins ( $\chi < 0.2$ ), our scenario reproduces the distributions of  $\chi_{\text{eff}}$  and  $\chi_p$  inferred from current observations. In particular, this model can explain the peak at positive  $\chi_{\text{eff}}$  with a tail at negative  $\chi_{\text{eff}}$ , and the broad peak at  $\chi_p \sim 0.2$ . This is in sharp contrast with the predictions of the isolated and the dynamical scenarios: the first fails to produce negative  $\chi_{\text{eff}}$ , while the second predicts a symmetric distribution around  $\chi_{\text{eff}} \sim 0$ .

In [Trani et al. \(2021a\)](#) we examined merging compact-object binaries in hierarchical triple systems. We first obtained a large sample of triples formed in low mass clusters through dynamical interactions, simulated using direct N-body methods ([Rastello et al. 2021](#)). Because we selected only stable triples, we evolved them using OKINAMI. We obtained the merger properties of binary black holes, black hole–neutron stars, and black hole–white dwarfs. The rates for binary black holes, black hole–neutron stars are about 100 times lower than those of binary mergers from the same clusters. This is caused by the lower merger efficiency of triple systems, which is about 100 times lower than that of binaries. Nonetheless, compact objects merging from triples have unique properties that can be used to discriminate them from other formation channels.

Compared to binary black-hole mergers from open clusters, mergers from triples have more massive primaries, with a mass distribution peaking at around  $30 M_{\odot}$  rather than  $10 M_{\odot}$ . The mass ratio also peaks at smaller values of 0.3, in contrast to the cluster binaries pathway, which favors equal-mass binaries. This is caused by the von Zeipel-Kozai-Lidov mechanism, whose eccentricity-pumping effect is enhanced at low mass ratios.

TSUNAMI is also ideal to model compact objects and stars in proximity to massive black holes. We investigated the impact of three-body encounters around massive black holes on binary black hole coalescence in [Trani et al. \(2019b\)](#), see also [Trani et al. 2019a](#) and [Trani 2020](#)

### 3.3. Statistical solutions to the three-body problem

The gravitational three-body problem is chaotic and has no general analytic solution, and only partial statistical solutions have been achieved so far (see [Stone and Leigh 2019](#), and references therein). The main idea behind these statistical escape theories is to leverage chaos to predict the evolution of a three-body problem only in a statistical sense, using the assumption of thermodynamical ergodicity. Recently, [Kol \(2021\)](#) introduced a novel statistical theory based on the flux of phase space, rather than on phase-space volume like all the previous theories.

In a series of papers, we have been testing the statistical theories by simulating a large ensembles of three-body systems with TSUNAMI and comparing the final outcome distributions to the theoretical predictions. Our results in [Manwadkar et al. \(2020\)](#) and [Manwadkar et al. \(2021\)](#) show that the flux-based theory is in tighter agreement with the outcome of three-body simulations, compared to the previous statistical theories. We are now in the process of further testing the potential of this theory with the aim of providing a complete, accurate statistical description of the three-body problem.

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