On the Mathematical Structure of Turbulence

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ONE is able to formulate in a very general way the mathematical problem of turbulence in the following form.

One has a system of second-order nonlinear partial differential equations with three space variables and one time variable. One looks for solutions sufficiently complex and nonstationary to represent irregular oscillations.

I propose to discuss the problem on the basis of the simple example of the equation,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

introduced by Burgers as a model for turbulence: $u(t)$ plays the role of a velocity component of a turbulent viscous fluid.

I. ALMOST PERIODIC SOLUTIONS

I will first show that Eq. (1) possesses solutions which can be developed in a Fourier series, generally nonperiodic. The general solution of (1) can be written

$$u=-\frac{1}{2}\frac{1}{Z}\frac{\partial Z}{\partial x}, \tag{2}$$

if Z is a solution of the heat transfer equation,

$$\frac{\partial Z}{\partial t}=\frac{1}{2}\frac{\partial^2 Z}{\partial x^2}. \tag{3}$$

Equation (3) has the solution

$$Z=\exp[-(1+i)\lambda x+i\lambda^2 t], \tag{4}$$

where λ is a real parameter. As Eq. (3) is linear, it has the more general solution

$$Z=1-\sum_{k=1}^{p} A_k \exp[-(1+i)\lambda kt+i\lambda_k^2 t], \tag{5}$$

that is to say a trigonometric polynomial which is, in general, nonperiodic. As a result, Eq. (1) admits a solution

$$u=\frac{i+1}{2}\frac{\lambda_1 A_1 \exp[-(1+i)\lambda_1 x+i\lambda_1^2 t]+\lambda_2 A_2 \exp[-(1+i)\lambda_2 x+i\lambda_2^2 t]}{1-A_1 \exp[-(1+i)\lambda_1 x+i\lambda_1^2 t]-A_2 \exp[-(1+i)\lambda_2 x+i\lambda_2^2 t]}, \tag{6}$$

taking for simplicity $p=2$. If, for example, $x\geq 0$, one can write

$$u=\frac{1+i}{2}\sum_{\alpha,\beta=1}^{\infty}\frac{(\alpha+\beta-1)!}{\alpha!\beta!}(\lambda_1\alpha+\lambda_2\beta)A_1^{\alpha}A_2^{\beta}$$
$$\times\exp[-(1+i)x(\lambda_1\alpha+\lambda_2\beta)+it(\lambda_1^2\alpha+\lambda_2^2\beta)]. \tag{7}$$

One sees that u is in general an almost periodic function of time, of which the pulsations are of the form

$$\lambda_1^2\alpha+\lambda_2^2\beta,$$

where λ_1 and λ_2 are two given numbers and α, β are two entirely arbitrary numbers. This is an almost periodic function in the sense of Esclangon. It is essentially complex, and since Eq. (1) is not linear, no real solution directly results. But the method generalizes easily, and permits one to obtain real solutions having the desired form.

II. SOLUTIONS UNABLE TO BE DEVELOPED IN A FOURIER SERIES

The local aspect of a function of the type given by Eq. (6) or Eq. (7) agrees well with the local form of experimental curves. But, on these curves, one is in the habit of computing time means and these means have at least the following properties:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(t)dt=0; \tag{8}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T [u(t)]^2 dt \text{ has a finite positive value.} \tag{9}$$

If one puts [autocorrelation function of $u(t)$],

$$\gamma(h)=\lim_{T\to\infty}\frac{1}{T}\int_0^T u(t)u(t+h)dt, \tag{10}$$

so $\gamma(h)$ will be a continuous function and

$$\lim_{h\to\infty}\gamma(h)=0. \tag{11}$$

The almost periodic continuous functions satisfy the properties (8), (9), (10), but not property (11). Thus, $\gamma(h)$ is an almost periodic function of h, which oscillates indefinitely without tending to zero. Expressing it

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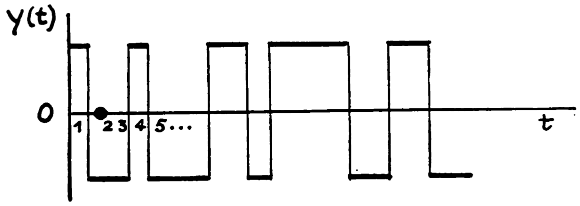

FIG. 1.

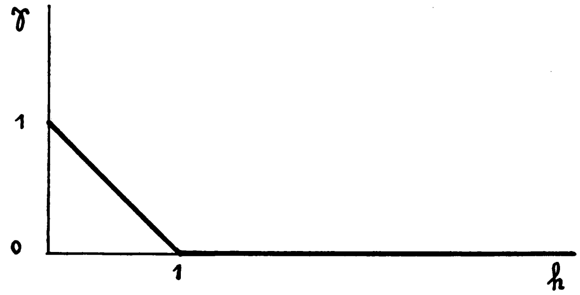

FIG. 2.

otherwise, a function of the form (7), which is capable of representing locally a turbulent velocity, does not have the properties of irregularity which correspond to the general structure of the phenomenon; it repeats itself almost exactly in nearly regular intervals. It does not satisfy the property that what happens at time $t+h$ is practically independent of what happens at time t if h is sufficiently large.

But it is difficult to construct functions which satisfy simultaneously properties (8) and (11). The ordinary Fourier integral

$$u(t)=\int_0^\infty f(\omega)e^{i\omega t}d\omega, \tag{12}$$

with very general $f(\omega)$, does not satisfy property (9). One finds in effect that $\gamma(0)=0$. It is in general incompatible with the idea of a statistically stationary phenomenon, since $u(t)$ tends to 0 as $t\rightarrow\infty$.

In order to make the solutions of Eq. (1) respond to the imposed requirements, one may proceed from an example of a function given by Wiener defined as follows.

One considers on the t axis the set of integers 0, 1, $2\cdots, n, \cdots$ (see Fig. 1). The desired function $Y(t)$ is 0 for $t<0$, is 1 or -1 for $t>0$. It jumps from one value to another at the points in the above sequence. If the points are not too regularly spaced (an extremely exceptional case) one finds that

$$\lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T Y(t)dt=0,$$

$$\gamma(h)=\lim\frac{1}{T}\int_0^T Y(t)Y(t+h)dt=\begin{cases}1-\text{h}; & 0\le h<1\\ 0 & ;\quad h\ge 1\end{cases}. \tag{13}$$

In particular, $\gamma(0)=1$, which is *a priori* evident (see Fig. 2).

This function $Y(t)$ permits one to construct numerous functions $u(t)$ having the desired properties. Here are two examples.

(i) If $f(t)$ is continuous and nearly periodic or

$$f(t)=\sum_{-\infty}^{\infty}c_k e^{i\omega_k t}, \tag{14}$$

the product

$$u(t)=Y(t)f(t) \tag{15}$$

is a function of the type sought except perhaps for some particular choices of coefficients c_k.

(ii) If $\varphi(\omega)$ is a function defined for $\omega\ge 0$ and such that the integral $\int_0^\infty|\varphi(\omega)|d\omega$ is convergent, the function

$$u(t)=\int_0^\infty Y(t+\omega)\varphi(\omega)d\omega \tag{16}$$

is of the type sought. One may easily form the autocorrelation function $\gamma(h)$. If $\varphi(\omega)$ tends to 0 sufficiently fast as $\omega\rightarrow\infty$, $u(t)$ is continuous [which is not the case for $Y(t)$] and has derivatives. One can, finally, show that the partial differential equation (1) admits solutions of the form

$$u(x,t)=\int_0^\infty Y(t+\omega)\varphi(\omega,x)d\omega. \tag{17}$$

This type of function is capable of representing turbulence. It would be interesting on the one hand to find the general form of such functions,[1] and, on the other hand, to examine if it is compatible, by means of suitable generalization, with the Navier-Stokes equations.

[1] Cf. J. Bass, Compt. rend. **245**, 1217 (1957).

DISCUSSION

J. M. BURGERS, *University of Maryland, College Park and Baltimore, Maryland:* I may be allowed to remark that in my own work this equation has been treated in a way greatly differing from that which was presented by Bass. As he mentioned, the equation can be reduced to a linear one and an explicit solution is possible for any given initial state. This solution in itself does not involve any random processes or random quantities; it is completely definite. To introduce a probabilistic element, one must therefore consider an ensemble of initial states; one will then obtain an ensemble of solutions, and averages can be taken over

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the ensemble. It is possible to give the solution in such a form that the calculation of double, triple, and higher correlation functions can be reduced to a problem of geometrical probability. My results lead me to believe that the statistical problems which present themselves here can be solved in explicit terms when one considers the limiting case $\nu \rightarrow 0$, $t \rightarrow \infty$. The application of the expressions obtained requires the evaluation of a complicated series of definite integrals.

By this process, all statistical problems referring to an ensemble of solutions of the equation are related to statistical properties of the ensemble of initial data. It is into these data that such quantities as measure functions must be introduced. The question whether fourth-order correlations can be expressed with the aid of second-order correlations, can be investigated in a direct way.

My opinion is that solutions of this equation can give information concerning hydrodynamic turbulence for a medium, the motion of which is restricted to one dimension, which is infinitely compressible, and which is endowed with a cooling mechanism such that the temperature always remains zero. This follows because the equation has no pressure term, so that it would be applicable only to a gas at zero temperature, in which no adiabatic heating can occur.

J. BASS, *Ecole Nationale Supérieure de l'Aeronautique, Paris, France:* I would only note that I wished to avoid the use of ensembles of functions and, if possible, to avoid the use of statistics—I should like to find the exact, not the statistical, form of the solutions.