On the Riesz-Riemann-Liouville Integral

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§ 1. *Introduction.*

In a lecture at the Oslo Congress in 1936, Marcel Riesz [1] introduced an important generalisation of the Riemann-Liouville integral of fractional order. Riesz's integral $I^\alpha f$ of order α is a multiple integral in m variables which converges uniformly when the real part of α exceeds $m-2$ and so represents an analytic function of the complex variable α. This integral is important in the theory of the generalised wave equation, for it provides a direct method of solving Cauchy's initial-value problem. The most recent developments [2] show that it is likely to be also of great importance in quantum electrodynamics.

Let us denote the m variables which appear in Riesz's integral by $(t, x_1, x_2, \ldots x_n)$, or, more briefly, by (t, x_i), so that $m = n + 1$. The variable t is treated separately because it is to be the time-variable in the applications of the theory. Let us denote the "interval" in the sense of special relativity between the two point-events $P(T; X_i)$ and $Q(t; x_i)$ by s, so that

$$s^2 = (T - t)^2 - \sum_{i=1}^{n} (X_i - x_i)^2. \tag{1.1}$$

When P is fixed and Q varies, the equation $s = 0$ represents the light-cone of special relativity; it is the characteristic cone of the generalised wave equation

$$L(u) \equiv \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0. \tag{1.2}$$

Inside the characteristic cone, s^2 is positive, outside it, negative. The half of the cone on which $t < T$ is called the retrograde cone and will be denoted by $D(P)$.

In the problem of Cauchy for the wave equation (1.2), we have to find the solution, given the values taken by u and its first partial derivatives on an n-dimensional manifold or hypersurface S. S is

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[1] *Comptes rendus du congrès international des mathématiciens* (*Oslo*, 1936). *Tome* 2, pp. 44-45.

[2] See a letter in *Nature*, 157, 734 (1946), by T. Gustafson.

spatially-directed in the sense of special relativity; that is, the tangent-plane to S at any point R cuts the characteristic cone with vertex R only at R. The surface S and the retrograde cone $D(P)$ bound a hypervolume which we denote by $D(P,S)$. The Riesz integral is then defined to be

$$I^{\alpha} f(P) = \frac{1}{H_m(\alpha)} \int_{D(P,S)} f(Q)\, s^{\alpha - m}\, dQ \tag{1.3}$$

where dQ denotes the element of hypervolume at Q and

$$H_m(\alpha) = \pi^{\frac{1}{2}m-1}\, 2^{\alpha-1}\Gamma(\tfrac{1}{2}\alpha)\Gamma(\tfrac{1}{2}\alpha - \tfrac{1}{2}m + 1). \tag{1.4}$$

The integral is then an analytic function of α, regular when $\mathrm{Rl}\,\alpha > \mathrm{Max}\,(m-2, 0)$, provided that f is continuous. Its characteristic properties are expressed by the equations

$$I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f, \qquad L\, I^{\alpha+2} f = I^{\alpha} f,$$

where L now refers to the variables (T, X_i).

In all applications, the fundamental problem is the analytical continuation of $I^{\alpha} f$ into the half-plane $\mathrm{Rl}\,\alpha > 0$. Riesz himself has published no solution of this problem. The cases of greatest physical importance are those for which $m = 3$ or 4 and S is the hyperplane $t = 0$; in these cases it has been shown [1] that the analytical continuation can be carried out by a simple change of variable and integration by parts. Quite recently, the general case has been discussed by Fremberg,[2] who uses a rather complicated change of variable which makes his work difficult to follow. In the present paper it is shown that the methods used in the simpler cases are also applicable in the general case and lead to the desired results. It is shown incidentally that, when S is the hyperplane $t = -\infty$, $I^{m-2} f$ is simply the retarded potential.

§2. *A transformation of Riesz's Integral.*

Since $I^{\alpha} f(P)$ is to be defined for all positive values of T, the hypersurface S and the retrograde cone $D(P)$ must bound a hypervolume no matter how large T is. Hence S cannot be a closed surface. We write the equation of S in the form $S(t, x_i) = 0$ where the function

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[1] Baker and Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, 1939), pp. 60-61. Copson, *Proc. Roy. Soc. Edin.* (A) 71, 260-272 (1943).

[2] *Kungl. Fysiografiska Sällskapets i Lund Forhandlingar*, Bd. 15, Nr. 27 (1945).

S and its first partial derivatives are assumed to be continuous. Then since S is spatially-directed,

$$\left(\frac{\partial S}{\partial t}\right)^2 - \sum_i \left(\frac{\partial S}{\partial x_i}\right)^2 > 0 \tag{2.1}$$

everywhere on S, a relation which implies that S is not a closed surface and has no singular points. The analytical continuation with respect to a is to be carried out for any fixed values of T and X_i. It follows that the expression on the left-hand side of (2.1) has a positive lower bound δ, depending on T and X_i, on the portion S' of S cut off by the retrograde cone; thus on S',

$$\left(\frac{\partial S}{\partial t}\right)^2 - \sum_i \left(\frac{\partial S}{\partial x_i}\right)^2 \geqq \delta > 0. \tag{2.2}$$

No radius vector from P inside $D(P)$ can touch S since S is spatially-directed. Moreover, every such radius vector cuts S in the same number of distinct points, since S has no singular points. But S and $D(P)$ bound a hypervolume; hence every radius vector from P inside $D(P)$ cuts S in one point. We shall denote the point where the radius vector from $P(T;X_i)$ through $Q(t;x_i)$ cuts S by $R(\tau;\xi_i)$. Similar considerations show that any line parallel to the axis of t cuts S in one point only, so that we may write the equation of S in the form $t = S(x_i)$. By (2.2), the inequality

$$\sum_i \left(\frac{\partial S}{\partial x_i}\right)^2 \leqq 1 - \delta < 1 \tag{2.3}$$

then holds everywhere on S'.

We now change to new variables $(x_1, x_2, \ldots x_m)$ defined by

$$x_i = x_i \qquad (i = 1, 2, 3, \ldots, n = m - 1)$$
$$x_m = +\sqrt{\{(T-t)^2 - \sum_i (x_i - X_i)^2\}}$$

where summation with respect to i is always over the range $1, 2, 3, \ldots n$. This is a (1,1) transformation which maps the interior of the retrograde cone on $x_m > 0$. Since

$$\frac{\partial(x_1, x_2, \ldots\ldots x_n, t)}{\partial(x_1, x_2, \ldots x_{m-1}, x_m)} = -\frac{x_m}{r},$$

Riesz's integral becomes

$$I^a f(T, X_i) = \frac{1}{H_m(a)} \int f(T - r, x_i) \frac{x_m{}^{a-m+1}}{r} dx_1\, dx_2 \ldots dx_m \tag{2.4}$$

where integration is over the hypervolume in $x_m > 0$ bounded by the hyperplane $x_m = 0$ and the hypersurface Σ whose equation is

$$\{T-S\,(x_i)\}^2 = \sum_i (x_i - X_i)^2 + x_m^2. \tag{2.5}$$

Evidently every radius vector from $P'(X_1, X_2, \ldots X_n, 0)$, which is the image of P, cuts Σ in one point R', since this radius vector is the image of a radius vector through P. It is convenient to use spherical polar coordinates $(r, \theta, \phi_1, \phi_2, \ldots \phi_{n-1})$ defined by

$$x_i = X_i + l_i\, r \sin\theta, \qquad x_m = r\cos\theta \tag{2.6}$$

where
$$\begin{aligned} l_n &= \cos\phi_1, \\ l_{n-1} &= \sin\phi_1 \cos\phi_2, \\ l_{n-2} &= \sin\phi_1 \sin\phi_2 \cos\phi_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ l_2 &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \cos\phi_{n-1}, \\ l_1 &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \sin\phi_{n-1}. \end{aligned}$$

If we make this change in (2.4), the angle θ varies from 0 to $\frac{1}{2}\pi$, whilst the angles ϕ vary so that the line whose direction-cosines in n-dimensional space are $(l_1, l_2, \ldots l_n)$ sweeps out the whole solid angle Ω_n. Hence $\phi_1, \phi_2, \ldots \phi_{n-2}$ vary from 0 to π, ϕ_{n-1} from 0 to 2π. The coordinates of R' are then

$$\xi_i = X_i + l_i \rho \sin\theta, \qquad \xi_m = \rho\cos\theta \tag{2.7}$$

where ρ is a continuous function of the variables θ and ϕ. With this change of variable, we have[1]

$$I^\alpha f = \frac{1}{H_m(\alpha)} \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \int_0^\rho f(T-r, x_i)\, r^{\alpha-1} \cos^{\alpha-m+1}\theta \sin^{m-2}\theta \, dr\, d\theta\, d\Omega_n \tag{2.8}$$

where

$$d\Omega_n = \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \ldots \sin\phi_{n-2}\, d\phi_1\, d\phi_2 \ldots d\phi_{n-1}.$$

The total solid angle is

$$\Omega_n = \int_{\Omega_n} d\Omega_n = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}. \tag{2.9}$$

The analytical continuation of $I^\alpha f$ depends on the formula (2.8), which shows that, so far, $I^\alpha f$ is regular when Rl $\alpha > \text{Max}\,(m-2, 0)$.

§ 3. *A lemma.*

We shall carry out the analytical continuation of $I^\alpha f$ by integrating (2.8) by parts with respect to θ. In doing so, we shall need the following lemma.[2]

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[1] Cf. Baker and Copson, *loc. cit.*, p. 60, equation (7.41).

[2] Cf. Fremberg, *loc. cit.*, p. 270.

If the first partial derivatives of $S(\xi_i)$ are continuous, $\partial\rho / \partial\theta = \sigma\rho \cos\theta$

where $$\sigma = -\frac{\Sigma l_i \partial S/\partial\xi_i}{1+\sin\theta\, \Sigma l_i\, \partial S/\partial\xi_i}.$$

The function σ is continuous on Σ.

If ρ is a continuous function of θ, ϕ satisfying an equation $\psi(\rho, \theta, \phi) = 0$, where ψ has continuous first partial derivatives, then $\partial\rho / \partial\theta$ exists and is given by

$$\frac{\partial\rho}{\partial\theta} = -\frac{\partial\psi}{\partial\theta}\Big/\frac{\partial\psi}{\partial\rho}$$

at any point where $\partial\psi / \partial\rho$ is not zero. In the present case it follows from (2.5) that ρ satisfies the equation $\psi = 0$ where

$$\psi = \rho - T + S(\xi_i), \qquad \xi_i = X_i + l_i\, \rho \sin\theta,$$

and so

$$\frac{\partial\rho}{\partial\theta} = -\frac{p\,\rho\cos\theta}{1+p\sin\theta}$$

where $$p = \Sigma\, l_i \frac{\partial S}{\partial\xi_i}.$$

To complete the proof of the lemma, we have to show that $1 + p\sin\theta$ does not vanish on Σ. Now

$$p^2 = \left(\Sigma\, l_i \frac{\partial S}{\partial\xi_i}\right)^2 \leqq \Sigma l_i^2 \,.\, \Sigma\left(\frac{\partial S}{\partial\xi_i}\right)^2 = \Sigma\left(\frac{\partial S}{\partial\xi_i}\right)^2 \leqq 1-\delta$$

by (2.3). Hence

$$1 + p\sin\theta \geqq 1 - |p| \geqq 1 - \sqrt{(1-\delta)} > \tfrac{1}{2}\delta$$

which was to be proved.

§ 4. *The analytical continuation of $I^\alpha f$.*

If f and S have continuous partial derivatives of order $k < \frac{1}{2}(m-1)$ and if Rl $\alpha >$ Max $(0, m-2k-2)$, *then $I^\alpha f$ is equal to*

$$\frac{1}{K_m(\alpha, k)}\Bigg[\int_{\Omega_n}\int_0^{\frac{1}{2}\pi}\int_0^{\rho} f_k(T-r, x_i)\; r^{\alpha-1}\cos^{\alpha-m+2k+1}\theta \sin^{m-2k-2}\theta\; dr d\theta d\Omega_n$$
$$+\int_{\Omega_n}\int_0^{\frac{1}{2}\pi} g_k(\rho, \xi_i)\,\rho^\alpha \cos^{\alpha-m+2k+1}\theta \sin^{m-2k-1}\theta\; d\theta d\Omega_n\Bigg] \tag{4.1}$$

where $$K_m(\alpha, k) = \pi^{\frac{1}{2}m-1}\, 2^{\alpha+k-1}\,\Gamma(\tfrac{1}{2}\alpha)\,\Gamma(\tfrac{1}{2}\alpha - \tfrac{1}{2}m + k + 1)$$

and $f_0 = f$

$$f_{k+1} = (m-2k-3)\, f_k + \Sigma\,(x_i - X_i)\frac{\partial f_k}{\partial x_i} \tag{4.2}$$

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$$g_0 = 0$$

$$g_{k+1} = f_k(T-\rho,\ \xi_i)\,\sigma + \sigma\rho\sin\theta\frac{\partial g_k}{\partial\rho} + (1+\sigma\sin\theta)\,\Sigma\,(\xi_i - X_i)\frac{\partial g_k}{\partial\xi_i}$$
$$+ \alpha\sigma g_k\sin\theta + (m-2k-2)\,g_k. \quad (4.3)$$

There are two points about the recurrence formulæ which must be emphasised. In (4.2), we regard f_k as a function of the independent variables $x_1, x_2, \ldots x_n, r$ and the spherical polar coordinates are introduced only when we substitute for f_k in (4.1). Similarly in (4.3) g_k is a function of the independent variables $\xi_1, \xi_2, \ldots \xi_n, \rho$; it would be more correct but more cumbrous to write

$$\sigma = -\frac{\rho\,\Sigma l_i\,\partial S/\partial\xi_i}{\rho+\Sigma\,(\xi_i - X_i)\,\partial S/\partial\xi_i}, \quad \sigma\sin\theta = -\frac{\Sigma\,(\xi_i - X_i)\,\partial S/\partial\xi_i}{\rho + \Sigma\,(\xi_i - X_i)\,\partial S/\partial\xi_i}.$$

The theorem is evidently true when $k = 0$. We shall prove it in the general case by induction. The functions f_k and g_k are continuous if the kth partial derivatives of f and S are continuous; for, in the construction of f_k and g_k by the recurrence formulæ starting with f_0 and g_0, a discontinuity could arise only from the successive derivatives of σ and $\sigma\sin\theta$, and these are all continuous since $1 + p \sin\theta > \frac{1}{2}\delta > 0$.

We assume then that $I^\alpha f$ is equal to the expression (4.1) that the derivatives of order $k+1$ are continuous and that Rl $\alpha > m - 2k - 2 > 1$. The expression (4.1) consists of two terms which we treat separately. The first term is

$$I_1 = \frac{1}{K_m(\alpha, k)}\int_{\Omega_n}\int_0^{\frac{1}{2}\pi} F_k(\rho,\theta)\cos^{\alpha-m+2k+1}\theta\sin^{m-2k-2}\theta\,d\theta\,d\Omega_n,$$

where

$$F_k(\rho,\theta) = \int_0^\rho f_k(T-r,\ x_i)\,r^{\alpha-1}\,dr.$$

If we integrate by parts with respect to θ, we obtain

$$I_1 = \frac{1}{K_m(\alpha, k+1)}\int_{\Omega_n}\int_0^{\frac{1}{2}\pi}\cos^{\alpha-m+2k+2}\theta\,\frac{d}{d\theta}\left[F_k(\rho,\theta)\sin^{m-2k-3}\theta\right]d\theta\,d\Omega_n,$$

the terms at the limits vanishing[1] since Rl$\alpha > m - 2k - 2 > 1$. Moreover

$$\frac{d}{d\theta}\left[F_k(\rho,\theta)\sin^{m-2k-3}\theta\right]$$

[1] If $m = 2k+3$, there is a contribution from the lower limit $\theta = 0$, and this is important in § 6.

$$= \left\{ \frac{\partial F_k}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \frac{\partial F_k}{\partial \theta} \right\} \sin^{m-2k-3} \theta + (m - 2k - 3) \, F_k \sin^{m-2k-4} \theta \cos \theta$$

$$= \left\{ f_k (T-\rho, \xi_i) \, \sigma \rho^\alpha \cos \theta + \int_0^\rho \Sigma \frac{\partial f_k (T-r, x_i)}{\partial x_i} \frac{\partial x_i}{\partial \theta} r^{\alpha-1} \, dr \right\} \sin^{m-2k-3} \theta$$
$$+ (m - 2k - 3) \, F_k \sin^{m-2k-4} \theta \cos \theta$$

$$= \left\{ \bar{f}_k \, \sigma \rho^\alpha \cos \theta + \int_0^\rho \Sigma l_i r \cos \theta \frac{\partial f_k}{\partial x_i} r^{\alpha-1} \, dr \right\} \sin^{m-2k-3} \theta$$
$$+ (m - 2k - 3) \, F_k \sin^{m-2k-4} \theta \cos \theta$$

$$= \bar{f}_k \sigma \rho^\alpha \cos \theta \sin^{m-2k-3} \theta + \int_0^\rho \Sigma \, (x^i - X_i) \frac{\partial f_k}{\partial x_i} r^{\alpha-1} \, dr \sin^{m-2k-4} \theta \cos \theta$$
$$+ (m - 2k - 3) \, F_k \sin^{m-2k-4} \theta \cos \theta$$

$$= \bar{f}_k \, \sigma \, \rho^\alpha \cos \theta \sin^{m-2k-3} \theta + \int_0^\rho f_{k+1} (T-r, x_i) \, r^{\alpha-1} \, dr \sin^{m-2k-4} \theta \cos \theta$$

where $\bar{f}_k$ denotes $f_k \, (T-\rho, \xi_i)$ and where f_{k+1} is defined by (4.2). We have thus proved that I_1 is equal to

$$\frac{1}{K_m(\alpha, k+1)} \left[\iiint_0^\rho f_{k+1}(T-r, x_i) \, r^{\alpha-1} \cos^{\alpha-m+2k+3}\theta \sin^{m-2k-4}\theta \, dr d\theta d\Omega_n \right.$$
$$\left. + \iint \bar{f}_k \, \sigma \rho^\alpha \cos^{\alpha-m+2k+3} \theta \sin^{m-2k-3} \theta \, d\theta \, d\Omega_n \right].$$

The second term in (4.1) is

$$I_2 = \frac{1}{K_m(\alpha, k)} \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} g_k \, (\rho, \xi_i) \, \rho^\alpha \cos^{\alpha-m+2k+1} \theta \sin^{m-2k-1} \theta \, d\theta \, d\Omega_n$$
$$= \frac{1}{K_m(\alpha, k+1)} \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \cos^{\alpha-m+2k+2} \theta \frac{d}{d\theta} \left[g_k(\rho, \xi_i) \rho^\alpha \sin^{m-2k-2} \theta \right] d\theta \, d\Omega_n$$

on integrating by parts, the terms at the limits vanishing as before. Remembering that $\xi_i = X_i + l_i \rho \sin \theta$, we have

$$\frac{d}{d\theta} \left[g_k \, (\rho, \xi_i) \, \rho^\alpha \sin^{m-2k-2} \theta \right]$$

$$= \left\{ \frac{\partial g_k}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \Sigma \frac{\partial g_k}{\partial \xi_i} \frac{\partial \xi_i}{\partial \theta} + \frac{\alpha}{\rho} \, g_k \frac{\partial \rho}{\partial \theta} \right\} \rho^\alpha \sin^{m-2k-2} \theta$$
$$+ (m - 2k - 2) \, g_k \, \rho^\alpha \sin^{m-2k-3} \theta \cos \theta$$

$$= \left\{ \frac{\partial g_k}{\partial \rho} \sigma \rho \cos \theta + \Sigma \, l_i \frac{\partial g_k}{\partial \xi_i} (\rho \cos \theta + \sigma \rho \sin \theta \cos \theta) \right.$$
$$\left. + g_k \, \alpha \, \sigma \cos \theta \right\} \rho^\alpha \sin^{m-2k-2} \theta + (m-2k-2) \, g_k \, \rho^\alpha \sin^{m-2k-3} \theta \cos \theta$$

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$$= \rho^{\alpha} \sin^{m-2k-3}\theta \cos\theta \left\{ \sigma\rho \frac{\partial g_k}{\partial \rho} \sin\theta + (1+\sigma\sin\theta) \Sigma (\xi_i - X_i) \frac{\partial g_k}{\partial \xi_i} + \alpha\sigma g_k \sin\theta + (m-2k-2) g_k \right\}$$

$$= \rho^{\alpha} \sin^{m-2k-3}\theta \cos\theta \{ g_{k+1} - \sigma \bar{f}_k \}$$

where g_{k+1} is defined by (4.3). Hence

$$I_2 = \frac{1}{K_m(\alpha, k+1)} \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \{ g_{k+1} - \sigma \bar{f}_k \} \rho^{\alpha} \cos^{\alpha-m+2k+3}\theta \sin^{m-2k-3}\theta \, d\theta \, d\Omega_n.$$

Adding the expressions for I_1 and I_2, we obtain

$$I^{\alpha} f = \frac{1}{K_m(\alpha, k+1)} \left[\int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \int_0^{\rho} f_{k+1} \, r^{\alpha-1} \cos^{\alpha-m+2k+3}\theta \sin^{m-2k-4}\theta \, dr d\theta d\Omega_n + \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} g_{k+1} \, \rho^{\alpha} \cos^{\alpha-m+2k+3}\theta \sin^{m-2k-3}\theta \, d\theta \, d\Omega_n \right] \quad (4.4)$$

which completes the induction, apart from the consideration of the region in which (4.4) is valid. But both integrals converge uniformly with respect to α when Rl $\alpha >$ Max $(0, m-2k-4)$, and so the result is proved.

§ 5. *The limit of $I^{\alpha} f$ as $\alpha \to +0$ when m is even.*

By § 4, $I^{\alpha} f$ is regular in the half-plane Rl $\alpha >$ Max $(0, m-2k-2)$, provided that the derivatives of f and S of order $k < \frac{1}{2}(m-1)$ are continuous. It is necessary to consider separately the cases when m is even or odd. If $m = 2p+1$, the largest possible value of k is $p-1$; and if the derivatives of order $p-1$ are continuous, $I^{\alpha} f$ is regular in Rl $\alpha > 1$. If $m = 2p+2$, the largest possible value of k is p; and if the derivatives of order p are continuous, $I^{\alpha} f$ is regular in Rl $\alpha > 0$. In the latter case, $I^{\alpha} f$ is equal to

$$\frac{\alpha^2}{\pi^p \, 2^{\alpha+p+1} \, \Gamma(\frac{1}{2}\alpha+1) \, \Gamma(\frac{1}{2}\alpha+1)} \left[\int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \int_0^{\rho} f_p \, r^{\alpha-1} \cos^{\alpha-1}\theta \, dr \, d\theta \, d\Omega_n + \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} g_p \, \rho^{\alpha} \cos^{\alpha-1}\theta \sin\theta \, d\theta \, d\Omega_n \right]. \quad (5.1)$$

By the recurrence formulæ (4.2) and (4.3), f_p is independent of α, whereas g_p is a polynomial in α of degree $p-1$.

The second term in square brackets in (5.1) evidently contributes nothing to the limit of $I^{\alpha} f$ as $\alpha \to +0$. To deal with the first term, we use the following lemma of Fremberg.[1]

[1] Fremberg, *loc. cit.*, p. 274. I am grateful to a referee for pointing out that Fremberg's lemma, which omits reference to any "unspecified parameters", is really insufficient. The unspecified parameters are to be the angle-variables $\phi_1, \phi_2, \ldots \phi_{n-1}$.

Let $g(x, y)$ be a function of x and y and certain unspecified parameters, which is absolutely integrable over $0 \leqq x \leqq a$, $0 \leqq y \leqq b$ and which is continuous at $(0,0)$ uniformly with respect to the parameters.

Let $\int_0^a \left| g(x,y) \right| dx$ be bounded uniformly with respect to the parameters and with respect to y in $0 \leqq y \leqq \eta \leqq b$.

Le $\int_0^b \left| g(x, y) \right| dy$ be bounded uniformly with respect to the parameters and with respect to x in $0 \leqq x \leqq \xi \leqq a$. Then, as $\alpha \to +0$,

$$\alpha^2 \int_0^a \int_0^b g(x, y)\, x^{\alpha-1} y^{\alpha-1}\, dx\, dy \to g(0, 0),$$

uniformly with respect to the parameters.

We have assumed throughout this work that the surface S is spatially-directed. It is readily seen that this implies that ρ, which is a continuous function of the angle-variables θ and ϕ, has a positive lower bound a. Moreover, the part of the range of integration with respect to r between a and ρ contributes nothing to the limit of $I^\alpha f$. Hence we apply the lemma to

$$J = \alpha^2 \int_0^{\frac{1}{2}\pi} \int_0^a f_p\, r^{\alpha-1} \cos^{\alpha-1}\theta\, dr\, d\theta$$

$$= \alpha^2 \int_0^1 \int_0^a f_p\, r^{\alpha-1}\, y^{\alpha-1}\, \frac{dr\, dy}{\sqrt{(1-y^2)}}$$

where $y = \cos\theta$. Evidently f_p, being a continuous function of x_i and r, is a function of r and θ, and therefore of r and y, which fulfils the conditions of the lemma; the "unspecified parameters" are the angle-variables ϕ. Hence, as $\alpha \to +0$,

$$J \to \Big[f_p(T - r, x_i) \Big]_{\substack{r=0 \\ \theta=\frac{1}{2}\pi}} = f_p(T, X_i).$$

But since the limit of J is independent of the variables ϕ, we have

$$\lim_{\alpha \to +0} I^\alpha f(T, X_i) = \frac{\Omega_n}{2^{p+1}\pi^p} f_p(T, X_i) = \frac{\sqrt{\pi}}{2^p\,\Gamma(p+\frac{1}{2})} f_p(T, X_i) \tag{5.2}$$

by (2.9) with $n = 2p + 1$.

From (4.2), we have

$$f_k(T, X_i) = (m - 2k - 1) f_{k-1}(T, X_i) = \frac{2^k\,\Gamma(\frac{1}{2}m - \frac{1}{2})}{\Gamma(\frac{1}{2}m - \frac{1}{2} - k)} f(T, X_i). \tag{5.3}$$

In the present case, $m = 2p + 2$, so that

$$f_p(T, X_i) = \frac{2^p\,\Gamma(p + \frac{1}{2})}{\sqrt{\pi}} f(T, X_i).$$

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Substituting in (5.2), we obtain the final result that

$$\lim_{\alpha \to +0} I^\alpha f(T, X_i) = f(T, X_i)$$

when $m = 2p + 2$, provided that the partial derivatives of f and S of order p are continuous.

§ 6. *The limit of $I^\alpha f$ as $\alpha \to +0$ when m is odd.*

If $m = 2p + 1$ and the derivatives of f and S of order $p - 1$ are continuous, $I^\alpha f$ is regular in Rl $\alpha > 1$ and is equal to

$$\frac{1}{\pi^{p-\frac{1}{2}}\, 2^{\alpha+p-2}\, \Gamma(\frac{1}{2}\alpha)\, \Gamma(\frac{1}{2}\alpha - \frac{1}{2})} \left[\int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \int_0^\rho f_{p-1}\, r^{\alpha-1} \cos^{\alpha-2}\theta \sin\theta \, dr\, d\theta\, d\Omega_n \right.$$
$$\left. + \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} g_{p-1}\, \rho^\alpha \cos^{\alpha-2}\theta \sin^2\theta \; d\theta\, d\Omega_n \right]. \quad (6.1)$$

To continue the function into Rl $\alpha > 0$, we have to assume that the derivatives of order p are continuous and again integrate by parts with respect to θ. The first term inside the square brackets is

$$-\frac{1}{\alpha - 1} \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} F_{p-1} \frac{d \cos^{\alpha-1}\theta}{d\theta}\, d\theta\, d\Omega_n,$$

so that, when we integrate by parts, the terms at the limits do not vanish: there is a non-vanishing term arising from the lower limit $\theta = 0$.

Carrying out the integration by parts, we obtain for $I^\alpha f$

$$\frac{1}{\pi^{p-\frac{1}{2}}\, 2^{\alpha+p-1}\, \Gamma(\frac{1}{2}\alpha)\, \Gamma(\frac{1}{2}\alpha + \frac{1}{2})} \left[\int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \int_0^\rho r^\alpha \cos^\alpha\theta \,\Sigma\, l_i \frac{\partial f_{p-1}}{\partial x_i}\, dr\, d\theta\, d\Omega_n \right.$$
$$+ \int_{\Omega_n} \left\{ \int_0^\rho f_{p-1}(T - r, x_i)\, r^{\alpha-1}\, dr \right\}_{\theta=0} d\Omega_n + \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \sigma\rho^\alpha \cos^\alpha\theta\, \bar{f}_{p-1}\, d\theta\, d\Omega_n$$
$$+ \int_{\Omega_n} \int_0^{\frac{1}{2}\pi} \rho^\alpha \cos^\alpha\theta \left\{ \sigma\rho \sin\theta \frac{\partial g_{p-1}}{\partial\rho} \right.$$
$$\left.\left. + (1 + \sigma\sin\theta)\Sigma(\xi_i - X_i)\frac{\partial g_{p-1}}{\partial \xi_i} + \alpha\sigma\, g_{p-1}\sin\theta + g_{p-1} \right\} d\theta\, d\Omega_n \right]$$

and this provides the analytical continuation into Rl $\alpha > 0$. When $\alpha \to +0$, all the terms inside the square brackets, except the second, remain finite, and so contribute nothing to the limit of $I^\alpha f$. As for the second term, we recall that ρ is a continuous function of θ and ϕ with positive lower bound a, and the part of the range of integration with respect to r between a and ρ again contributes nothing. Hence, since $x_i = X_i$ when $\theta = 0$,

$$\lim_{\alpha \to +0} I^\alpha f(T, X_i)$$

$$= \lim_{\alpha \to +0} \frac{\alpha}{\pi^{p-\frac{1}{2}}\, 2^{\alpha+p}\, \Gamma(\frac{1}{2}\alpha + \frac{1}{2})\Gamma(\frac{1}{2}\alpha + 1)} \int_{\Omega_n} \int_0^a f_{p-1}(T - r, X_i)\, r^{\alpha-1}\, dr\, d\Omega_n$$

$$= \frac{\Omega_n}{2^p \pi^p} \lim_{\alpha \to +0} \alpha \int_0^a f_{p-1}(T - r, X_i)\, r^{\alpha-1}\, dr,$$

the integrand now being independent of the angle-variables ϕ.

To calculate this limit, we use the result that, if $g(x)$ is continuous at 0,

$$\lim_{\alpha \to +0} \alpha \int_0^a g(x)\, x^{\alpha-1}\, dx = g(0).$$

Hence we have

$$\lim_{\alpha \to +0} I^\alpha f(T, X_i) = \frac{\Omega_{2p}}{2^p \pi^p} f_{p-1}(T, X_i) = f(T, X_i)$$

by (2.9) and (5.3). We have thus proved that

$$\lim_{\alpha \to +0} I^\alpha f(T, X_i) = f(T, X_i)$$

when $m = 2p + 1$ provided that the partial derivatives of f and S of order p are continuous. The results of § 5 and § 6 agree with those of Fremberg.

§ 7. *Retarded potentials.*

The transformation (2.4) of Riesz's integral leads to an interesting generalisation of the ordinary retarded potential. Let us suppose that $m \geqq 4$, that f and $\partial f / \partial t$ are continuous and that S is the hyperplane $t = -a$. Then, when $\mathrm{Rl}\,\alpha > m - 2$,

$$I^\alpha f = \frac{1}{H_m(\alpha)} \int_V f(T - r, x_i)\, \frac{x_m^{\alpha-m+1}}{r}\, dx_1\, dx_2 \ldots dx_m,$$

where integration is over the hypervolume V bounded by $x_m = 0$, $r = T + a$. We shall denote the boundary of V by Σ and the direction-cosines of the outward normal by $(\lambda_1, \ldots \lambda_m)$.

Integrating by parts, we have

$$I^\alpha f = \frac{1}{(\alpha - m + 2)\, H_m(\alpha)} \left\{ \int_\Sigma f(T - r, x_i)\, \frac{x_m^{\alpha-m+2}}{r}\, \lambda_m\, d\Sigma \right.$$
$$\left. - \int_V x_m^{\alpha-m+2}\, \frac{\partial}{\partial x_m}\left(\frac{f}{r}\right) dx_1\, dx_2 \ldots dx_m \right\}.$$

But since Rl $a > m - 2$, the portion Σ_1 of Σ on which x_m vanishes makes no contribution, and we may replace integration over Σ by integration over Σ_2, the curved part of Σ. The resulting formula provides the analytical continuation into Rl$a > m - 4$, as may be seen by introducing spherical polar coordinates.

In particular, we have

$$I^{m-2} f = \frac{1}{2^{m-2} \pi^{\frac{1}{2}m-1} \Gamma(\frac{1}{2}m-1)} \left[\int_{\Sigma_2} \frac{f(T-r, x_i)}{r} \lambda_m \, d\Sigma \right.$$
$$\left. - \int_{\Gamma} \frac{\partial}{\partial x_m} \left\{ \frac{f(T-r, x_i)}{r} \right\} dx_1 \, dx_2 \ldots dx^m \right].$$

If we integrate by parts again and remember that $\lambda_m = -1$ on Σ_1, we obtain

$$I^{m-2} f = \frac{1}{2^{m-2} \pi^{\frac{1}{2}m-1} \Gamma(\frac{1}{2}m-1)} \int_{\Sigma_1} \frac{f(T-r, x_i)}{r} \, d\Sigma.$$

On Σ_1, $d\Sigma = dx_1 \, dx_2 \ldots dx_n$ and integration is over

$$r^2 = \sum_1^n (x_i - X_i)^2 \leqq (T + a)^2.$$

Lastly, if we make $a \to +\infty$, we have

$$I^{m-2} f(T, X_i) = \frac{1}{2^{m-2} \pi^{\frac{1}{2}m-1} \Gamma(\frac{1}{2}m-1)} \int \frac{f(T-r, x_i)}{r} \, dx_1 \, dx_2 \ldots dx_n,$$

where integration is over the whole n-dimensional space. This formula is a generalisation of the ordinary retarded potential, to which it reduces when $m = 4$. For in the latter case we have

$$I^2 f = \frac{1}{4\pi} \int \frac{f(T-r, x_i)}{r} \, dx_1 \, dx_2 \, dx_3$$

which is a solution of $Lu = f$, since $LI^2 f = I^0 f = f$.

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