ON THE RELATION BETWEEN BOUNDEDNESS AND OSCILLATION OF DIFFERENTIAL EQUATIONS OF SECOND ORDER

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1. In this paper we are dealing with differential equations of the forms:

$$(E_i) \qquad \ddot{x} + p_i(t)g_i(x, \dot{x}) = 0 , \quad i = 1, 2 ,$$

where the functions p_i are positive.

By a solution of an equation of the above forms, we mean a function $x(t) \in C^2[c, +\infty)$ where c is a non-negative constant, which satisfies the corresponding equation on the whole interval $[c, +\infty)$. By an oscillatory solution of (E_i), we mean a solution with arbitrarily large zeros.

In the second section (g_i homogeneous with respect to both variables together) we give a "semi-comparison" theorem relating the character of the solutions of the equation (E_1) to those of the equation (E_2), and in the third section we extend the results of the second section to the case $g_i(x, y) = g_i(x)$ where the g_i's are not necessarily homogeneous. Sufficient smoothness of the functions p_i, g_i , $i = 1, 2$, for the existence of solutions on an interval of the form $[c, +\infty)$, will be assumed without mention.

2. THEOREM 1. Consider the differential equation (E_i) $i = 1, 2$ with the following assumptions:

(1, i) $p_i : I \to \mathbb{R}_+$, $I = [t_0, +\infty)$, $t_0 \geq 0$, $\mathbb{R}_+ = (0, +\infty)$, continuous and such that $p_1(t) \geq p_2(t)$ for every $t \in I$;

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(1, ii) $g_i : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $\mathcal{R} = (-\infty, +\infty)$, continuous, $xg_i(x, y) > 0$ for every $x \neq 0$ and $g_i(\lambda x, \lambda y) = \lambda^{2n_i+1} g_i(x, y)$ for any $(\lambda, x, y) \in \mathcal{R}^3$, where n_i are positive integers;

(1, iii) $g_2(1, y) \leq k_2$ for every $y \in \mathcal{R}_+$, where k_2 is a positive constant;

(1, iv) there exists an oscillatory solution $y(t) \not\equiv 0$ of (E_2) which is bounded on I (i.e. there exists a constant $L > 0$ such that $|y(t)| \leq L$, $t \in I$);

then every solution of (E_1) is bounded or oscillatory.

Proof. Suppose that there exists a solution $x(t)$ of (E_1) which is non-oscillatory. Then without any loss of generality, we may (and do) assume that $x(t)$ is defined and positive on I. Now, if $\dot{x}(t_1) < 0$ for some $t_1 \geq t_0$, then since $\ddot{x} < 0$ for every $t \in [t_1, +\infty)$, we must have:

$$(1) \qquad x(t) = x(t_1) + \int_{t_1}^{t} \dot{x}(s)ds \leq x(t_1) + \dot{x}(t_1)(t-t_1) \to -\infty \text{ as } t \to +\infty,$$

a contradiction. Thus, $x(t)$ is strictly increasing on I, while its derivative $\dot{x}(t)$ is positive and strictly decreasing on I. Here we distinguish two cases:

Case I. $\lim_{t\to+\infty} x(t) < +\infty$ and $\lim_{t\to+\infty} \dot{x}(t) = \alpha \geq 0$. Then if $\alpha > 0$ we obtain

$$(2) \qquad x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s)ds \geq x(t_0) + \alpha(t-t_0) \to +\infty \text{ as } t \to +\infty,$$

a contradiction. Thus $\alpha = 0$, and consequently $\lim_{t\to+\infty} \dot{x}(t)/x(t) = 0$.

Case II. $\lim_{t\to+\infty} x(t) = +\infty$. Then since $\dot{x}(t)$ is bounded on I, $\lim_{t\to+\infty} \dot{x}(t)/x(t) = 0$.

Now, by use of the continuity of g_1 and the fact that $\lim_{t\to+\infty} \dot{x}(t)/x(t) = 0$, given a fixed positive $\varepsilon < g(1, 0)$, there exists a $t^* \geq t_0$ such that:

(3) $0 < k_1 = g_1(1, 0) - \varepsilon < g_1(1, x(t)/x(t)) < g_1(1, 0) + \varepsilon$

for every $t \geq t^*$.

Since the function $y(t)$ is oscillatory on I, there exists an interval $(a_1, b_1)(t^* < a_1 < b_1)$ such that: $y(a_1) = y(b_1) = 0$, $\dot{y}(a_1) > 0$, $\dot{y}(b_1) < 0$ and $y(t) > 0$ for every $t \in (a_1, b_1)$. From (E_i) $i = 1, 2$, after multiplication of (E_1) by $y(t)$ and (E_2) by $x(t)$, $t \in (a_1, b_1)$, we find

$$x(b_1)\dot{y}(b_1) - x(a_1)\dot{y}(a_1) = \int_{a_1}^{b_1} xy \left[p_1(t) \frac{g_1(x, \dot{x})}{x} - p_2(t) \frac{g_2(y, \dot{y})}{y} \right] dt$$

(4)

$$= \int_{a_1}^{b_1} xy \left[p_1(t) x^{2n_1} g_1(1, \dot{x}/x) - p_2(t) y^{2n_2} g_2(1, \dot{y}/y) \right] dt$$

The first member of (4) is negative, so that there must exist a point $t_1 > t^*$ such that $t_1 \in (a_1, b_1)$ and

(5) $$p_1(t_1) x^{2n_1}(t_1) \left[g_1(1, \dot{x}/x) \right]_{t_1} < p_2(t_1) y^{2n_2}(t_1) \left[g_2(1, \dot{y}/y) \right]_{t_1} .$$

Now, taking into account our assumptions, we obtain

(6) $$x(t_1) < C_1 y^{n_2/n_1}(t_1) \quad \text{where} \quad C_1 = (k_2/k_1)^{1/2n_1} > 0.$$

Proceeding in the same way, we construct a sequence of points $\{t_n\}$, $t_n \in (a_n, b_n)$ where (a_n, b_n) are suitable intervals, $b_n < +\infty$, such that $\lim_{n\to\infty} t_n = +\infty$, $y(t_n) > 0$ and

(7) $$x^{2n_1}(t_n) < C_n y^{2n_2}(t_n) \qquad n = 1, 2, \ldots$$

where C_n are positive constants. It follows that $\liminf_{t\to+\infty} x(t) \leq L$, which implies that $x(t) \leq L$, $t \in I$. Thus, every non-oscillatory solution of (E_1) is bounded on I.

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COROLLARY. Let the equations (E_i) $i = 1, 2$, be such that:

(i) the assumptions (1, i) - (1, iii) are satisfied;

(ii) there exists a solution $y(t) \not\equiv 0$, $t \in I$ of (E_2) which is oscillatory on I and tending to zero as $t \to \infty$; then every non-trivial solution of (E_1) is oscillatory.

Proof. The proof can be easily derived from Theorem 1, for in this case (7) implies $\liminf_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(t) = 0$, which contradicts the increasing character of $x(t)$.

3. THEOREM 2. Consider the equations (E_i) $i = 1, 2$, under the following assumptions:

(2, i) $p_i(t)$ as in (1, i) of Theorem 1;

(2, ii) $g_i(x, y) \equiv g_i(x)$: $\mathbb{R} \to \mathbb{R}$, continuous, $x g_i(x) > 0$ for $x \neq 0$, $g_i(-x) = -g_i(x)$, and

$$\lim_{x\to 0} g_i(x)/x = 0, \quad \lim_{x\to+\infty} g_1(x)/x = +\infty;$$

(2, iii) there exists a bounded oscillatory solution $y(t) \not\equiv 0$ of (E_2); then every solution of (E_1) is bounded or oscillatory.

Proof. Assume the existence of a solution $x(t)$, $t \in I$ of (E_1), which is positive on I. Let $y(t) \not\equiv 0$ be a bounded oscillatory solution of (E_2). Then by (4) of Theorem 1, we have

$$(8) \qquad x(b_1)\dot{y}(b_1) - x(a_1)\dot{y}(a_1) = \int_{a_1}^{b_1} xy[p_1(t)g_1(x)/x - p_2(t)g_2(y)/y]\, dt.$$

The left hand member of (8) is negative, so that there exists a point $t_1 \in (a_1, b_1)$ for which we have

$$(9) \qquad p_1(t_1)g_1(x(t_1))/x(t_1) < p_2(t_1)g_2(y(t_1))/y(t_1),$$

from which it follows that:

$$(10) \qquad g_1(x(t_1))/x(t_1) < g_2(y(t_1))/y(t_1);$$

thus, continuing in the same manner, we construct a sequence of points $\{t_n\}$ $n = 1, 2, \ldots,$ such that:

$$(11) \qquad g_1(x(t_n))/x(t_n) < g_2(y(t_n))/y(t_n),$$

and $\lim_{n\to\infty} t_n = +\infty$.

Since every eventually positive solution of (E_1) is strictly increasing, (11) implies that $\lim_{t\to+\infty} x(t) < +\infty$.

Thus, every non-oscillatory solution of (E_1) is bounded.

REMARK. If instead of $\lim_{x\to\infty} g_1(x)/x = +\infty$, we suppose in Theorem 2 that $\liminf_{x\to\infty} g_1(x)/x > 0$ and $y(t) \to 0$ as $t \to +\infty$, then it is obvious that every solution of (E_1) must be oscillatory since (11) implies now $\lim_{n\to\infty} x(t_n) = 0$.

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