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EISENSTEIN SERIES TO THE TREDECIC BASE

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Abstract

We employ a modular method to establish the new result that two types of Eisenstein series to the tredecic base may be parametrised in terms of the eta quotients $\eta^{13}(\tau)/\eta(13\tau)$ and $\eta^2(13\tau)/\eta^2(\tau)$. The method can also be used to give short and simple proofs for the analogous cubic, quintic and septic theories.

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1. Introduction

Let $q = e^{2\pi i\tau}$ where $\operatorname{Im}\tau > 0$. Dedekind's eta function is defined by

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j).$$

For any positive integer m, let η_m be defined by

$$\eta_m = \eta(m\tau) = q^{m/24}\prod_{j=1}^{\infty}(1-q^{mj}).$$

The identity

$$\sum_{j=1}^{\infty}\frac{j(q^j-q^{2j}-q^{3j}+q^{4j})}{1-q^{5j}} = \frac{\eta_5^5}{\eta_1} \tag{1.1}$$

occurs on several pages in Ramanujan's lost notebook [17, pages 139, 354, 357]. It is famous for being used by Ramanujan to prove the identity

$$\sum_{n=0}^{\infty}p(5n+4)q^n = 5\prod_{j=1}^{\infty}\frac{(1-q^{5j})^5}{(1-q^j)^6}$$

where $p(n)$ is the number of partitions of n. See [1, page 144] for more information and references to proofs of (1.1). Another proof of (1.1) has been given by Venkatachaliengar [18, pages 50–53].

In unpublished work, Z.-G. Liu and R. P. Lewis discovered that

$$\sum_{j=1}^{\infty} \frac{j^3(q^j - q^{2j} - q^{3j} + q^{4j})}{1 - q^{5j}} = \frac{\eta_5^{10}}{\eta_1^2} \left(\frac{\eta_1^6}{\eta_5^6} + 22 + 125 \frac{\eta_5^6}{\eta_1^6} \right)^{1/2}. \tag{1.2}$$

Examples (1.1) and (1.2) prompted Chan and Liu [7] to seek and prove a generalisation that involves the series

$$\sum_{j=1}^{\infty} \frac{j^{2k-1}(q^j - q^{2j} - q^{3j} + q^{4j})}{1 - q^{5j}} = \sum_{j=1}^{\infty} \frac{j^{2k-1}}{1 - q^{5j}} \sum_{\ell=1}^{4} \left(\frac{\ell}{5} \right) q^{j\ell} \tag{1.3}$$

for any positive integer k, where $(\cdot/p)$ is the Legendre symbol, and eta quotients.

The goal of this work is to establish level 13 analogues of (1.1)–(1.3). For example, the level 13 analogue of (1.1), proved in [11, (25 i, ii)], is

$$\sum_{j=1}^{\infty} \frac{j(q^j - q^{2j} + q^{3j} + q^{4j} - q^{5j} - q^{6j} - q^{7j} - q^{8j} + q^{9j} + q^{10j} - q^{11j} + q^{12j})}{1 - q^{13j}}$$
$$= \eta_1 \eta_{13}^3 \left(\frac{\eta_1^2}{\eta_{13}^2} + 5 + 13 \frac{\eta_{13}^2}{\eta_1^2} \right)^{2/3}. \tag{1.4}$$

We will prove the following analogue of (1.2):

$$\sum_{j=1}^{\infty} \frac{j^3(q^j - q^{2j} + q^{3j} + q^{4j} - q^{5j} - q^{6j} - q^{7j} - q^{8j} + q^{9j} + q^{10j} - q^{11j} + q^{12j})}{1 - q^{13j}}$$
$$= \eta_1^3 \eta_{13}^5 \left(\frac{\eta_1^2}{\eta_{13}^2} + 5 + 13 \frac{\eta_{13}^2}{\eta_1^2} \right)^{1/3} \left(\frac{\eta_1^2}{\eta_{13}^2} + 6 + 13 \frac{\eta_{13}^2}{\eta_1^2} \right)^{1/2} \left(\frac{\eta_1^2}{\eta_{13}^2} + 9 + 29 \frac{\eta_{13}^2}{\eta_1^2} \right). \tag{1.5}$$

We shall also establish an analogue of the generalisation of (1.3) for the series

$$\sum_{j=1}^{\infty} \frac{j^{2k-1}(q^j - q^{2j} + q^{3j} + q^{4j} - q^{5j} - q^{6j} - q^{7j} - q^{8j} + q^{9j} + q^{10j} - q^{11j} + q^{12j})}{1 - q^{13j}}$$
$$= \sum_{j=1}^{\infty} \frac{j^{2k-1}}{1 - q^{13j}} \sum_{\ell=1}^{12} \left(\frac{\ell}{13} \right) q^{j\ell}$$

for any positive integer k.

Our method of proof is different from the method used by Chan and Liu [7] and has the advantage that it can be used for any level p for which p is an odd prime and $(p-1)|24$, that is, for $p = 3, 5, 7$ and 13.

2. Statement of results

Let p be an odd prime and let k be a positive integer that satisfies

$$k \equiv \frac{p-1}{2} \pmod 2. \tag{2.1}$$

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The integers p and k are called the level and weight, respectively. The generalised Bernoulli numbers $B_{k,p}$ are defined by

$$\frac{x}{e^{px}-1}\sum_{\ell=1}^{p-1}\left(\frac{\ell}{p}\right)e^{\ell x}=\sum_{k=0}^{\infty}B_{k,p}\frac{x^k}{k!}.$$

The generalised Eisenstein series $E_k^0(\tau;\chi_p)$ and $E_k^\infty(\tau;\chi_p)$ are defined by

$$E_k^0(\tau;\chi_p)=-\delta_{k,1}\frac{B_{k,p}}{2k}+\sum_{j=1}^{\infty}\frac{j^{k-1}}{1-q^{pj}}\sum_{\ell=1}^{p-1}\left(\frac{\ell}{p}\right)q^{j\ell}$$

and

$$E_k^\infty(\tau;\chi_p)=-\frac{B_{k,p}}{2k}+\sum_{j=1}^{\infty}\left(\frac{j}{p}\right)\frac{j^{k-1}q^j}{1-q^j},$$

where $\delta_{m,n}$ is the Kronecker delta function, defined by

$$\delta_{m,n}=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases}$$

We are now ready to state the results. The main result is Theorem 2.4, which is new. To put the results into context, we recount the known results for the cubic, quintic and septic theories in Theorems 2.1–2.3, respectively. We begin with the cubic theory, which was initiated by Ramanujan [16, page 257], and has been developed further in [4–6, 9].

THEOREM 2.1 (Cubic theory). *Let y and z be defined by*

$$y=\frac{\eta_3^{12}}{\eta_1^{12}}=q\prod_{j=1}^{\infty}\frac{(1-q^{3j})^{12}}{(1-q^j)^{12}}\quad\text{and}\quad z=\frac{\eta_1^3}{\eta_3}=\prod_{j=1}^{\infty}\frac{(1-q^j)^3}{(1-q^{3j})}.$$

Let k be a positive integer. There exist polynomials p_{2k-2} and s_{2k-2}, each of degree exactly $2k-2$, such that

$$(E_{2k+1}^0(\tau;\chi_3))^3=z^{6k+3}y^3p_{2k-2}(y)\tag{2.2}$$

and

$$(E_{2k+1}^\infty(\tau;\chi_3))^3=z^{6k+3}s_{2k-2}(y).$$

When $k=0$, we have

$$(E_1^0(\tau;\chi_3))^3=(E_1^\infty(\tau;\chi_3))^3=\frac{z^3}{216}(1+27y).$$

The corresponding quintic theory, established by Chan and Liu [7], is stated in the following theorem.

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THEOREM 2.2 (Quintic theory). *Let y and z be defined by*

$$y = \frac{\eta_5^6}{\eta_1^6} = q\prod_{j=1}^{\infty}\frac{(1-q^{5j})^6}{(1-q^j)^6} \quad \text{and} \quad z = \frac{\eta_1^5}{\eta_5} = \prod_{j=1}^{\infty}\frac{(1-q^j)^5}{(1-q^{5j})}.$$

Let k be a positive integer. There exist polynomials p_{2k-2} and s_{2k-2}, each of degree exactly $2k-2$, such that

$$(E_{2k}^{0}(\tau;\chi_5))^2 = z^{2k}\, y^2\, p_{2k-2}(y) \tag{2.3}$$

and

$$(E_{2k}^{\infty}(\tau;\chi_5))^2 = z^{2k}\, s_{2k-2}(y). \tag{2.4}$$

The septic theory has been studied by Chan and Cooper [8], and is stated next.

THEOREM 2.3 (Septic theory). *Let y and z be defined by*

$$y = \frac{\eta_7^4}{\eta_1^4} = q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^4}{(1-q^j)^4} \quad \text{and} \quad z = \frac{\eta_1^7}{\eta_7} = \prod_{j=1}^{\infty}\frac{(1-q^j)^7}{(1-q^{7j})}.$$

Let k be a positive integer. There exist polynomials p_{4k-1} and s_{4k-1}, each of degree exactly $4k-1$, such that

$$(E_{2k+1}^{0}(\tau;\chi_7))^3 = z^{2k+1} y^3 p_{4k-1}(y)$$

and

$$(E_{2k+1}^{\infty}(\tau;\chi_7))^3 = z^{2k+1} s_{4k-1}(y). \tag{2.5}$$

When $k=0$, we have

$$(E_1^{\infty}(\tau;\chi_7))^3 = (E_1^{0}(\tau;\chi_7))^3 = \frac{z}{8}(1+13y+49y^2). \tag{2.6}$$

The main result of this work is the following tredecic analogue of Theorems 2.1–2.3.

THEOREM 2.4 (Tredecic theory). *Let y and z be defined by*

$$y = \frac{\eta_{13}^2}{\eta_1^2} = q\prod_{j=1}^{\infty}\frac{(1-q^{13j})^2}{(1-q^j)^2} \quad \text{and} \quad z = \frac{\eta_1^{13}}{\eta_{13}} = \prod_{j=1}^{\infty}\frac{(1-q^j)^{13}}{(1-q^{13j})}.$$

Let k be a positive integer. There exist polynomials p_{14k-6} and s_{14k-6}, each of degree exactly $14k-6$, such that

$$(E_{2k}^{0}(\tau;\chi_{13}))^6 = z^{2k}\, y^6\, p_{14k-6}(y) \tag{2.7}$$

and

$$(E_{2k}^{\infty}(\tau;\chi_{13}))^6 = z^{2k}\, s_{14k-6}(y). \tag{2.8}$$

The reader who wishes to skim ahead to the proof of Theorem 2.4 may refer to Section 4.

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3. Discussion

Observe that the cubic and septic theories in Theorems 2.1 and 2.3 have a different form from the quintic and tredecic theories in Theorems 2.2 and 2.4. This is because the Eisenstein series for cubic and septic analogues have odd weight, whereas the quintic and tredecic Eisenstein series have even weight; this goes back to (2.1).

For any fixed value of k, the coefficients in the polynomials in Theorems 2.1–2.4 can be determined by equating coefficients in the q-expansions.

The first few instances of the first part of Theorem 2.1 are equivalent to results given by Ramanujan [16, page 257]. Taking $k = 1, 2, 3$ and 4 in (2.2) and taking the cube root of each result gives

$$\sum_{j=1}^{\infty} \frac{j^2(q^j - q^{2j})}{1 - q^{3j}} = z^3 y,$$
$$\sum_{j=1}^{\infty} \frac{j^4(q^j - q^{2j})}{1 - q^{3j}} = z^5 y(1 + 27y)^{2/3},$$
$$\sum_{j=1}^{\infty} \frac{j^6(q^j - q^{2j})}{1 - q^{3j}} = z^7 y(1 + 63y)(1 + 27y)^{1/3}$$

and

$$\sum_{j=1}^{\infty} \frac{j^8(q^j - q^{2j})}{1 - q^{3j}} = z^9 y(1 + 270y + 7281y^2),$$

where y and z are as for Theorem 2.1. If we let a and x be defined by

$$a = (1 + 27y)^{1/3} z \quad \text{and} \quad x = \frac{27y}{1 + 27y}$$

then the results above may be rephrased as

$$\sum_{j=1}^{\infty} \frac{j^2 q^j}{1 + q^j + q^{2j}} = \frac{a^3 x}{27},$$
$$\sum_{j=1}^{\infty} \frac{j^4 q^j}{1 + q^j + q^{2j}} = \frac{a^5 x}{27},$$
$$\sum_{j=1}^{\infty} \frac{j^6 q^j}{1 + q^j + q^{2j}} = \frac{a^7 x}{27}\left(1 + \frac{4x}{3}\right)$$

and

$$\sum_{j=1}^{\infty} \frac{j^8 q^j}{1 + q^j + q^{2j}} = \frac{a^9 x}{27}\left(1 + 8x + \frac{720x^2}{729}\right),$$

as given by Ramanujan [16, page 257] apart from a misprint in the last result which has been corrected here. These results have been analysed (with the misprint corrected) in [4–6, 9].

Identities (1.1) and (1.2) are special cases of Theorem 2.1: take $k = 1$ and $k = 2$ in (2.3), respectively. Taking $k = 1$ in (2.4) gives another identity of Ramanujan (see [16, Ch. 19, Entry 9(v)], [17, pages 354, 357]):

$$1 - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j} = \prod_{j=1}^{\infty}\frac{(1-q^j)^5}{(1-q^{5j})}.$$

For proofs and references, see [3, pages 257–261], [12, 13].

Identity (2.6) was given by Ramanujan [16, Ch. 21, Entry 5(i)]. The case $k = 1$ of (2.5) was also given by Ramanujan [17, pages 53, 355, 357] and the case $k = 2$ of (2.5) was given by Liu [15, Equation (1.18)].

Let us discuss some specific instances of Theorem 2.4. Taking $k = 1$ in (2.7) gives

$$(E_2^0(\tau;\chi_{13}))^6 = z^2\, y^6\, (1 + 5y + 13y^2)^4.$$

Taking sixth roots and rearranging gives

$$E_2^0(\tau;\chi_{13}) = (z\, y^5)^{1/3}\left(\frac{1}{y} + 5 + 13y\right)^{2/3}.$$

This is equivalent to (1.4). In a similar way, taking $k = 2$ in (2.7) gives (1.5). Taking $k = 1$ in (2.8) leads to

$$1 - \sum_{j=1}^{\infty}\left(\frac{j}{13}\right)\frac{jq^j}{1-q^j} = \eta_1^3\eta_{13}\left(\frac{\eta_1^2}{\eta_{13}^2} + 5 + 13\frac{\eta_{13}^2}{\eta_1^2}\right)^{2/3}.$$

This is the companion to identity (1.4). Similarly, taking $k = 2$ in (2.8) gives the following companion identity to (1.5):

$$\begin{aligned} 29 &+ \sum_{j=1}^{\infty}\left(\frac{j}{13}\right)\frac{j^3q^j}{1-q^j} \\ &= \eta_1^5\eta_{13}^3\left(\frac{\eta_1^2}{\eta_{13}^2} + 5 + 13\frac{\eta_{13}^2}{\eta_1^2}\right)^{1/3}\left(\frac{\eta_1^2}{\eta_{13}^2} + 6 + 13\frac{\eta_{13}^2}{\eta_1^2}\right)^{1/2}\left(29\frac{\eta_1^2}{\eta_{13}^2} + 117 + 169\frac{\eta_{13}^2}{\eta_1^2}\right). \end{aligned}$$

4. Proofs

Let

$$\Gamma = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\right\}$$

and

$$\Gamma_0(p) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \ (\mathrm{mod}\ p)\right\}.$$

We will require the following three lemmas.

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Lemma 4.1. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Then

$$\eta^{24}\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{12}\eta^{24}(\tau)$$

and

$$\eta\Big(\frac{-1}{\tau}\Big) = \sqrt{\frac{\tau}{i}}\,\eta(\tau).$$

Proof. See [2, pages 48–52]. □

Lemma 4.2. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p).$$

Then

$$E_k^0\Big(\frac{a\tau + b}{c\tau + d};\chi_p\Big) = \Big(\frac{d}{p}\Big)(c\tau + d)^k E_k^0(\tau;\chi_p),$$
$$E_k^\infty\Big(\frac{a\tau + b}{c\tau + d};\chi_p\Big) = \Big(\frac{d}{p}\Big)(c\tau + d)^k E_k^\infty(\tau;\chi_p),$$
$$E_k^0\Big(\frac{-1}{p\tau};\chi_p\Big) = \frac{\sqrt{p}}{c_p}\tau^k E_k^\infty(\tau;\chi_p)$$

and

$$E_k^\infty\Big(\frac{-1}{p\tau};\chi_p\Big) = \frac{1}{c_p\sqrt{p}}(p\tau)^k E_k^0(\tau;\chi_p),$$

where

$$c_p = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. See [10, Theorem 6.1] or [14]. □

Lemma 4.3. *Let $f(\tau)$ be analytic and bounded in the upper half plane* $\mathrm{Im}(\tau) > 0$*, and suppose it satisfies the transformation property*

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = f(\tau) \quad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p).$$

Then f is constant.

Proof. See [2, Theorem 4.4, page 79]. □

We are now in a position to prove our new result.

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Proof of Theorem 2.4. We begin by proving (2.7). Recall that $q = e^{2\pi i\tau}$ and let $q_{13} = e^{-2\pi i/13\tau}$. Let f and g be defined by

$$f(\tau) = \frac{(E^0_{2k}(\tau;\chi_{13}))^6}{z^{2k}(q)y^6(q)} \quad \text{and} \quad g(\tau) = y(q).$$

Observe that f and g are analytic in the upper half plane $\mathrm{Im}(\tau) > 0$. By the definitions of $y(q)$ and $z(q)$ we have

$$f(\tau) = \frac{(E^0_{2k}(\tau;\chi_{13}))^6 y^{k-6}(q)}{\eta^{24k}(\tau)}.$$

By Lemmas 4.1 and 4.2 it follows that $f(\tau)$ and $g(\tau)$ are each invariant under $\Gamma_0(13)$. Let us examine the behaviour at $\tau = 0$. By Lemmas 4.1 and 4.2,

$$\begin{aligned} f\Big(\frac{-1}{13\tau}\Big) &= \frac{(E^0_{2k}(-1/13\tau;\chi_{13}))^6 y^{k-6}(q_{13})}{\eta^{24k}(-1/13\tau)} \\ &= \frac{(E^\infty_{2k}(\tau;\chi_{13}))^6}{13^{13k-9}\eta^{24k}(13\tau)y^{k-6}(q)} \\ &= \frac{1}{13^{13k-9}}\Big\{\Big(\frac{B_{2k,13}}{4k}\Big)^6 \frac{1}{q^{14k-6}} + \cdots + O(q)\Big\} \end{aligned}$$

and

$$g\Big(\frac{-1}{13\tau}\Big) = y(q_{13}) = \frac{1}{13y(q)} = \frac{1}{13}\Big\{\frac{1}{q} - 2 + O(q)\Big\}.$$

Thus, $f(-1/13\tau)$ and $g(-1/13\tau)$ have poles of orders $14k-6$ and 1, respectively, at $q = 0$. It follows that there are unique constants, $a_{14k-6}, a_{14k-7}, \ldots, a_0$, that can be determined by successively comparing coefficients of $q^{-(14k-6)}, q^{-(14k-7)}, \ldots, q^{-1}, q^0$, such that $a_{14k-6} \neq 0$ and

$$f\Big(\frac{-1}{13\tau}\Big) - \sum_{i=0}^{14k-6} a_i g^i\Big(\frac{-1}{13\tau}\Big) = O(q).$$

Replacing τ with $-1/13\tau$, we deduce that the function $h(\tau)$ defined by

$$\begin{aligned} h(\tau) &:= f(\tau) - \sum_{i=0}^{14k-6} a_i g^i(\tau) \\ &= \frac{(E^0_{2k}(\tau;\chi_{13}))^6}{z^{2k}(q)y^6(q)} - \sum_{i=0}^{14k-6} a_i g^i(\tau) \end{aligned}$$

is analytic at the cusp $\tau = 0$. Clearly, $h(\tau)$ is also analytic in the upper half plane $\mathrm{Im}(\tau) > 0$, and

$$h(i\infty) = \frac{(E^0_{2k}(i\infty;\chi_{13}))^6}{z^{2k}(0)y^6(0)} - \sum_{i=0}^{14k-6} a_i g^i(i\infty) = 1 - a_0.$$

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It follows that h is bounded. Since h is invariant under $\Gamma_0(13)$, Lemma 4.3 implies that h is constant. Since $h(i\infty) = 1 - a_0$, it follows that

$$h(\tau) = 1 - a_0,$$

that is,

$$\frac{(E^0_{2k}(\tau;\chi_{13}))^6}{z^{2k}(q)y^6(q)} = 1 + \sum_{i=1}^{14k-6} a_i g^i(\tau).$$

Hence, there is a polynomial $p_{14k-6}(y)$ in y of degree exactly $14k - 6$ such that

$$(E^0_{2k}(\tau;\chi_{13}))^6 = z^{2k}y^6 p_{14k-6}(y).$$

This completes the proof of (2.7). Identity (2.8) can be proved by the same method starting with

$$f(\tau) = \frac{(E^\infty_{2k}(\tau;\chi_{13}))^6}{z^{2k}(q)} \quad \text{and} \quad g(\tau) = y(q). \qquad \square$$

The method used in the proof of Theorem 2.4 can also be employed to prove each of Theorems 2.1–2.3 by selecting appropriate functions $f(\tau)$ and $g(\tau)$. For example, to prove the first identity in Theorem 2.3, let

$$f(\tau) = \frac{(E^0_{2k+1}(\tau;\chi_7))^3}{z^{2k+1}(q)y^3(q)} \quad \text{and} \quad g(\tau) = y(q).$$

Then follow the steps in the proof of (2.7). The essence of this method is that $g(\tau)$ is a generator for the function field of $\Gamma_0(p)$ and $f(\tau)$ is modular on $\Gamma_0(p)$ with no poles in its fundamental domain. Therefore $f(\tau)$ must be a polynomial in $g(\tau)$.

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