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TOEPLITZ OPERATORS ON BERGMAN SPACES

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1. Introduction and definitions. Let G be a bounded, open, connected, non-empty subset of the complex plane $\mathbf{C}$. We put the usual two dimensional (Lebesgue) area measure on G and consider the Hilbert space $L^2(G)$ that consists of the complex-valued, measurable functions defined on G that are square integrable. The inner product on $L^2(G)$ is given by $\langle h, g\rangle = \int_G h\bar{g}$; the norm $\|h\|_2$ of a function h in $L^2(G)$ is given by $\|h\|_2 = (\int_G |h|^2)^{1/2}$.

The Bergman space of G, denoted $L_a^2(G)$, is the set of functions in $L^2(G)$ that are analytic on G. The Bergman space $L_a^2(G)$ is actually a closed subspace of $L^2(G)$ (see [**12**, Section 1.4]) and thus it is a Hilbert space.

Let $\bar{G}$ denote the closure of G and let $C(\bar{G})$ denote the set of continuous, complex-valued functions defined on $\bar{G}$. For $f \in C(\bar{G})$, the norm $\|f\|_\infty$ is given by

$$\|f\|_\infty = \sup\{|f(z)|: \quad z \in G\}.$$

Let P denote the orthogonal projection of $L^2(G)$ onto $L_a^2(G)$. For $f \in C(\bar{G})$ the Toeplitz operator with symbol f, denoted T_f, is the linear map from $L_a^2(G)$ to $L_a^2(G)$ defined by $T_f h = P(fh)$. It is clear that T_f is a bounded operator and $\|T_f\| \leqq \|f\|_\infty$. For the special case where f is the function $f(z) = z$ (we shall call this function z), the Toeplitz operator T_z is just multiplication by z on $L_a^2(G)$. The operator T_z is called the Bergman shift on G. (This terminology arises from the special case in which G is the open unit disk, because then T_z is unitarily equivalent to a unilateral weighted shift.)

Let $\mathscr{B}(L_a^2(G))$ denote the Banach algebra of bounded linear operators from $L_a^2(G)$ to $L_a^2(G)$. An operator $T \in \mathscr{B}(L_a^2(G))$ is called Fredholm if the range of T is closed, the kernel of T has finite dimension, and the range of T has finite codimension. Letting $\mathscr{K}(G)$ denote the ideal of compact operators from $L_a^2(G)$ to $L_a^2(G)$, it turns out that T is Fredholm if and only if $T + \mathscr{K}(G)$ is invertible in $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$. The essential spectrum of T, denoted $\sigma_e(T)$, is defined to be the set of complex numbers λ such that $T - \lambda$ is not Fredholm. Knowing the essential spectrum of an operator often helps give an idea of what the operator looks like;

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conversely, trying to determine the essential spectrum of a concrete operator often leads to interesting problems in analysis.

This paper is a study of Toeplitz operators whose symbol is in $C(\bar{G})$. Among other results, we answer the following questions: What is the essential spectrum of T_f? When is T_f compact? The answers to these questions involve eliminating those boundary points of G at which no function in $L_a^2(G)$ has a singularity. More precisely, a point $\lambda \in \partial G$ is said to be removable with respect to $L_a^2(G)$ if there exists an open neighborhood V of λ such that every function in $L_a^2(G)$ can be extended to an analytic function defined on $G \cup V$. The set of all points of ∂G which are removable with respect to $L_a^2(G)$ is denoted by $\partial_{2-r}G$. The Bergman essential boundary of G, denoted $\partial_{2-e}G$, is the set of all points of ∂G which are not removable with respect to $L_a^2(G)$; so

$$\partial_{2-e}G = \partial G \sim \partial_{2-r}G.$$

If G is finitely connected, then the Bergman essential boundary of G is just $\partial G \sim \{$isolated points of $\partial G\}$ (see Propositions 1 and 14).

Here is an outline of the main results of the paper: In Theorem 5 we determine the essential spectrum of the Bergman shift T_z. (Actually, this is a special case of Corollary 10, but Theorem 5 is a key step in the chain of results that leads to Corollary 10.) In Theorem 7 we determine precisely which Toeplitz operators with symbol in $C(\bar{G})$ are compact. Theorem 9 gives a description of the C^*-algebra generated by $\{T_f: f \in C(\bar{G})\}$. In Corollary 10 we find the essential spectrum of an arbitrary Toeplitz operator with symbol in $C(\bar{G})$. Theorem 16 gives a description of the Bergman essential boundary $\partial_{2-e}G$ in terms of logarithmic capacity.

Note that it is easy to determine the spectrum of the Bergman shift T_z (it is just $\bar{G}$), but that it is hard to determine the essential spectrum of the same operator (see Theorem 5). However in studying T_f for an arbitrary function $f \in C(\bar{G})$, Corollaries 10 and 12, when compared to Examples 11 and 13, show that it is far more natural to work in $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$ than in $\mathscr{B}(L_a^2(G))$.

For the special case where G is the open unit disk, Theorems 7 and 9 (and their corollaries) were proved by Coburn [**10**, Lemma 2 and Theorem 1]. Coburn's proof uses the explicit orthonormal basis and the explicit reproducing kernels that it is possible to write down when G is the open unit disk; thus his proofs do not carry over to more general regions. Also, when G is the open unit disk, then $\partial G = \partial_{2-e}G$, and thus Coburn had no need to define the concept of the Bergman essential boundary. For additional results when G is the open unit disk, see [**20**], [**21**], and [**18**].

For $\lambda \in \mathbf{C}$, we will let $B(\lambda; \delta)$ denote the open disk centered at λ which has radius δ.

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2. Spectral properties. The main results of the paper are in this section. We prove that T_f is compact if and only if $f\,|\partial_{2-e}G = 0$ (Theorem 7), that the abelianization of the C^*-algebra generated by the Toeplitz operators is $C(\partial_{2-e}G)$ (Theorem 9), and that the essential spectrum of T_f is $f(\partial_{2-e}G)$ (Corollary 10). None of these results is particularly surprising, but the proofs are far from trivial. Part of the difficulty lies in proving that the essential spectrum of the Bergman shift T_z is what we guess it should be (Theorem 5). We note that the proof of Theorem 9 depends partly on Theorem 7, and that the proof of Theorem 7 depends partly on Theorem 5.

Before beginning the proof of Theorem 5, we need three propositions concerning the Bergman essential boundary $\partial_{2-e}G$ (or its complement $\partial_{2-r}G$ in ∂G). A more complete description of the Bergman essential boundary is given in Section 3.

A bounded analytic function cannot have an isolated singularity. The following proposition says that the same is true if the analytic function is merely square integrable.

PROPOSITION 1. *If λ is an isolated point of ∂G, then $\lambda \in \partial_{2-r}G$.*

Proof. If λ is an isolated point of ∂G, then there exists a positive number δ such that $B(\lambda;\delta) \sim \{\lambda\} \subset G$. Suppose that $h \in L_a^2(G)$. Then h has a Laurent series expansion in $B(\lambda;\delta) \sim \{\lambda\}$;

$$h(z) = \sum_{-\infty}^{\infty} a_n (z-\lambda)^n.$$

Now

$$\infty > \int_{B(\lambda;\delta)\sim\{\lambda\}} |h|^2 = \int_0^{\delta} r \int_0^{2\pi} \left| \sum_{-\infty}^{\infty} a_n r^n e^{in\theta} \right|^2 d\theta dr$$
$$= 2\pi \sum_{-\infty}^{\infty} |a_n|^2 \int_0^{\delta} r^{2n+1} dr.$$

Since $\int_0^{\delta} r^{2n+1}\,dr = \infty$ if $n < 0$, the above inequality shows that $a_n = 0$ for all $n < 0$. Thus h has an analytic extension to $B(\lambda;\delta)$ and so $\lambda \in \partial_{2-r}G$.

It is clear from the definition that $\partial_{2-r}G$ is a relatively open subset of ∂G, and thus $\partial_{2-e}G$ is compact. In particular, $\partial_{2-r}G$ is a measurable subset of the plane. It is possible for the boundary of an open set in the plane to have positive area. The following proposition shows that the removable part of the boundary always has zero area.

PROPOSITION 2. *$\partial_{2-r}G$ has zero area.*

Proof. Let K be a compact subset of $\partial_{2-r}G$. Suppose that K had positive area. Then (see [**14**, p. 2]) there would exist a non-constant bounded analytic function h defined on $\mathbf{C} \sim K$. In particular, $h|G \in L_a^2(G)$, and so h would extend to a non-constant bounded analytic function defined on all of $\mathbf{C}$. This contradicts Liouville's Theorem, and and so K must have zero area. Since every compact subset of $\partial_{2-r}G$ has zero area, we can conclude that ∂_{2-r} has zero area.

The following proposition shows that each removable boundary point must lie in the interior of the closure of G.

PROPOSITION 3. *$\partial\bar{G} \subset \partial_{2-e}G$. Furthermore, $G \cup \partial_{2-r}G$ is an open subset of* $\mathbf{C}$.

Proof. Let $\lambda \in \partial\bar{G}$. Let V be an open neighborhood of λ. Then V contains a point γ which is not in $\bar{G}$. The function $(z - \gamma)^{-1}|G$ is in $L_a^2(G)$ but clearly it cannot be analytically extended to $G \cup V$. Thus $\lambda \in \partial_{2-e}G$ and so $\partial\bar{G} \subset \partial_{2-e}G$.

Let $\lambda \in \partial_{2-r}G$. Since $\partial_{2-r}G$ is a relatively open subset of ∂G, there is a positive number δ such that

$$B(\lambda; \delta) \cap \partial G \subset \partial_{2-r}G.$$

Since $\partial\bar{G} \subset \partial_{2-e}G$, we know that λ is in the interior of $\bar{G}$. Thus we can assume that δ was chosen small enough so that we also have

$$B(\lambda; \delta) \subset \bar{G} = G \cup \partial G.$$

It is now clear that

$$B(\lambda; \delta) \subset G \cup \partial_{2-r}G.$$

Thus $G \cup \partial_{2-r}G$ is an open subset of $\mathbf{C}$.

Since $G \cup \partial_{2-r}G$ is open (Proposition 3), it makes sense to consider $L_a^2(G \cup \partial_{2-r}G)$. Proposition 2 and the definition of $\partial_{2-r}G$ show that

$$L_a^2(G) = L_a^2(G \cup \partial_{2-r}G),$$

where the equality means that there is an obvious isometry between the two spaces. For $\lambda \in G \cup \partial_{2-r}G$, the linear functional on $L_a^2(G \cup \partial_{2-r}G) = L_a^2(G)$ which takes h to $h(\lambda)$ is bounded (see [**8**, p. 5]).

The following lemma is used in the proof of Theorem 5. Later, we will see that the conclusion of Lemma 4 can be strengthened. In fact, if $0 \in \partial G$ and T_z has closed range, then $1/z \notin L_a^2(G)$; see the remark following the proof of Corollary 6 (which depends on Lemma 4).

LEMMA 4. *Suppose that $0 \in \partial G$ and that T_z has closed range. Then there exists a positive integer n such that $z^{-n} \notin L_a^2(G)$.*

Proof. Since T_z is an injective operator which has closed range there is a constant $c > 0$ such that

$$c\|h\|_2 \leqq \|T_z h\|_2 \quad \text{for all } h \in L_a^2(G).$$

Let $g \in L^2(G)$ be the function defined on G which is 1 on $B(0; c/2) \cap G$ and 0 elsewhere on G.

Suppose that $z^{-n} \in L_a^2(G)$ for each positive integer n. Then

$$\begin{aligned} |\langle g, 1\rangle| &= |\langle g, T_{z^n} z^{-n}\rangle| \\ &= |\langle T_{z^n}^* g, z^{-n}\rangle| \\ &\leqq \|z^n g\|_2 \, \|z^{-n}\|_2 \\ &\leqq (c/2)^n \, \|g\|_2 \, \|z^{-n}\|_2. \end{aligned}$$

Note the definition of c shows that

$$\|z^{-n}\|_2 \leqq c^{-1}\|T_z z^{-n}\|_2 = c^{-1}\|z^{-n+1}\|_2.$$

Iterating this inequality shows that $\|z^{-n}\|_2 \leqq c^{-n}\|1\|_2$. Applying this estimate for $\|z^{-n}\|_2$ to the above estimate for $|\langle g, 1\rangle|$ shows that

$$|\langle g, 1\rangle| \leqq 2^{-n} \, \|g\|_2 \, \|1\|_2.$$

Letting $n \to \infty$, we see that $\langle g, 1\rangle = 0$. However, it is clear from the definition of g that $\langle g, 1\rangle$ equals the area of $B(0; c/2) \cap G$. But $B(0; c/2) \cap G$ is a non-empty (because $0 \in \partial G$) open set and hence cannot have zero area. This contradiction completes the proof.

If $T \in \mathscr{B}(L_a^2(G))$ is Fredholm, then the index of T is defined to be the dimension of the kernel of T minus the codimension of the range of T. In the following theorem we determine the essential spectrum of the Bergman shift T_z.

THEOREM 5. *The essential spectrum of the Bergman shift on G is the Bergman essential boundary of G; that is, $\sigma_e(T_z) = \partial_{2-e}G$. Furthermore the index of $T_{z-\gamma}$ is -1 if γ is in $G \cup \partial_{2-r}G$, and $T_{z-\gamma}$ is invertible if γ is in $\mathbf{C} \sim \bar{G}$.*

Proof. First suppose that $\lambda \in G \cup \partial_{2-r}G$. Then it is easy to verify that the range of $T_{z-\lambda}$ is equal to the kernel of the linear functional on $L_a^2(G) = L_a^2(G \cup \partial_{2-r}G)$ which sends h to $h(\lambda)$. In particular, the range of $T_{z-\lambda}$ is a closed subspace of $L_a^2(G)$ of codimension 1. Since the kernel of $T_{z-\lambda}$ is $\{0\}$, we conclude that $T_{z-\lambda}$ is Fredholm (with index -1) and so $\lambda \notin \sigma_e(T_z)$. Thus

$$\sigma_e(T_z) \subset \mathbf{C} \sim (G \cup \partial_{2-r}G).$$

If $\lambda \notin \bar{G}$, then clearly $T_{1/(z-\lambda)}$ is an inverse for $T_{z-\lambda}$, and so $T_{z-\lambda}$ is Fredholm. Thus $\sigma_e(T_z) \subset \bar{G}$. Combining the last two inclusions shows that $\sigma_e(T_z) \subset \partial_{2-e}G$.

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The proof of the inclusion in the opposite direction is more difficult. Suppose that $\lambda \notin \sigma_e(T_z)$. We need to prove that $\lambda \notin \partial_{2-e}G$. For convenience, we will assume that $\lambda = 0$; translations of the complex plane show that there is no loss of generality in this assumption. Thus we are assuming that T_z is Fredholm. We can also assume that $0 \in \partial G$, because if $0 \notin \partial G$, then $0 \notin \partial_{2-e}G$ and we are through.

Since T_z is Fredholm, the range of T_z, which we will denote by ran T_z, is closed. Thus by Lemma 4, there exists a smallest non-negative integer N such that $z^{-N} \notin \text{ran } T_z$. (We will see at the end of the proof that $N = 0$.)

Since $z^{-N+1} \in \text{ran } T_z$, there exists a function $g \in L_a^2(G)$ such that $z^{-N+1} = T_z g$. Clearly g must equal z^{-N}; thus $z^{-N} \in L_a^2(G)$. Let ker T_z^* denote the kernel of T_z^*. Since

$$z^{-N} \notin \text{ran } T_z = (\ker T_z^*)^\perp,$$

there exists a function $k \in \ker T_z^*$ such that $\langle z^{-N}, k \rangle = 1$.

For $h \in L_a^2(G)$, let $\hat{h}(0) = \langle h, k \rangle$. The intuition behind the rest of the proof is that if it made any sense to evaluate the function $z^N h(z)$ at $z = 0$, the result should be $\hat{h}(0)$. (To see where this intuition comes from, consider the case where $N = 0$, so $z^N = 1$, and suppose that 0 were in G, rather than in ∂G. Then the element k of ker T_z^* which satisfies $\langle 1, k \rangle = 1$ is just the reproducing kernel associated with point evaluation at 0; that is, $\hat{h}(0) = \langle h, k \rangle$ for all $h \in L_a^2(G)$.) We will use this intuition to find a formula which shows that $z^N h$ has a Taylor series expansion in a neighborhood of 0.

For $h \in L_a^2(G)$, let $\tilde{h}$ be the function in $L_a^2(G)$ such that

$$h - \hat{h}(0)z^{-N} = z\tilde{h}.$$

To see that such a function exists in $L_a^2(G)$, recall that at the beginning of this proof, we showed that $T_{z-\lambda}$ has index -1 for each λ in G. We are assuming that $0 \in \partial G$ and that T_z is Fredholm; since the index of $T_{z-\lambda}$ is a continuous function of λ (where defined), we see that the index of T_z is also -1. Thus ker T_z^* is one dimensional and must consist of scalar multiples of k. Since

$$\langle h - \hat{h}(0)z^{-N}, k \rangle = 0,$$

we conclude that

$$h - \hat{h}(0)z^{-N} \in (\ker T_z^*)^\perp = \text{ran } T_z.$$

Thus there is a function $\tilde{h}$ in $L_a^2(G)$ such that

$$h - \hat{h}(0)z^{-N} = T_z\tilde{h},$$

as promised.

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For $h \in L_a^2(G)$, we now inductively define a sequence $h_0, h_1, h_2, \ldots$ in $L_a^2(G)$ by $h_0 = h$ and $h_n = \tilde{h}_{n-1}$ for $n = 1, 2, \ldots$. Since T_z has closed range and a trivial kernel, there is a constant c such that

$$\|g\|_2 \leqq c\|T_z g\|_2 \text{ for all } g \text{ in } L_a^2(G).$$

Thus

$$\|h_n\|_2 \leqq c\|T_z h_n\|_2 = c\|T_z \tilde{h}_{n-1}\|_2 = c\|h_{n-1} - \hat{h}_{n-1}(0)z^{-N}\|_2$$
$$\leqq \|h_{n-1}\|_2\, c(1 + \|k\|_2\, \|z^{-N}\|_2).$$

Iterating this estimate, and recalling that $h_0 = h$, we see that

$$\|h_n\|_2 \leqq \|h\|_2\, c^n(1 + \|k\|_2\, \|z^{-N}\|_2)^n.$$

The definitions of $\tilde{h}$ and h_1 imply that

$$z^N h = \hat{h}(0) + z z^N h_1.$$

Iterating this equation shows that

$$z^N h = \hat{h}(0) + \hat{h}_1(0)z + \hat{h}_2(0)z^2 + \ldots + \hat{h}_{n-1}(0)z^{n-1} + z^n z^N h_n.$$

To estimate the last term of the above equation, let

$$\delta = \tfrac{1}{2}[c(1 + \|k\|_2\, \|z^{-N}\|_2)]^{-1},$$

and for $z \in G$ let c_z denote the norm of the linear functional on $L_a^2(G)$ which takes h to $h(z)$. For fixed $z \in B(0; \delta) \cap G$, we have

$$|z^n z^N h_n(z)| \leqq \delta^N \delta^n \|h_n\|_2 c_z;$$

the choice of δ and the estimate on $\|h_n\|_2$ show that this quantity goes to 0 as $n \to \infty$. Thus

$$z^N h = \sum_0^\infty \hat{h}_n(0)z^n \quad \text{for each } z \in B(0; \delta) \cap G.$$

The estimate on $\|h_n\|_2$ shows that

$$|\hat{h}_n(0)| \leqq \|k\|_2\, \|h\|_2 c^n(1 + \|k\|_2\, \|z^{-N}\|_2)^n;$$

combining this estimate with the choice of δ shows that $\Sigma_0^\infty \hat{h}_n(0)z^n$ converges pointwise for $z \in B(0; \delta)$. Thus each function $h \in L_a^2(G)$ has an analytic extension to $B(0; \delta) \sim \{0\}$; the extended function (also denoted h) is defined by

$$h(z) = z^{-N} \sum_0^\infty \hat{h}_n(0)z^n \quad \text{for } z \in B(0; \delta) \sim \{0\}.$$

Thus

$$B(0; \delta) \sim \{0\} \subset G \cup \partial_{2-r}G.$$

So if $h \in L_a^2(G)$, then $h|(B(0; \delta) \sim \{0\})$ is in $L_a^2(B(0, \delta) \sim \{0\})$, because $\partial_{2-r}G$ has area zero (by Proposition 2). However, Proposition 1 shows that a function in $L_a^2(B(0;\delta) \sim \{0\})$ extends to $B(0;\delta)$. Thus each function in $L_a^2(G)$ extends to be analytic on $B(0;\delta)$. This shows that $0 \notin \partial_{2-e}G$ (and also that $N = 0$) and the proof of Theorem 5 is complete.

The following corollary will be used in the proof of Theorem 7. To prove it, we need to make use of the notion of a left Fredholm operator. An operator $T \in \mathscr{B}(L_a^2(G))$ is called left Fredholm if the kernel of T is finite dimensional and the range of T is closed. (It turns out that T is left Fredholm if and only if $T + \mathscr{K}(G)$ is left invertible in $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$.) If T is left Fredholm, the index of T is defined to be the dimension of the kernel of T minus the codimension of the range of T. It is clear that if T is left Fredholm and the index of T is not equal to $-\infty$, then T is Fredholm. In the proof of the next corollary we use the following fact: The index mapping from $\{\lambda \in \mathbf{C}: T - \lambda \text{ is left Fredholm}\}$ to $\mathbf{Z} \cup \{-\infty\}$ is continuous. A clean statement of this result is in [**22**, Proposition 1.17]; a proof can be found in [**15**, Theorem *V*. 1.6].

COROLLARY 6. *If* $\lambda \in \partial_{2-e}G$, *then* $T_{z-\lambda}$ *does not have closed range.*

Proof. Let $\lambda \in \partial_{2-e}G$. Clearly $T_{z-\lambda}$ has a trivial kernel. If $T_{z-\lambda}$ had closed range, then $T_{z-\lambda}$ would be left invertible, and hence left Fredholm. Thus the index of $T_{z-\lambda}$ would be well defined. Since the index mapping is continuous and $\lambda \in \partial_{2-e}G$, Theorem 5 would show that the index of $T_{z-\lambda}$ equals -1. In partciular, $T_{z-\lambda}$ would be a Fredholm operator and so $\lambda \notin \sigma_e(T_z)$. However, this contradicts the hard part of Theorem 5. Thus we can conclude that $T_{z-\lambda}$ does not have closed range.

Remark. We can now give a strengthened version of Lemma 4, as promised. If $0 \in \partial G$ and T_z has closed range, then, by Corollary 6, $0 \in \partial_{2-r}G$. The function $1/z$ clearly does not extend to be analytic at 0, and so $1/z \notin L_a^2(G)$. Robert Olin has given an independent proof of this remark, using the Cauchy transform (private communication).

We can now precisely characterize the functions in $C(\bar{G})$ for which the corresponding Toeplitz operator is compact.

THEOREM 7. *Let* $f \in C(\bar{G})$. *Then* T_f *is compact if and only if* $f\,|\partial_{2-e}G = 0$.

Proof. First suppose that $f\,|\partial_{2-e}G = 0$. Let $\epsilon > 0$ and pick $g \in C(\bar{G})$ such that $\|f - g\|_\infty < \epsilon$ and g is 0 on a neighborhood of $\partial_{2-e}G$. We will show that T_g takes every weakly convergent sequence to a norm convergent sequence, and thus T_g is compact. So suppose $h_n \to 0$ weakly in $L_a^2(G) = L_a^2(G \cup \partial_{2-r}G)$. Let

$$K = \overline{\{z \in \bar{G}: \quad g(z) \neq 0\}}.$$

Then K is a compact subset of $G \cup \partial_{2-r}G$ and so $h_n \to 0$ uniformly on K

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(this follows from the Cauchy integral formula and the fact that the point evaluation functions are uniformly bounded on K [**8**, p. 5]). Now

$$\|T_g h_n\|_2 = \|P(gh_n)\|_2 \leqq \|gh_n\|_2 \leqq \|g\|_2 \, \|h_n|K\|_\infty \to 0,$$

and so T_g is compact. Since $\|T_f - T_g\| < \epsilon$ and ϵ can be made arbitrarily small, we can conclude that T_f is compact.

To prove the other direction of this theorem suppose that T_f is a compact operator. Let $\lambda \in \partial_{2-e}G$. We need to show that $f(\lambda) = 0$.

By Corollary 6, $T_{z-\lambda}$ does not have closed range, and so it is not bounded below. Thus there exists a sequence $\{h_n\} \subset L_a^2(G)$ such that

$$\|h_n\|_2 = 1 \quad \text{and} \quad \|T_{z-\lambda}h_n\|_2 \to 0.$$

Passing to a subsequence, we can assume that $h_n \to h$ weakly in $L_a^2(G)$ for some $h \in L_a^2(G)$. Thus $T_{z-\lambda}h_n \to T_{z-\lambda}h$ weakly. However,

$$\|T_{z-\lambda}h_n\|_2 \to 0,$$

so we must have $\|T_{z-\lambda}h\|_2 = 0$. Thus $h = 0$ and so $h_n \to 0$ weakly.

Let $\epsilon > 0$, and let U be neighborhood of λ such that

$$\|(f - f(\lambda))|G \cap U\|_\infty < \epsilon.$$

Then

$$\begin{aligned} \|(f - f(\lambda))h_n\|_2^2 &= \int_{G \cap U} |f - f(\lambda)|^2 |h_n|^2 + \int_{G \sim U} |f - f(\lambda)|^2 |h_n|^2 \\ &\leqq \epsilon^2 + \int_{G \sim U} \left| \frac{f - f(\lambda)}{z - \lambda} \right|^2 |z - \lambda|^2 |h_n|^2 \\ &\leqq \epsilon^2 + \left\| \frac{f - f(\lambda)}{z - \lambda} \middle| G \sim U \right\|_\infty^2 \|T_{z-\lambda}h_n\|_2^2. \end{aligned}$$

For n sufficiently large, the right hand side of the above inequality is less than $2\epsilon^2$; we conclude that

$$\|(f - f(\lambda))h_n\|_2 \to 0 \quad \text{as } n \to \infty.$$

Now

$$\begin{aligned} |f(\lambda)| &= \|f(\lambda)h_n\|_2 \\ &\leqq \|T_f h_n - f(\lambda)h_n\|_2 + \|T_f h_n\|_2 \\ &= \|P(fh_n - f(\lambda)h_n)\|_2 + \|T_f h_n\|_2 \\ &\leqq \|(f - f(\lambda))h_n\|_2 + \|T_f h_n\|_2. \end{aligned}$$

We just saw that $\|(f - f(\lambda))h_n\|_2 \to 0$ as $n \to \infty$. Since T_f is compact and $h_n \to 0$ weakly, we also know that $\|T_f h_n\|_2 \to 0$ as $n \to \infty$. The above inequality thus shows that $f(\lambda) = 0$, completing the proof.

The work of Berger and Shaw on selfcommutators of hyponormal operators is the key ingredient in the proof of the following proposition.

PROPOSITION 8. *Let f and g be functions in $C(\bar{G})$. Then $T_{fg} - T_f T_g$ and $T_f T_g - T_g T_f$ are compact operators.*

Proof. For $f \in C(\bar{G})$, define operators

$$H_f: L_a^2(G) \to L^2(G) \ominus L_a^2(G) \text{ and}$$
$$S_f: L^2(G) \ominus L_a^2(G) \to L^2(G) \ominus L_a^2(G)$$

by

$$H_f h = (1 - P)(fh) \quad \text{and} \quad S_f h = (1 - P)(fh).$$

The operator H_f is called the Hankel operator with symbol f; clearly $\|H_f\| \leqq \|f\|_\infty$. It is easy to verify that the adjoint operator

$$H_f^*: L^2(G) \ominus L_a^2(G) \to L_a^2(G)$$

is defined by $H_f^* h = P(\bar{f}h)$. Straightforward calculations now show that

$$T_{fg} - T_f T_g = H_{\bar{f}}^* H_g,$$
$$H_{fg} = S_f H_g + H_f T_g.$$

Let

$$B = \{f \in C(\bar{G}): H_f \text{ is compact}\}.$$

Clearly B is a closed subspace of $C(\bar{G})$; the above equation for H_{fg} shows that B is, in fact, a closed subalgebra of $C(\bar{G})$. Berger and Shaw [**7**, Theorem 5.1] (also see [**5**, Theorem 7] and [**6**, Corollary 1]) have proved that $T_{\bar{z}} T_z - T_z T_{\bar{z}}$ is a trace class operator. Since every trace class operator is compact, and since

$$T_{\bar{z}} T_z - T_z T_{\bar{z}} = T_{\bar{z}z} - T_z T_{\bar{z}} = H_{\bar{z}}^* H_{\bar{z}},$$

we can conclude that $H_{\bar{z}}$ is compact. Thus $\bar{z} \in B$.

It is obvious from the definition of a Hankel operator that $H_1 = H_z = 0$. So B is a closed subalgebra of $C(\bar{G})$ which contains 1, z, and $\bar{z}$. The Stone-Weierstrass Theorem now implies that $B = C(\bar{G})$, so H_f is compact for every $f \in C(\bar{G})$. If $f, g \in C(\bar{G})$, then our formula for $T_{fg} - T_f T_g$ in terms of Hankel operators now shows that it is compact. Since

$$T_g T_f - T_f T_g = (T_g T_f - T_{gf}) + (T_{fg} - T_f T_g),$$

the proof is complete.

Let $\mathscr{T}(G)$ denote the norm closed subalgebra of $\mathscr{B}(L_a^2(G))$ generated by $\{T_f: f \in C(\bar{G})\}$. The commutator ideal of $\mathscr{T}(G)$ is defined to be the smallest norm closed two sided ideal of $\mathscr{T}(G)$ containing $\{TS - ST: T,$

$S \in \mathscr{T}(G)\}$. The following theorem gives an essentially complete description of the C^*-algebra $\mathscr{T}(G)$.

THEOREM 9. *The commutator ideal of $\mathscr{T}(G)$ is $\mathscr{K}(G)$. Furthermore $\mathscr{T}(G)/\mathscr{K}(G)$ and $C(\partial_{2-e}G)$ are isometrically isomorphic C^*-algebras with an isomorphism that maps $T_f + \mathscr{K}(G)$ to $f|\partial_{2-e}G$ for each $f \in C(\bar{G})$.*

Proof. Suppose that $Q \in \mathscr{B}(L_a^2(G))$ is a projection which commutes with T_z. By [**23**, Lemma 6] (actually, [**23**] assumes that G is the open unit disk, but this assumption is never used), there is an analytic function h defined on G such that Q is multiplication by h. The equation $Q^2 = Q$ implies that $h^2 = h$; since h is analytic this implies that $Q = 0$ or $Q = 1$. In particular, the C^*-algebra $\mathscr{T}(G)$ is irreducible.

By Proposition 8, $T_zT_{\bar{z}} - T_{\bar{z}}T_z$ is compact, and it is easy to see that $T_zT_{\bar{z}} - T_{\bar{z}}T_z \neq 0$. Thus $\mathscr{T}(G)$ is an irreducible C^*-algebra which contains a non-zero compact operator. By [**4**, p. 18, Corollary 2], we can conclude that $\mathscr{K}(G) \subset \mathscr{T}(G)$.

Let $\mathscr{J}(G)$ denote the commutator ideal of $\mathscr{T}(G)$. Proposition 8 implies that $\mathscr{J}(G) \subset \mathscr{K}(G)$. Since $\mathscr{J}(G)$ is a two-sided ideal of $\mathscr{T}(G)$ and $\mathscr{K}(G) \subset \mathscr{T}(G)$, it is clear that $\mathscr{J}(G)$ is a two-sided ideal of $\mathscr{K}(G)$. Furthermore $\mathscr{J}(G)$ is not equal to $\{0\}$ because $0 \neq T_zT_{\bar{z}} - T_{\bar{z}}T_z \in \mathscr{J}(G)$, and so by [**4**, p. 18, Corollary 1], we can conclude that $\mathscr{J}(G) = \mathscr{K}(G)$.

Consider the map α of $C(\bar{G})$ into $\mathscr{T}(G)/\mathscr{K}(G)$ defined by $\alpha(f) = T_f + \mathscr{K}(G)$. By Proposition 8, α is a homomorphism, and hence its range is a subalgebra of $\mathscr{T}(G)/\mathscr{K}(G)$. The definition of $\mathscr{T}(G)$ now implies that $\alpha(C(\bar{G}))$ is dense in $\mathscr{T}(G)/\mathscr{K}(G)$.

Let

$$Z(G) = \{f \in C(\bar{G}): f|\partial_{2-e}G = 0\}.$$

By Theorem 7, the kernel of α is precisely $Z(G)$. Thus there is a homomorphism $\tilde{\alpha}$ from $C(\bar{G})/Z(G)$ into $\mathscr{T}(G)/\mathscr{K}(G)$ defined by

$$\tilde{\alpha}(f + Z(G)) = T_f + \mathscr{K}(G).$$

Now $\tilde{\alpha}$ is an injective C^*-homomorphism, and so it must be an isometry [**11**, Proposition 4.67]. In particular, the range of $\tilde{\alpha}$ must be closed. We already noted that $\alpha(C(\bar{G}))$ is dense in $\mathscr{T}(G)/\mathscr{K}(G)$, and since α and $\tilde{\alpha}$ have the same range, we see that $\tilde{\alpha}$ is a C^*-isomorphism of $C(\bar{G})/Z(G)$ onto $\mathscr{T}(G)/\mathscr{K}(G)$. The proof of the theorem is now completed by noting that there is an obvious C^*-isomorphism of $C(\bar{G})/Z(G)$ onto $C(\partial_{2-e}G)$, namely

$$f + Z(G) \mapsto f|\partial_{2-e}G.$$

We can now determine the essential spectrum of an arbitrary Toeplitz operator whose symbol is in $C(\bar{G})$.

COROLLARY 10. *Let* $f \in C(\bar{G})$. *Then* $\sigma_e(T_f) = f(\partial_{2-e}G)$.

Proof. The spectrum of $f\,|\,\partial_{2-e}G$ in $C(\partial_{2-e}G)$ is $f(\partial_{2-e}G)$. Thus by Theorem 9, the spectrum of $T_f + \mathscr{K}(G)$ in $\mathscr{T}(G)/\mathscr{K}(G)$ is $f(\partial_{2-e}G)$. However, the spectrum of an element of a C^*-algebra does not change when the C^*-algebra is enlarged [**11**, Theorem 4.28], and so the spectrum of $T_f + \mathscr{K}(G)$ in $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$ is $f(\partial_{2-e}G)$.

Corollary 10 might lead one to suspect that if $f \in C(\bar{G})$, then $\sigma(T_f) = f(\bar{G})$; here $\sigma(T_f)$ denotes the spectrum of T_f. By analogy with the theory of Toeplitz operators on the classical Hardy space H^2 (see [**11**, Corollary 7.7]) one might expect to have at least a spectral inclusion theorem: $f(\bar{G}) \subset \sigma(T_f)$. However, Keough [**19**, Theorem 2.1] proves that there is always a function $f \in C(\bar{G})$ such that $f(\bar{G}) \not\subset \sigma(T_f)$. Here we give an explicit example for the case when G is the open unit disk.

EXAMPLE 11. *Suppose that G is the open unit disk. Then there exists a function $f \in C(\bar{G})$ such that neither the inclusion $f(\bar{G}) \subset \sigma(T_f)$ nor the inclusion $\sigma(T_f) \subset f(\bar{G})$* holds.

Proof. Define f on the closed unit disk $\bar{G}$ by

$$f(re^{i\theta}) = \exp\,(2\pi i r^2).$$

Clearly $f\,|\,\partial G = 1$, and so $T_f - 1$ is compact by Theorem 7. In particular, the spectrum of T_f must be countable. Since $f(\bar{G})$ is uncountable, we cannot have $f(\bar{G}) \subset \sigma(T_f)$.

To show that the opposite inclusion also does not hold, note that if n is a non-negative integer, then

$$\langle T_f 1, z^n \rangle = \langle f, z^n \rangle = \int f\bar{z}^n$$
$$= \left(\int_0^1 r^{n+1} \exp\,(2\pi i r^2) dr\right)\left(\int_0^{2\pi} e^{in\theta} d\theta\right).$$

The $d\theta$ integral above equals zero except when $n = 0$. When $n = 0$, the dr integral above equals zero. Thus

$$\langle T_f 1, z^n \rangle = 0 \text{ for all } n \geqq 0.$$

Since for this particular choice of G the linear span of $\{z^n: n \geqq 0\}$ is dense in $L_a^2(G)$, we can conclude that $T_f 1 = 0$. Thus T_f is not invertible and so $0 \in \sigma(T_f)$. Clearly $0 \notin f(\bar{G})$, and thus we do not have $\sigma(T_f) \subset f(\bar{G})$.

The essential norm of an operator is its distance from the compact operators. More precisely, if $T \in \mathscr{B}(L_a^2(G))$, then the essential norm of T, denoted $\|T\|_e$, is defined to be the norm of $T + \mathscr{K}(G)$ in $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$. The following corollary gives a formula for the essential norm of a Toeplitz operator on a Bergman space.

COROLLARY 12. *Let* $f \in C(\bar{G})$. *Then* $\|T_f\|_e = \|f|\partial_{2-e}G\|_\infty$.

Proof. By Proposition 8, $T_f + \mathscr{K}(G)$ is a normal element of $\mathscr{B}(L_a^2(G))/\mathscr{K}(G)$. Thus the norm of $T_f + \mathscr{K}(G)$ is equal to its spectral radius. The result now follows from Corollary 10.

Corollary 12 and analogy with the theory of Toeplitz operators on the classical Hardy space H^2 (see [**11**, Corollary 7.8]) might lead one to suspect that if $f \in C(\bar{G})$, then $\|T_f\| = \|f\|_\infty$. However, Keough [**19**, Theorem 2.1] proves that there is always a function $f \in C(\bar{G})$ such that $\|T_f\| < \|f\|_\infty$. We will give an explicit example which shows, in the case where G is the open unit disk, that $\|T_f\|$ and $\|f\|_\infty$ are not even equivalent norms.

Suppose (temporarily) that G is the open unit disk. Consider the map

$$\beta: C(\bar{G}) \rightarrow \mathscr{B}(L_a^2(G))$$

defined by $\beta(f) = T_f$. If $f \in C(\bar{G})$ and $T_f = 0$, then for all non-negative integers m and n

$$0 = \langle T_f z^m, z^n \rangle = \langle f z^m, z^n \rangle = \int_G f z^m \bar{z}^n.$$

The linear span of $\{z^m \bar{z}^n: m, n \geqq 0\}$ is dense in $C(\bar{G})$. Thus the above equation shows that if $T_f = 0$, then $f = 0$. Thus the map β is injective. The following example shows that β does not have closed range. Note that $\{T_f: f \in C(\bar{G})\} + \mathscr{K}(G)$ is a closed subspace of $\mathscr{B}(L_a^2(G))$ (because by Theorem 9 it is equal to $\mathscr{T}(G)$), and so it is somewhat curious that $\{T_f: f \in C(\bar{G})\}$ is not closed.

EXAMPLE 13. *Suppose that* G *is the open unit disk. Then there does not exist a constant* c *such that* $\|f\|_\infty \leqq c\|T_f\|$ *for every* $f \in C(\bar{G})$.

Proof. Suppose that $c > 1$. Let $b = 1/\sqrt{c}$ and define a function f on the closed unit disk $\bar{G}$ by

$$f(re^{i\theta}) = \begin{cases} 1 - r/b & \text{if } r \leqq b \\ 0 & \text{if } r > b. \end{cases}$$

Clearly $f \in C(\bar{G})$ and $f|\partial G = 0$, so by Theorem 7 T_f is compact. Since f is real-valued, T_f is self-adjoint (it is always true that $T_f^* = T_{\bar{f}}$), so to determine the norm of the compact self-adjoint operator T_f we need only find the largest eigenvalue.

So suppose that λ is an eigenvalue of T_f. Let $h \in L_a^2(G)$ be such that $h \neq 0$ and $T_f h = \lambda h$. In the unit disk G, h has a power series expansion:

$$h(z) = \sum_0^\infty a_n z^n.$$

If $n \geqq 0$ then

$$0 = \langle T_{f-\lambda}h, z^n\rangle = \langle (f-\lambda)h, z^n\rangle$$

$$= \int_0^1 r^{n+1}(f(r) - \lambda)\left[\int_0^{2\pi} h(re^{i\theta})e^{-in\theta}d\theta\right]dr$$

$$= 2\pi a_n \int_0^1 r^{2n+1}(f(r) - \lambda)dr$$

$$= 2\pi a_n\left[\int_0^b r^{2n+1}(1 - r/b)dr - \lambda\int_0^1 r^{2n+1}dr\right]$$

$$= \frac{2\pi a_n}{2n+2}\left[\frac{b^{2n+2}}{2n+3} - \lambda\right].$$

Since $h \neq 0$, there is some $n \geqq 0$ such that $a_n \neq 0$. The above equation then shows that $\lambda = b^{2n+2}/(2n+3)$. Since $0 < b < 1$, this shows that $|\lambda| \leqq b^2/3$. Thus $\|T_f\| \leqq b^2/3$ and so $c\|T_f\| \leqq 1/3$. Since $\|f\|_\infty = 1$, we cannot have $\|f\|_\infty \leqq c\|T_f\|$.

3. The Bergman essential boundary. The main result of this section is Theorem 16, which gives a local necessary and sufficient condition for a boundary point of G to be removable.

The following proposition shows that $\partial_{2-r}G$ is totally disconnected.

PROPOSITION 14. *Let $\lambda \in \partial G$. If the connected component of ∂G containing λ contains more than one point, then $\lambda \in \partial_{2-e}G$.*

Proof. Suppose that the connected component of ∂G containing λ contains more than one point but that $\lambda \in \partial_{2-r}G$. Since $\partial_{2-r}G$ is a relatively open subset of ∂G, there is a positive number δ such that

$$\overline{B(\lambda;\delta)} \cap \partial G \subset \partial_{2-r}G.$$

Let K denote the connected component of $\overline{B(\lambda;\delta)} \cap \partial G$ that contains λ. Then K contains more than one point and so there is a conformal map h of $\mathbf{C} \cup \{\infty\} \sim K$ onto the open unit disk. Clearly $h|G \in L_a^2(G)$, and since $K \subset \partial_{2-r}G$, we see that h extends to a non-constant bounded analytic function defined on all of $\mathbf{C}$. This contradicts Liouville's Theorem, and proves that $\lambda \in \partial_{2-e}G$.

A compact set $K \subset \mathbf{C}$ is said to have zero logarithmic capacity if

$$\sup_{z\in\mathbf{C}}\left\{\int_K \log\frac{1}{|z-w|}\,du(w)\right\} = \infty$$

for every probability measure u supported on K. There are many equivalent definitions scattered throughout the literature.

We will need to use the following lemma. It is just a special case of Theorem 1.4 of [**16**] and also of Theorem B of [**1**], where the differential operator in those theorems is taken to be $\partial/\partial\bar{z}$. To see that the sets of zero logarithmic capacity, as defined above, are the same as the sets of zero capacity defined by the two theorems just referred to, see [**1**, Theorem A], [**16**, Theorem 2.1], and [**17**, Lemma 1 and Lemma 2]. However, an independent proof based on [**9**, page 73] is also possible. This proof is presented for completeness.

LEMMA 15. *Let K be a compact subset of $\mathbf{C}$ and let U be an open subset of $\mathbf{C}$ such that $K \subset U$. Then K has zero logarithmic capacity if and only if every function in $L_a^2(U \sim K)$ has an analytic extension to U.*

Proof. According to [**9**, page 73], K has logarithmic capacity 0 if and only if $L_a^2(\mathbf{C} \sim K) = \{0\}$.

So suppose that every function in $L_a^2(U \sim K)$ has an analytic extension to U. If $f \in L_a^2(\mathbf{C} \sim K)$, then this implies f has an analytic extension to $\mathbf{C}$; that is, f is entire and in $L_a^2(\mathbf{C})$. A power series argument shows that $L_a^2(\mathbf{C}) = \{0\}$. Hence $L_a^2(\mathbf{C} \sim K) = \{0\}$ and, by [**9**, page 73], K has zero logarithmic capacity.

For the converse, suppose that $L_a^2(\mathbf{C} \sim K) = \{0\}$ and let $h \in L_a^2(U \sim K)$. By standard arguments, $h = h_1 + h_2$ where h_1 is analytic on U, h_2 is analytic in the complement of K in the extended plane, and $h_2(\infty) = 0$. The proof will be accomplished by showing that h is analytic on U.

Let $R > 0$, $B = \{z : |z| > R\}$, and let g be analytic on $B \cup \{\infty\}$. By examining the power series development of g at ∞, $g(z) = a_0 + \sum_{n=1}^{\infty} a_n/z^n$, it is easy to show that $g \in L_a^2(B)$ if and only if

$$g(\infty) = a_0 = 0 \text{ and } a_1 = g'(\infty) = 0.$$

With this in mind, note that for w in $\mathbf{C} \sim K$,

$$\lim_{z\to\infty} z\left[\frac{h_2(z) - h_2(w)}{z - w}\right] = -h_2(w).$$

If $h_2 \neq 0$, choose distinct points a and b from $\mathbf{C} \sim K$ such that $h_2(a) \neq 0 \neq h_2(b)$. Put

$$g(z) = \frac{1}{h_2(a)}\left[\frac{h_2(z) - h_2(a)}{z - a}\right] - \frac{1}{h_2(b)}\left[\frac{h_2(z) - h_2(b)}{z - b}\right].$$

It is easy to see that

$$g(\infty) = \lim_{z\to\infty} g(z) = 0 \quad \text{and} \quad g'(\infty) = \lim_{z\to\infty} zg(z) = 0.$$

From the preceding paragraph,

$$g \in L_a^2(\{z : |z| > R\}) \text{ if } K \subseteq \{z : |z| < R\}.$$

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Also, if V is an open set containing K such that $\bar{V} \subset U$, then

$$h_2 = h - h_1 \in L_a^2(V \sim K).$$

Hence $g \in L_a^2(V \sim K)$. It follows that $g \in L_a^2(\mathbf{C} \sim K)$. By hypothesis, $g = 0$.

This implies

$$\frac{1}{h_2(a)}\left[\frac{h_2(z) - h_2(a)}{z - a}\right] = \frac{1}{h_2(b)}\left[\frac{h_2(z) - h_2(b)}{z - b}\right].$$

Solving for $h_2(z)$, we see that h_2 is a rational function with precisely 1 pole, call it c. If $c \notin U$, then h is analytic on U as desired. If $c \in U$, then h is analytic on $U \sim \{c\}$ with a pole at c. But $h \in L_a^2(U \sim K)$ (this forces $c \in K$) and K has zero area, so $h \in L_a^2(U \sim \{c\})$. But by Proposition 1, h extends to be analytic on U.

The reader may wish to compare Lemma 15 with [**3**, Theorem 3], which shows that every bounded analytic function defined on $U \sim K$ can be analytically extended to U if and only if K has zero analytic capacity. A set may have zero analytic capacity without having zero logarithmic capacity; an example is provided by the usual Cantor set (see [**9**, page 31]).

Lemma 15 can be used to give an example of an open set G such that $\partial_{2-r}G$ is uncountable. To do this, let K be an uncountable set with zero logarithmic capacity (for examples, see [**9**, page 31]), let U be an open disk containing K, and let $G = U \sim K$. Then Lemma 15 (and Proposition 14) show that $\partial_{2-r}G = K$.

A set with zero logarithmic capacity is very small. It must have zero area; in fact, it must meet every line and every circle in a set of linear Lebesgue measure zero (see [**2**, p. 29 and Theorem 2.7]). The following theorem says that a boundary point of G is removable if and only if almost every point (in the sense of logarithmic capacity) near it is in G: compare to Proposition 1. Theorem 16 is essentially a local version of Lemma 15.

THEOREM 16. *Suppose* $\lambda \in \partial G$. *Then* $\lambda \in \partial_{2-r}G$ *if and only if there exists a positive number* δ *such that* $\overline{B(\lambda;\delta)} \sim G$ *has zero logarithmic capacity.*

Proof. First suppose that $\lambda \in \partial_{2-r}G$. Then by Proposition 3 there is a positive number δ such that $\overline{B(\lambda;\delta)}$ is contained in G. Since $\partial_{2-r}G$ is a relatively open subset of ∂G, we can choose δ also to satisfy

$$\overline{B(\lambda;\delta)} \cap \partial G \subset \partial_{2-r}G.$$

Thus $\overline{B(\lambda;\delta)} \subset G \cup \partial_{2-r}G$ and so $\overline{B(\lambda;\delta)} \sim G \subset \partial_{2-r}G$. Let $U = G \cup \partial_{2-r}G$ and let $K = \overline{B(\lambda;\delta)} \sim G$. Then U is open and K is a compact

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set contained in U. Since $K \subset \partial_{2-r}G$, every function in $L_a^2(U \sim K)$ has an analytic extension to U. Lemma 15 now allows us to conclude that $\overline{B(\lambda;\delta)} \sim G$ has zero logarithmic capacity.

To prove the other direction of this theorem, suppose that δ is a positive number such that $\overline{B(\lambda;\delta)} \sim G$ has zero logarithmic capacity. Thus $\overline{B(\lambda;\delta)} \sim G$ is totally disconnected. (One way to show this is to use [**3**, Theorem 3] and Lemma 15 to conclude that every set with zero logarithmic capacity is a Painlevé null set and hence must be totally disconnected [**13**, p. 198].) Thus there is an open set $V \subset \mathbf{C}$ such that $\lambda \in V \subset B(\lambda;\delta)$ and $V \cap (\overline{B(\lambda;\delta)} \sim G)$ is compact. Let $U = G \cup V$ and let

$$K = V \cap (\overline{B(\lambda;\delta)} \sim G).$$

Then K is a compact set contained in the open set U. Furthermore $U \sim K = G$. Since K has zero logarithmic capacity, by Lemma 15 we see that every function in $L_a^2(U \sim K) = L_a^2(G)$ has an analytic extension to $U = G \cup V$. Thus $\lambda \in \partial_{2-r}G$ and the proof is complete.

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