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# LEBESGUE MEASURE OF SUM SETS – THE BASIC RESULT FOR COIN-TOSSING

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**Abstract.** Let $\mu_p$ be the distribution of a random variable on the interval [0, 1), each digit of whose binary expansion is 0 or 1 with probability $p$ or $1-p$. Thus $\mu_p = *_{n=1}^{\infty}(p\delta_0 + (1-p)\delta_{\frac{1}{2^n}})$. We show that for any Borel subsets $E$, $F$ of [0, 1) we have

$$\lambda(E+F) \geq \mu_p(E)^{\alpha}\mu_q(F)^{\beta},$$

where $0<\alpha, \beta<1$ with $\alpha\log a + \beta\log b = \log 2$ and $a = [\max\{p, 1-p\}]^{-1}$, $b = [\max\{q, 1-q\}]^{-1}$. Here $\lambda = \mu_{1/2}$ denotes Lebesgue measure.

**1. Introduction.** We define the sum set $E+F$ of subsets $E$, $F$ of [0, 1) by

$$E+F = \{x+y \pmod 1) | x \in E,\ y \in F\}.$$

For many years, the measure of algebraic sums of sets has been of interest to mathematicians. (See for example [**4**], [**5**], [**6**] and [**7**].) This is because the sum of “thin” sets can be “thick”. In fact, in 1947, Marshall Hall, Jr. [**4**] proved that under certain condition, the sum of two Cantor-type set contains an interval.

It took a surprisingly long time to establish precise measure estimates. After contributions by Haydon, Talagrand, Hall and Woodall, the basic symmetric results for the Lebesgue singular measure $\nu$ on the Cantor middle-third set were established independently by Brown and Moran [**2**] and Hajela and Seymour [**5**]. This result states that, for Borel sets $E$ and $F$, we have

$$\lambda(E+F) \geq \nu(E)^{\alpha}\nu(F)^{\alpha},$$

where $\alpha = \log 3/\log 4$.

The first named author set up some analytic inequalities in [**1**] and we developed these further to establish several inequalities for the Lebesgue measure of sum sets where the summands are non-null with respect to singular measures which are uniformly distributed over a set of numbers missing certain digits in their base 3 or base 4 expansions. An account of these can be found in the University of Adelaide Ph.D. thesis of the second named author [**8**]. These results include the basic asymmetric version of the Cantor middle-third case. Namely

$$\lambda(E+F) \geq \nu(E)^{\alpha}\nu(F)^{\beta},$$

provided that $\alpha+\beta \geq \frac{\log 3}{\log 2}$, $3(\alpha^{-1}+\beta^{-1}) \leq 8$ and $\alpha,\ \beta \geq \frac{\log 3}{\log 2}-1$.

In this paper we establish the basic result for the case in which the singular measures are determined by coin-tossing. For $0<p<1$, we let

$$\mu_p = \mathop{*}_{n=1}^{\infty}\left(p\delta_0+(1-p)\delta_{\frac{1}{2^n}}\right),$$

where $\delta_x$ is the probability measure concentrated on the point $x$. Note that $\mu_p$ is the distribution of the random variable, the $n$-th digit of whose binary expansion is 0 with probability $p$ and 1 with probability $1-p$.

Brown and Williamson [**3**] studied sum sets and coin-tossing showing that some $n$-fold sum of any Borel set with positive $\mu_p$ measure must have positive Lebesgue measure. The main result of [**3**] is as follows.

THEOREM. (Brown and Williamson). *Let* $a=[\max\{p, 1-p\}]^{-1}$. *Suppose that* $a \geq 2^{1/n}$ *and* $\alpha = \log 2/n \log a$. *Suppose that* $E_1,\ E_2,\ldots,\ E_n$ *are Borel subsets of* $[0, 1]$. *Then*

$$\lambda(E_1+E_2+\cdots+E_n) \geq \mu_p(E_1)^{\alpha}\mu(E_2)^{\alpha}\cdots\mu_p(E_n)^{\alpha}.$$

The technique used to prove the Brown-Williamson theorem is to reduce the measure theoretical problem to a combinatorial problem. Notice that in the above theorem, they consider the same measure $\mu_p$ and the same value of $\alpha$ for all subsets $E_i$. The natural generalization is to consider different measures $\mu_{p_i}$ and different values of $\alpha_i$ for each subset $E_i$.

In this paper we consider the Lebesgue measure of a sum set $E+F$ of two subsets $E, F$ with $E$ and $F$ having positive $\mu_p$ and $\mu_q$ measures respectively, where in general $p \neq q$. We set up the basic result of the type

$$\lambda(E+F) \geq \mu_p(E)^{\alpha}\mu_q(F)^{\beta},$$

where in general $\alpha \neq \beta$. We follow the pattern of proof of the above Brown-Williamson theorem in [**3**] to reduce the measure theoretical problem to a counting problem and obtain the related combinatorial result.

We state our main results in Section 2. The proofs will be given in Sections 3 and 4. In Section 5, we consider the size of sum sets in terms of a general coin-tossing measure $\mu_r$, $(0<r<1)$, rather than only using the specific one $\lambda=\mu_{1/2}$.

## 2. Main results

THEOREM 1. *Let* $a=[\max\{p,\ 1-p\}]^{-1}$, $b=[\max\{q,\ 1-q\}]^{-1}$, *where* $0<p,\ q<1$. *Let* $0<\alpha, \beta \leq 1$ *with*

$$\alpha\log a+\beta\log b = \log 2.$$

*Then for any Borel subsets E, F of* $[0, 1]$ *one has*

$$\lambda(E+F) \geq \mu_p(E)^{\alpha}\mu_q(E)^{\beta}. \tag{1}$$

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To prove Theorem 1, we need the following combinatorial result. To shorten our notation, we shall use $(u, v)$ to denote $\max\{u, v\}$ from now on.

THEOREM 2. *Let $p, q, \alpha, \beta$ be the same as in Theorem 1. Then, for any $0 \le x, y \le 1$, we have*

$$\left(\left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}, \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta}\right) + \left(\left[\frac{x}{p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta}, \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}\right) \ge 2. \tag{2}$$

The proof of the results above will be given in Sections 3 and 4. In Section 3 we convert the measure theoretical inequality (1) to the combinatorial inequality (2). Theorem 2 will be proved in Section 4.

**3. Reduction process.** The aim of this section is to show that in order to prove Theorem 1, it is sufficient to prove Theorem 2. In this section, when we consider the numbers $u, v$ as elements of $[0, 1)$, by $u + v$ we mean $u + v(\text{mod } 1)$.

Since $\mu_p$ and $\mu_q$ are regular, we may assume that $E$ and $F$ are closed. In fact, for any Borel subsets $A$, $B$ and given $\epsilon > 0$, there exist closed $A_\epsilon \subseteq A$ and $B_\epsilon \subseteq B$ such that $\mu_p(A_\epsilon) \ge (1-\epsilon)\mu_p(A)$ and $\mu_q(B_\epsilon) \ge (1-\epsilon)\mu_q(B)$. If Theorem 1 holds for closed subsets then

$$\begin{aligned}\lambda(A+B) \ge \lambda(A_\epsilon + B_\epsilon) &\ge \mu_p(A_\epsilon)^{\alpha}\mu_q(B_\epsilon)^{\beta}\\ &\ge (1-\epsilon)^{\alpha+\beta}\mu_p(A)^{\alpha}\mu_q(B)^{\beta}.\end{aligned}$$

Let

$$S_n = \left\{\sum_{i=1}^{n} \frac{\epsilon_i}{2^i} | \epsilon_i = 0,\ 1\right\}.$$

Define probability measures $\mu_p^{(n)}, \mu_q^{(n)}$ on $S_n$ by

$$\mu_p^{(n)} = \mathop{*}_{k=1}^{n}\left(p\delta_0 + (1-p)\delta_{\frac{1}{2^k}}\right)$$

and

$$\mu_q^{(n)} = \mathop{*}_{k=1}^{n}\left(q\delta_0 + (1-q)\delta_{\frac{1}{2^k}}\right).$$

Assume that $E$ and $F$ are closed subsets of $[0, 1)$. It is easy to see that $E + F$ is also closed. Define

$$A_n = \left\{\sum_{k=1}^{n} \frac{\epsilon_k}{2^k} | \text{there exist } x \in E, \text{ with } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{2^k},\ \epsilon_k = 0 \text{ or } 1\right\},$$

and

$$B_n = \left\{\sum_{k=1}^{n} \frac{\epsilon_k}{2^k} | \text{there exist } x \in F, \text{ with } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{2^k},\ \epsilon_k = 0 \text{ or } 1\right\}.$$

Let $E_n = A_n + [0, \frac{1}{2^n}]$ and $F_n = B_n + [0, \frac{1}{2^n}]$. We have the following facts.

$$E = \bigcap_{n=1}^{\infty} E_n \quad \text{and} \quad F = \bigcap_{n=1}^{\infty} F_n. \tag{3}$$

$$\mu_p^{(n)}(A_n) = \mu_p(E_n) \quad \text{and} \quad \mu_q^{(n)}(B_n) = \mu_q(F_n). \tag{4}$$

It is easy to see that

$$E + F = \bigcap_{n=1}^{\infty} (E_n + F_n). \tag{5}$$

From (3), (4) and (5) we obtain

$$\lambda(E + F) = \lim_{n\to\infty} \lambda(E_n + F_n),$$

$$\mu_p(E) = \lim_{n\to\infty} \mu_p(E_n) = \lim_{n\to\infty} \mu_p^{(n)}(A_n)$$

and

$$\mu_q(F) = \lim_{n\to\infty} \mu_q(F_n) = \lim_{n\to\infty} \mu_q^{(n)}(B_n).$$

If we can show that

$$\lambda(E_n + F_n) \geq \mu_p^{(n)}(A_n)^\alpha \mu_q^{(n)}(B_n)^\beta,$$

then we obtain

$$\lambda(E + F) = \lim_{n\to\infty} \lambda(E_n + F_n) \geq \mu_p(E)^\alpha \mu_q(F)^\beta.$$

However

$$E_n + F_n \supseteq A_n + B_n + \left[0, \frac{1}{2^n}\right]$$

and

$$\lim_{n\to\infty} \lambda\left(A_n + B_n + \left[0, \frac{1}{2^n}\right]\right) = \lim_{n\to\infty} \lambda^{(n)}(A_n + B_n),$$

where $\lambda^{(n)}$ is the measure which assigns mass $\frac{1}{2^n}$ to each member of $S_n$.

Now it will suffice to prove that, for all subsets $A$, $B$ of $S_n$, we have

$$\lambda^{(n)}(A + B) \geq \mu_p^{(n)}(A)^\alpha \mu_q^{(n)}(B)^\beta. \tag{6}$$

We prove (6) by induction. For $n = 1$, we have $\lambda^{(1)}(A + B) = 1/2$ if $\#(A) = \#(B) = 1$; or $\lambda^{(1)}(A + B) = 1$ otherwise. We need to check only the first case. Now

$$\mu_p^{(1)}(A)^\alpha \mu_q^{(1)}(B)^\beta \leq \frac{1}{a^\alpha b^\beta} = \frac{1}{2}.$$

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Assume that (6) holds for some $n$. We show that it holds also for $n+1$. In fact, for arbitrary subsets $A$, $B$ of $S_{n+1}$, we have

$$A+B=\left[(A^0+B^0)\cup\left(A^1+B^1+\frac{2}{2^{n+1}}\right)\right] \cup\left[\left(A^0+B^1+\frac{1}{2^{n+1}}\right)\cup\left(A^1+B^0+\frac{1}{2^{n+1}}\right)\right],$$

where

$$A^i=\left\{\sum_{k=1}^{n}\frac{\epsilon_k}{2^k}\,\middle|\,\sum_{k=1}^{n+1}\frac{\epsilon_k}{2^k}\in A \text{ with } \epsilon_{n+1}=i\right\},$$
$$B^i=\left\{\sum_{k=1}^{n}\frac{\epsilon_k}{2^k}\,\middle|\,\sum_{k=1}^{n+1}\frac{\epsilon_k}{2^k}\in B \text{ with } \epsilon_{n+1}=i\right\}.$$

The two sets in square brackets are clearly disjoint so that

$$\lambda^{(n+1)}(A+B)\geq\frac{1}{2}\big(\lambda^{(n)}(A^0+B^0),\lambda^{(n)}(A^1+B^1)\big)+\frac{1}{2}\big(\lambda^{(n)}(A^0+B^1),\lambda^{(n)}(A^1+B^0)\big).$$

By induction,

$$\lambda^{(n)}(A^i+B^j)\geq\mu_p^{(n)}(A^i)^\alpha\mu_q^{(n)}(B^j)^\beta.$$

On the other hand there exist $0\leq x\leq 1$, $0\leq y\leq 1$ such that

$$\mu_p^{(n)}(A^0)=\frac{1}{p}\mu_p^{(n+1)}(A^0)=\frac{x}{p}\mu_p^{(n+1)}(A),$$
$$\mu_q^{(n)}(B^0)=\frac{1}{q}\mu_q^{(n+1)}(B^0)=\frac{y}{q}\mu_q^{(n+1)}(B).$$

It follows that

$$\mu_p^{(n)}(A^1)=\frac{1-x}{1-p}\mu_p^{(n+1)}(A),$$
$$\mu_q^{(n)}(B^1)=\frac{1-y}{1-q}\mu_q^{(n+1)}(B).$$

Therefore,

$$\lambda^{(n+1)}(A+B)\geq\frac{1}{2}\left\{\left(\left[\frac{x}{p}\right]^\alpha\left[\frac{y}{q}\right]^\beta,\left[\frac{1-x}{1-p}\right]^\alpha\left[\frac{1-y}{1-q}\right]^\beta\right)+\left(\left[\frac{x}{p}\right]^\alpha\left[\frac{1-y}{1-q}\right]^\beta,\left[\frac{1-x}{1-p}\right]^\alpha\left[\frac{y}{q}\right]^\beta\right)\right\}\cdot\mu_p^{(n+1)}(A)^\alpha\mu_q^{(n+1)}(B)^\beta.$$

By Theorem 2, we have

$$\lambda^{(n+1)}(A+B)\geq\mu_p^{(n+1)}(A)^\alpha\mu_q^{(n+1)}(B)^\beta,$$

and this completes the induction.

Now it remains to prove Theorem 2 and this will be done in the next section.

**4. Proof of Theorem 2.** In this section, we prove Theorem 2. Without loss of generality, we assume that $x \geq p$, $y \geq q$. Then (2) becomes

$$\left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta} + \left(\left[\frac{x}{p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta}, \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}\right) \geq 2.$$

For fixed $p,\ q,\ \alpha$, and $\beta$, define

$$f(x, y) = \left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta} + \left[\frac{x}{p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta},$$

and

$$g(x, y) = \left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta} + \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}.$$

Then we have that $f''_{xx}(x, y) < 0, f''_{yy}(x, y) < 0$ and $g''_{xx}(x, y) < 0, g''_{yy}(x, y) < 0$ for $0 < x, y < 1$. By the concavity of $f(1, y)$ and the facts that

$$f(1, q) = \frac{2}{p^{\alpha}} > 2$$

and

$$f(1, 1) = \frac{1}{p^{\alpha}q^{\beta}} \geq a^{\alpha}b^{\beta} = 2$$

we see that

$$f(1, y) \geq 2 \quad \text{for all } q \leq y \leq 1.$$

Obviously, for all $q \leq y \leq 1$ we have

$$g(p, y) = 2\left(\frac{y}{q}\right)^{\beta} \geq 2.$$

Similarly we can show that

$$f(x, q) \geq 2 \quad \text{and} \quad g(x, 1) \geq 2,$$

for $p \leq x \leq 1$. For fixed $x \in (p, 1)$, if we have a $\phi(x)$ with $q \leq \phi(x) \leq 1$ such that

$$f(x, \phi(x)) \geq 2 \quad \text{and} \quad g(x, \phi(x)) \geq 2$$

then, by the concavity of $f(x, y)$ and $g(x, y)$ with respect to $y$, we can prove that

$$f(x, y) \geq 2 \quad \text{for } q \leq y \leq \phi(x) \tag{7}$$

and

$$g(x, y) \geq 2 \quad \text{for } \phi(x) \leq y \leq 1. \tag{8}$$

The combination of (7) and (8) will prove Theorem 2.

Define a function $\mu(x)$ for $p \leq x \leq 1$ in the following way: let $\mu(x) = y_0$ if $f(x, y_0) = 2$ and $f(x, y) < 2$ for $y_0 < y \leq 1$, and $\mu(x) = 1$ otherwise. Since $f(x, y)$ is concave with respect to $y$ and $f(x, q) \geq 2$ for all $p \leq x \leq 1$, we see that $\mu(x)$ is well defined and we have $f(x, y) \geq 2$ for $q \leq y \leq \mu(x)$ and $f(x, y) < 2$ for $\mu(x) < y \leq 1$ in the case $\mu(x) < 1$.

Similarly, define $\nu(x) = y_0$ if $g(x, y_0) = 2$ and $g(x, y) < 2$ for $q \leq y < y_0$, and $\nu(x) = q$ otherwise. Because of the concavity of $g(x, y)$ with respect to $y$ and the fact that $g(x, 1) \geq 2$ for $p \leq x \leq 1$, we know that $\nu(x)$ is well defined. Furthermore, we have $g(x, y) \geq 2$ for $\nu(x) \leq y \leq 1$ and $g(x, y) < 2$ for $q \leq y < \nu(x)$ in the case $\nu(x) > q$.

If we have $\nu(x) \leq \mu(x)$ then we can define $\phi(x)$ to be any number in the interval $[\nu(x), \mu(x)]$.

We prove that we have $\nu(x) \leq \mu(x)$ for all $p \leq x \leq 1$. First notice that $\nu(1) \leq 1 = \mu(1)$ and $\nu(p) = q \leq \mu(p)$. By the concavity of $f(x, y)$ and $g(x, y)$ with respect to $x$ we can see that $\mu(x)$ and $\nu(x)$ are non-decreasing and continuous. We take $\mu(x)$ as an example.

Let $x_1 < x_2$. By definition, $f(x_1, y) \geq 2$ for $q \leq y \leq \mu(x_1)$. Recall that we have $f(1, y) \geq 2$ for $q \leq y \leq 1$. Since $x_1 < x_2 \leq 1$, by the concavity of $f(x, y)$ with respect to $x$, we have $f(x_2, y) \geq 2$ for $q \leq y \leq \mu(x_1)$. Hence $\mu(x_2) \geq \mu(x_1)$. Therefore $\mu(x)$ is non-decreasing. If $\mu$ is not continuous then, because it is non-decreasing, we have $\mu(x_0-) < \mu(x_0)$ or $\mu(x_0) < \mu(x_0+)$ for some $p \leq x_0 \leq 1$. In the first case, we claim that for any $\mu(x_0-) < y \leq \mu(x_0)$ we have $f(x_0, y) = 2$. In fact, by the definition of $\mu(x_0)$, it is clear that $f(x_0, y) \geq 2$, for $\mu(x_0-) < y \leq \mu(x_0)$. If, for some $\mu(x_0-) < y_0 \leq \mu(x_0)$ we have $f(x_0, y_0) > 2$, then, since $f(x, y_0)$ is continuous, we have $\lim_{x \uparrow x_0} f(x, y_0) = f(x_0, y_0) > 2$. Thus there exists $x < x_0$ with $f(x, y_0) \geq 2$. Then, by definition, we must have $\mu(x) \geq y_0 > \mu(x_0-)$, a contradiction. On the other hand, by the definition of $f(x, y)$, it is impossible that $f(x_0, y)$ is *constant* for $y$ in an interval. Hence we must have $\mu(x_0-) = \mu(x_0)$. The second case is also impossible, since we have

$$2 \leq \lim_{x \downarrow x_0} f(x, \mu(x_0+)) = f(x_0, \mu(x_0+))$$

from which it follows that $\mu(x_0) \geq \mu(x_0+)$.

Assume that for some $p < x < 1$ we have $\mu(x) < \nu(x)$. Let

$$s = \inf\{x : \ \mu(x) < \nu(x)\}$$

and

$$t = \sup\{u : \ \mu(x) < \nu(x), s < x < u\}.$$

By the continuity of $f$, $g$, $\mu$ and $\nu$, we have the following facts:

$$\mu(s) = \nu(s), \quad \mu(t) = \nu(t),$$

and for all $s \leq x \leq t$

$$f(x, \mu(x)) = g(x, \nu(x)) = 2.$$

Then, for $s < x < t$, we have

$$\mu'(x) = \left[ \frac{\beta p}{2\alpha} \left( \frac{x}{p} \right)^{1+\alpha} \left( \frac{(1-\mu(x))^{\beta-1}}{(1-q)^{\beta}} - \frac{\mu(x)^{\beta-1}}{q^{\beta}} \right) \right]^{-1}$$

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and

$$\nu'(x) = \frac{\alpha q}{2\beta}\left(\frac{\nu(x)}{q}\right)^{1+\alpha}\left(\frac{(1-x)^{\alpha-1}}{(1-p)^{\alpha}} - \frac{x^{\alpha-1}}{p^{\alpha}}\right).$$

Since $\mu$ and $\nu$ are non-decreasing, we have $\mu'(x) \geq 0$ and $\nu'(x) \geq 0$. Because $0 < \alpha, \beta < 1$, we see that $\nu'(x)$ is a product of non-negative non-decreasing functions and so is the reciprocal of $\mu'(x)$. Thus $\nu'(x)$ is increasing and $\mu'(x)$ is decreasing, for $s < x < t$. Since we have $\mu(x) \geq \nu(x)$ for $x \leq s$ and $\mu(x) < \nu(x)$ for $s < x < t$, we must have $\mu'(s+) < \nu'(s+)$. Then, by the discussion above, we should have

$$\mu'(x) - \nu'(x) < 0$$

for $s < x < t$. But the fact that $\mu(s) = \nu(s)$ and $\mu(t) = \nu(t)$ implies that there exists $x_0 \in (s, t)$ such that $\mu'(x_0) - \nu'(x_0) = 0$. This contradiction implies that $\mu(x) < \nu(x)$ is impossible. Now we have shown that for all $p \leq x \leq 1$ we have $\mu(x) \geq \nu(x)$. The proof is complete.

**5. Generalization.** In Theorem 1, we considered the size of sum sets in terms of Lebesgue measure, $\lambda = \mu_{1/2}$. In this section, we use general coin-tossing measure $\mu_r$ to replace Lebesgue measure. Then Theorem 1 can be generalized to the following form.

THEOREM 3. *Let* $a = [\max\{p, \; 1-p\}]^{-1}$, $b = [\max\{q, \; 1-q\}]^{-1}$ *and* $c = [\min\{r, 1-r\}]^{-1}$, *where* $0 < p, \; q, \; r < 1$. *If there exist* $0 < \alpha, \; \beta \leq 1$ *such that*

$$\alpha \log a + \beta \log b = \log c, \tag{9}$$

*then for any Borel subsets E, F of* $[0, 1]$ *one has*

$$\mu_r(E + F) \geq \mu_p(E)^{\alpha}\mu_q(E)^{\beta}. \tag{10}$$

Using a similar argument as in the proof of Theorem 1, we can convert the proof of Theorem 3 to the proof of the following result.

THEOREM 4. *Assume that* $p, \; q, \; r$ *and* $\alpha, \; \beta$ *are the same as defined in Theorem* 3. *Then for any* $0 \leq x, \; y \leq 1$ *we have*

$$\begin{aligned} &r\left(\left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}, \; \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta}\right) \\ &\quad + (1-r)\left(\left[\frac{x}{p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta}, \; \left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}\right) \geq 1. \end{aligned} \tag{11}$$

Theorem 4 can be proved in the same way as Theorem 2. We need only change the definitions of $f(x, y)$ and $g(x, y)$ there to

$$f(x, y) = r\left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta} + (1-r)\left[\frac{x}{p}\right]^{\alpha}\left[\frac{1-y}{1-q}\right]^{\beta},$$

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and

$$g(x,y) = r\left[\frac{x}{p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta} + (1-r)\left[\frac{1-x}{1-p}\right]^{\alpha}\left[\frac{y}{q}\right]^{\beta}.$$

All the arguments there remain valid with some minor changes.

REMARK. Although Theorem 3 generalizes Theorem 1, we have not gone very far. Notice that the minimal possible value of $c$ is 2, and from (9) we see that the larger the value of $c$ is, the smaller the range of $a,\ b$ is. For example, if we let $p = q = 1/3$, then Theorem 1 holds for any $0 < \alpha,\ \beta < 1$ with $\alpha + \beta = \log 2/(\log 3 - \log 2) = 1.7095\cdots$. But for $r = 1/3$, we do not have $0 < \alpha,\ \beta < 1$ such that $(\alpha + \beta)(\log 3 - \log 2) = \log 3$. This illustrates the limitation of the generalization. In fact, if (9) holds for some $0 < \alpha,\ \beta < 1$, we must have $ab > c$, so that in any circumstances, we have $c < 4$; that is $1/4 < r < 3/4$.

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