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ON A THEOREM OF J. A. GREEN

MICHAEL F. O'REILLY

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Let G be a finite group, k a field of characteristic p and Γ the group algebra of G over k. Let $e = \Sigma_{g \in G} \alpha_g g$, $\alpha_g \in k$, be a primitive central idempotent of Γ; let *supp* $e = \{g \in G : \alpha_g \neq 0\}$. We provide a short proof of a slightly stronger version of Theorem 5 of Green (1968).

THEOREM. *Let P be any p-subgroup of G containing a defect group D of e. There exist p-regular elements $x, y, z \in$ supp e such that $xy = z$, D is a Sylow p-subgroup of $C(x)$, $C(y)$, $C(z)$ and $D = P \cap P^x = P \cap P^y = P \cap P^z$.*

For $X \subset G$ let $\bar{X} = \Sigma_{g \in X} g$. For $H \leqq G$ let $A_H = \{\gamma \in \Gamma : \gamma^h = \gamma$, all $h \in H\}$. For $\gamma \in A_H$ and $H \leqq L \leqq G$ write $\gamma_H^L = \Sigma_l \gamma^l$ where l ranges over a right transversal of H in L; then $\gamma_H^L \in A_L$. A_L has as basic the distinct L-orbit sums $\{g_{C_L(g)}^L : g \in G\}$. If $H \leqq C_L(x)$ and $K \leqq C_L(y)$, $x, y \in G$ then

$$(1) \qquad x_H^L y_K^L = \sum_b (x^b y)_{H^b \cap K}^L$$

where b ranges over a (H, K) double coset transversal in L. See Green (1968; Lemma 4f).

Let L be a p-subgroup of G. If $S \leqq L$ and $g \in C(S)$ then

$$g_S^L = [C_L(g) : S] g_{C_L(g)}^L$$

which is zero unless $S = C_L(g)$. Hence if $(x^b y)_{H^b \cap K}^L$ is a non-zero term in (1) then $H^b \cap K = C_L(x^b y), H = C_L(x)$ and $K = C_L(y)$. So $x^b, y \in C(C_L(x^b y))$. We deduce

(2) *if L is a p-subgroup and z_S^L a non-zero L-orbit sum occurring as a summand in* (1) *then $z = x^b y^c$ for some $b, c \in L$ with $x^b, y^c \in C(S)$.*

For such a term $C_L(z) = L$ if and only if $H = K = L$. It follows that the linear projection $\sigma_L : A_L \to A_L$ which annihilates terms $g^L_{C_L(g)}$ with $C_L(g) < L$ is an algebra homomorphism.

Let P be a p-subgroup of G, $a \in G$. Let $S(a) = C_P(a)$.

LEMMA. *If $a^P_{S(a)} \not\in rad\, A_P$ then $S(a) = P \cap P^a$.*

PROOF. If $S(a) < P \cap P^a$,

$$\bar{P} a^P_{S(a)} = [P \cap P^a : S(a)] \overline{PaP} = 0.$$

Thus for any $g \in G$, $\overline{PgP} a^P_{S(a)} = 0$ i.e. right multiplication $\lambda(a^P_{S(a)})$ by $a^P_{S(a)}$ annihilates an element of each indecomposable summand in the $P \times P$-module decomposition $\Gamma_{P\times P} = \bigoplus \Sigma [PgP]$; here the sum ranges over the $P \times P$-modules $[PgP]$ spanned by the double cosets PgP. Thus $\lambda(a^P_{S(a)}) \in rad\, End_{P\times P}\Gamma$, (Jacobson (1943; page 60, Theorem 8). Since λ embeds A_P in $End_{P\times P}\,\Gamma, a^P_{S(a)} \in A_P \cap \lambda^{-1}\{rad\, End_{P\times P}\Gamma\} \subset rad\, A_P$.

PROOF OF THEOREM. Writing e as the sum of P-orbit sums, $e = \Sigma_a \beta_a a^P_{S(a)}$ where $\beta_a \in k$, each a is p-regular, $D \geqq_G$ any Sylow p-subgroup of $C(a)$ and $S(a) = D$ for some a with $\beta_a \neq 0$. See Green (1968; Lemma 2d). Thus $e\sigma_D \neq 0$.

Let $\mathcal{K} = ker\, \sigma_D \cap A_P$, $R = \mathcal{K} + rad\, A_P$. The elements $e \bmod \mathcal{K}$ and $e \bmod R$ are non-zero idempotents. If $a^P_{S(a)} \not\in R$ then

(i) $a^P_{S(a)} \not\in \mathcal{K}$ whence $S(a) =_P D$ and by suitable choice of a, $S(a) = D$ and

(ii) $a^P_D \in rad\, A_P$ whence from the lemma $D = P \cap P^a$. Thus $e \equiv \Sigma \beta_a a^P_D \ (mod\, R)$ where $S(a) = D = P \cap P^a$.

Since $e \bmod R$ is idempotent any $z^P_D \not\in R$ in this summation—at least one must exist—occurs as a term in the product of some pair of terms $u^P_D, v^P_D \not\in R$. By (2) $z = u^b v^c$ where $b, c \in P$ and $u^b, v^c \in C(D)$. Choose $x = u^b, y = v^c$. Then $z = xy, D \leqq C(x) \cap C(y) \cap C(z)$. Since $D \geqq_G$ any Sylow p-subgroup of $C(x), C(y)$ or $C(z), D$ must be a Sylow p-subgroup of these groups. Since $x^P_D, y^P_D, z^P_D \not\in R, D = P \cap P^x = P \cap P^y = P \cap P^z$.

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Department of Mathematics
University of Papua New Guinea
Port Moresby
Papua New Guinea.