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A GENERALISATION OF A RECENT CHARACTERISATION OF PLANAR GRAPHS

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Planar graphs have recently been characterised as those which have no strict elegant odd ring of circuits. Here we generalise that result by showing that its dual yields a theorem that is valid for all graphs.

1. Introduction

A new characterisation of planar graphs appears in [2]. This characterisation hinges on the idea of a ring of circuits. If C is a collection of circuits of a graph G and the edges of G can be directed so that every circuit of C is a directed circuit, then we say that C is *consistently orientable*. The cyclic sequence $C = (C_0, C_1, \ldots, C_{n-1})$ of circuits, where $n \geq 3$, is a *ring* of circuits in the graph G if

(i) C is consistently orientable,

(ii) $EC_i \cap EC_j \neq \emptyset$ if and only if $i = j$, $i \equiv j + 1 \pmod{n}$ or $i \equiv j - 1 \pmod{n}$, and

(iii) no edge of G belongs to more than two circuits of C.

The ring C above is *odd* if n is odd, *strict* if $|VC_i \cap VC_j| \leq 1$ whenever $EC_i \cap EC_j = \emptyset$, and *elegant* if for each $i \in \{0, 1, \ldots, n-1\}$ there is a path M_i satisfying the conditions $EM_i = EC_{i+1} - EC_i$ and $|VM_i \cap VC_i| = 2$. (Here, and throughout this paper, subscripts are to be

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read modulo n .)

It is proved in [2] that a graph is planar if and only if it contains no strict elegant odd ring. The proof in [2] has been greatly simplified by Chen [1] using topological considerations. In this paper, we achieve further simplification by dualising some of Chen's ideas, thereby also generalising the result.

Let G be a graph and $X, Y \subseteq VG$. We denote by $[X, Y]$ the set of all edges joining a vertex of X and a vertex of Y . If $Y = VG - X$, then $[X, Y]$ is called an *edge cut* if $X \neq \emptyset$ and $Y \neq \emptyset$. We abbreviate $[X, Y]$ by $\delta(X)$ or $\delta(Y)$. A minimal non-empty edge cut is a *bond*. It is well known that if G is connected, then $[X, Y]$ is a bond if and only if $G[X]$ and $G[Y]$ are connected. Moreover, any non-empty edge cut is a union of disjoint bonds. In the case where G is a directed graph and every edge of $\delta(X)$ is directed toward the end in X , we say that $\delta(X)$ is a *directed edge cut*.

If $\mathcal{C}$ is a collection of bonds of a graph G and the edges of G can be directed so that every bond of $\mathcal{C}$ is directed, then we say that $\mathcal{C}$ is *consistently orientable*. The cyclic sequence $\mathcal{C} = (C_0, C_1, \ldots, C_{n-1})$ of bonds, where $n \geq 3$, is a *ring* of bonds in the graph G if

(i) $\mathcal{C}$ is consistently orientable,

(ii) $C_i \cap C_j \neq \emptyset$ if and only if $i = j$, $i \equiv j + 1 \pmod{n}$ or $j \equiv j - 1 \pmod{n}$, and

(iii) no edge of G belongs to more than two bonds of $\mathcal{C}$.

Let A and B be bonds of G , where $A = [X, Y]$. Then a proper subset P of B is a *B-chord* of A if P is a bond of $G[X]$ or $G[Y]$. Note that P , being a proper subset of B , cannot be a bond of G . Hence if $P = [S, T]$ and P is a bond of $G[X]$, then $[Y, S] \neq \emptyset$ and $[Y, T] \neq \emptyset$. Similarly if P is a bond of $G[Y]$, then $[X, S] \neq \emptyset$ and $[X, T] \neq \emptyset$.

The ring $\mathcal{C} = (C_0, C_1, \ldots, C_{n-1})$ of bonds is *odd* if n is odd, *even* if n is even, *strict* if there do not exist distinct bonds A, B, C satisfying the conditions $B \in \mathcal{C}$, $C \in \mathcal{C}$, $B \cap C = \emptyset$, $A \subseteq B \cup C$, and *elegant* if for each i there exists a unique C_{i+1}-chord of C_i . It is easily seen that if G is planar the dual of a strict elegant odd ring of circuits is a strict elegant odd ring of bonds. Hence no planar graph can contain a strict elegant odd ring of bonds. It is our purpose here to generalise this result to non-planar graphs by proving the following theorem.

THEOREM. *In any graph, every strict elegant ring of bonds is even.*

2. Preliminary lemmas

Throughout the rest of this paper, we let C be a ring $(C_0, C_1, \ldots, C_{n-1})$ of bonds in a graph G. Furthermore, for each i we will write $C_i = [A_i, B_i]$.

LEMMA 1. *There is a component D of G such that* $\bigcup_{i=0}^{n-1} C_i \subseteq ED$.

Proof. Let D be a component of G such that $ED \cap C_0 \neq \emptyset$. Since C_0 is a bond, we have $C_0 \subseteq ED$. Proceeding by induction, assume that $\bigcup_{i=0}^{k-1} C_i \subseteq ED$ for some $k > 0$. Since $C_{k-1} \cap C_k \neq \emptyset$, it follows that $ED \cap C_k \neq \emptyset$; hence $C_k \subseteq ED$ since C_k is a bond. The result follows by induction.

Thus we henceforth assume without loss of generality that G is connected.

LEMMA 2. *For any i and any $e \in C_{i+1}-C_i$, there exists a C_{i+1}-chord of C_i which contains e.*

Proof. Without loss of generality, let $e \in [A_i, A_i]$. Then $[A_i \cap A_{i+1}, A_i \cap B_{i+1}]$ is a non-empty edge cut of $G[A_i]$ and so a union of bonds of $G[A_i]$. Since $C_i \cap C_{i+1} \neq \emptyset$, these bonds are proper subsets of C_{i+1}, and hence C_{i+1}-chords of C_i. One of them contains e.

LEMMA 3. *If C is elegant, then for each i there is a unique C_i-chord of C_{i+1}.*

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Proof. Let $P = [X, Y]$ be the unique C_{i+1}-chord of C_i, and without loss of generality let $P \subseteq [A_i, A_i]$. By Lemma 2, any $e \in C_{i+1} - C_i$ must belong to P. Hence $P = C_{i+1} - C_i$, and it follows immediately that we may assume $C_{i+1} = P \cup [X, B_i]$ without loss of generality. Since $G[B_i]$ and $G[Y]$ are connected and $\emptyset \subset [Y, B_i] \subset C_i$, it follows that $[Y, B_i]$ is a C_i-chord of C_{i+1}, and it is clearly the only such chord.

LEMMA 4. *Let* C *be strict and elegant. Then for all* $i, j, k \in \{0, 1, \ldots, n-1\}$, *either* $(C_j \cup C_k) - C_i \subseteq [A_i, A_i]$ *or* $(C_j \cup C_k) - C_i \subseteq [B_i, B_i]$.

Proof. Without loss of generality, let $i = 0$ and $j \le k$. We shall show first that either $C_j - C_0 \subseteq [A_0, A_0]$ or $C_j - C_0 \subseteq [B_0, B_0]$. This statement is trivial if $j = 0$; suppose therefore that $j > 0$.

Case I. Suppose $1 < j < n-1$. Then $C_0 \cap C_j = \emptyset$ by (ii).

Suppose that $C_j \cap [A_0, A_0] \ne \emptyset$ and $C_j \cap [B_0, B_0] \ne \emptyset$. Then $A_0 \cap A_j$, $A_0 \cap B_j$, $B_0 \cap A_j$ and $B_0 \cap B_j$ are all non-empty. Let D be a component of $G[A_0 \cap A_j]$. Thus $\delta(VD) \subseteq C_0 \cup C_j$.

We show next that $\delta(VD)$ is a bond. Since D is connected, it suffices to demonstrate that $G - VD$ is connected. As C_j is a bond, $G[B_j]$ must be connected. Because $VD \subseteq A_j$, there must therefore be a component X of $G - VD$ for which $B_j \subseteq VX$. If D' is any component of $G[A_0 \cap A_j]$ other than D, then $[VD', A_0 \cap A_j] \ne \emptyset$ because $G[A_0]$ must be connected; hence $VD' \subseteq VX$. Finally, since $B_0 \cap B_j \ne \emptyset$ and $G[B_0]$ is connected, we must have $B_0 \subseteq VX$.

We infer that X is the only component of $G - VD$, and that $\delta(VD)$ is therefore a bond. Since $C_0 \in C$, $C_j \in C$, $C_0 \cap C_j = \emptyset$ and $\delta(VD) \subseteq C_0 \cup C_j$, the strictness of C is contradicted. Hence either $C_j - C_0 \subseteq [A_0, A_0]$ or $C_j - C_0 \subseteq [B_0, B_0]$.

Case II. Suppose $j \in \{1, n-1\}$. By Lemma 3, we may assume without loss of generality that $j = 1$. Since $C_0 \neq C_1$ (otherwise $C_0 \cap C_1 \cap C_2 \neq \emptyset$), we may further assume without loss of generality that $C_1 \cap [A_0, A_0] \neq \emptyset$. Then by Lemma 2, $[A_0, A_0]$ contains a C_1-chord of C_0. Similarly if $C_1 \cap [B_0, B_0] \neq \emptyset$, then $[B_0, B_0]$ contains a C_1-chord of C_0. Since $\mathcal{C}$ is elegant, there is only one such chord, and so we cannot have both $C_1 \cap [A_0, A_0] \neq \emptyset$ and $C_1 \cap [B_0, B_0] \neq \emptyset$. Thus either $C_1 - C_0 \subseteq [A_0, A_0]$ or $C_1 - C_0 \subseteq [B_0, B_0]$.

The lemma has now been proved if $k = j$; let us therefore assume as an induction hypothesis that $j < k < n$ and $(C_j \cup C_{k-1}) - C_0 \subseteq [A_0, A_0]$. Since $C_k \cap (C_{k-1}-C_0) \neq \emptyset$, we have $C_k \cap [A_0, A_0] \neq \emptyset$, and the previous result with j replaced by k shows that $C_k - C_0 \subseteq [A_0, A_0]$. Thus $(C_j \cup C_k) - C_0 \subseteq [A_0, A_0]$. Similarly, $(C_j \cup C_k) - C_0 \subseteq [B_0, B_0]$ if $(C_j \cup C_{k-1}) - C_0 \subseteq [B_0, B_0]$.

3. Proof of the theorem

Let the edges of G be oriented so that every bond in $\mathcal{C}$ is directed. Choose any $C_i \in \mathcal{C}$. By Lemma 4, we may assume without loss of generality that $C_{i+1} - C_i \subseteq [A_i, A_i]$. We shall define C_i to be *positive* if its edges are directed toward A_i and *negative* otherwise. Let us assume without loss of generality that C_i is positive. It now suffices to show that C_{i+1} is negative, for then the bonds in $\mathcal{C}$ must alternate in sign, so that $\mathcal{C}$ must be even. Without loss of generality, let $C_i - C_{i+1} \subseteq [A_{i+1}, A_{i+1}]$. Then $B_i \subseteq A_{i+1}$ since $C_{i+1} \cap [B_i, B_i] = \emptyset$. Since $C_i \cap C_{i+1} \neq \emptyset$ and C_i is positive, the edges of C_{i+1} must be directed toward B_{i+1}. Since $C_i - C_{i+1} \subseteq [A_{i+1}, A_{i+1}]$, Lemma 4 shows that $C_{i+2} - C_{i+1} \subseteq [A_{i+1}, A_{i+1}]$. It follows that C_{i+1} is negative and the theorem is proved.

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