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The Asymptotics of the Higher Dimensional Reidemeister Torsion for Exceptional Surgeries Along Twist Knots

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Abstract. We determine the asymptotic behavior of the higher dimensional Reidemeister torsion for the graph manifolds obtained by exceptional surgeries along twist knots. We show that all irreducible $SL_2(\mathbb{C})$-representations of the graph manifold are induced by irreducible metabelian representations of the twist knot group. We also give the set of the limits of the leading coefficients in the higher dimensional Reidemeister torsion explicitly.

1 Introduction

The purpose of this paper is to observe the asymptotic behavior of the higher dimensional Reidemeister torsion for graph manifolds. In particular, we are interested in graph manifolds whose $SL_2(\mathbb{C})$-representations of the fundamental groups are described by certain subsets of the $SL_2(\mathbb{C})$-representations of hyperbolic knot groups.

A closed orientable irreducible 3-manifold M is called a *graph manifold* if there exists disjoint incompressible tori $T_1^2, \dots, T_k^2$ in M such that each component of $M \setminus (T_1^2 \cup \dots \cup T_k^2)$ is a Seifert fibered space, and the whole space M does not admit any Seifert fibration. It has been shown in [Yam] that the higher dimensional Reidemeister torsion for a Seifert fibered space grows exponentially, and its logarithm has the same order as the dimension of representations. It is natural to expect that we have the same growth order in the case of a graph manifold. In this paper, we determine the growth order and the limit of the leading coefficient in the sequence given by the logarithm of the higher dimensional Reidemeister torsion for certain graph manifolds. We will see the difference in the limit of the leading coefficient between our graph manifolds and the Seifert fibered spaces studied in [Yam]. In the study of exceptional surgeries along a hyperbolic knot, the problem of finding incompressible tori that cut the resulting manifold into Seifert fibered spaces has been investigated. For example there exists a complete list of exceptional surgeries along two–bridge knots [BW01]. The torus decomposition of the resulting graph manifolds is also given in [Pat95, CT13, Ter13].

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When a manifold is obtained by a surgery along a knot, its fundamental group is given by a quotient group of the knot group. Therefore, we can pull-back $\mathrm{SL}_2(\mathbb{C})$-representations from the fundamental group of the resulting manifold to the knot group (for details, see Section 2.2). The $\mathrm{SL}_2(\mathbb{C})$-representation space of a hyperbolic knot group can be regarded as a parameter space for deformations of the hyperbolic structure of the knot exterior. Since exceptional surgeries along a hyperbolic knot yield non–hyperbolic manifolds, the resulting manifolds induce $\mathrm{SL}_2(\mathbb{C})$-representations of the hyperbolic knot group that correspond to degenerate hyperbolic structures. We are also motivated to see the asymptotic behavior of the higher dimensional Reidemeister torsion when we choose an $\mathrm{SL}_2(\mathbb{C})$-representation for a hyperbolic 3-manifold that is different from the holonomy representation. Here the holonomy representation is an $\mathrm{SL}_2(\mathbb{C})$-representation corresponding to the complete hyperbolic structure. We wish to investigate the asymptotic behavior of the higher dimensional Reidemeister torsion for degenerate hyperbolic structures through the $\mathrm{SL}_2(\mathbb{C})$-representations induced by an exceptional surgery.

For our purpose, we choose hyperbolic twist knots (see Fig. 2) with 4-surgeries. According to the torus decomposition in [Pat95], in the set of exceptional surgeries along two–bridge knots, only 4-surgeries along hyperbolic twist knots yield graph manifolds consisting of two Seifert fibered spaces that include a torus knot exterior. More precisely, 4-surgery along a twist knot K_n illustrated in Figure 2 yields the graph manifold M consisting of the torus knot exterior of type $(2, 2n+1)$, which will be denoted by $T(2, 2n+1)$, and the twisted I-bundle over the Klein bottle. We consider the asymptotic behavior of the higher dimensional Reidemeister torsion for M. When we choose a homomorphism $\overline{\rho}$ from $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$, we also have a sequence of homomorphisms $\sigma_{2N} \circ \overline{\rho}$ from $\pi_1(M)$ into $\mathrm{SL}_{2N}(\mathbb{C})$ by the composition with the irreducible representations σ_{2N} of $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SL}_{2N}(\mathbb{C})$. Our main theorem is stated as follows.

Theorem (Theorem 4.4 and Corollary 4.5) *The growth of* $\log|\mathrm{Tor}(M; \sigma_{2N} \circ \overline{\rho})|$ *has the same order as* $2N$ *for every irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation* $\overline{\rho}$ *of* $\pi_1(M)$*. The limits of the leading coefficients are expressed as*

$$\Big\{ \lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \sigma_{2N} \circ \overline{\rho})|}{2N} \;\Big|\; \overline{\rho} \textit{ is irreducible} \Big\} = \\ \Big\{ \frac{1}{p_k} \big(\log|\Delta_{T(2,2n+1)}(-1)| - \log 2 \big) \;\Big|\; p_k > 1, p_k \textit{ is a divisor of } |\Delta_{K_n}(-1)| \Big\},$$

where $\Delta_K(t)$ *is the Alexander polynomial of a knot* K*.*

In particular, the minimum in the limits of the leading coefficients is given by

$$\frac{1}{|\Delta_{K_n}(-1)|} \big(\log|\Delta_{T(2,2n+1)}(-1)| - \log 2 \big).$$

We will prove our main theorem using the following procedures. First, we will see that all irreducible $\mathrm{SL}_2(\mathbb{C})$-representations $\overline{\rho}$ of $\pi_1(M)$ are induced by irreducible metabelian representations ρ of a twist knot group $\pi_1(E_{K_n})$. Here, E_{K_n} is the knot exterior of a twist knot K_n. Concerning the decomposition of M as the union of $E_{T(2,2n+1)}$ and the twist I-bundle $N(Kb)$ over the Klein bottle Kb, the restriction of $\overline{\rho}$ to $\pi_1(E_{T(2,2n+1)})$ is abelian. On the other hand, the restriction to $\pi_1(N(Kb))$ is irreducible. We can also compute the Reidemeister torsion for M and $\overline{\rho}$ by the product of the Reidemeister torsions for $E_{T(2,2n+1)}$ and $N(Kb)$ in the JSJ decomposition of M. We will obtain the limits of the leading coefficients in our main theorem from the observation about the asymptotic behavior of the Reidemeister torsion for the torus knot exterior $E_{T(2,2n+1)}$ and abelian representations given by the restrictions of $\overline{\rho}$. We remark that, since $|\Delta_K(-1)|$ is always odd, these limits differ from the limit of the leading coefficient for the exterior of the torus knot $T(2, 2n+1)$ and an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation in [Yam], which is given by $(1-1/2-1/q')\log 2$ with a divisor $q'(>1)$ of $2n+1$. The maximum of $(1-1/2-1/q')$ is equal to $-\chi$ where χ is the Euler characteristic of the base orbifold in the Seifert fibration of the exterior $T(2, 2n+1)$.

From the viewpoint of hyperbolic structures, 4-surgery along a hyperbolic twist knot yields degenerate hyperbolic structures of the twist knot exterior. In this paper, we see that such degenerate hyperbolic structures are given by irreducible metabelian representations in the $\mathrm{SL}_2(\mathbb{C})$-representation space of a twist knot group. The above Theorem (Theorem 4.4 and Corollary 4.5) and the results in [MFP14, Por] imply that, in the case of a hyperbolic twist knot exterior, the growth order of the higher dimensional Reidemeister torsion for any irreducible metabelian representation decreases from that for the holonomy representation. Note that the Reidemeister torsion under our convention is the inverse of that of [MFP14] (for more details, see [Por]). We will observe that this degeneration occurs for any knot in the subsequent paper [TY]. In other words, we will observe that the growth order of the higher dimensional Reidemeister torsion for any irreducible metabelian representation of a hyperbolic knot group is less than that for the holonomy representation.

2 Preliminaries

2.1 The Higher Dimensional Reidemeister Torsion

For the Reidemeister torsion, we follow the notation and definition used in [Yam]. For the details and related topics, we refer the reader to the survey articles [Mil66, Por] by J. Milnor and J. Porti or the book [Tur01] by V. Turaev. We need a homomorphism from the fundamental group into $\mathrm{SL}_2(\mathbb{C})$ to observe the Reidemeister torsion for a manifold. Throughout this paper, a homomorphism from a group H into a linear group G will be referred to as *a G-representation* of H. The symbol σ_n denotes the right action of $\mathrm{SL}_2(\mathbb{C})$ on the vector space V_n, consisting of homogeneous polynomials $p(x, y)$ of degree $n-1$, defined as

$$\sigma_n(A)\cdot p(x,y) = p(x',y'), \quad \text{where } \begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

It is known that this action induces a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SL}_n(\mathbb{C})$, which is referred to as *the n-dimensional irreducible representation of* $\mathrm{SL}_2(\mathbb{C})$. For simplicity, we use the same symbol σ_n for the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We mainly use the $2N$-dimensional irreducible representation σ_{2N}. If $A \in \mathrm{SL}_2(\mathbb{C})$ has eigenvalues $\xi^{\pm 1}$, then $\sigma_{2N}(A)$ has eigenvalues $\xi^{\pm 1}, \xi^{\pm 3}, \dots, \xi^{\pm(2N-1)}$.

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This is due to the action

$$\sigma_{2N}(A)\cdot(x^{2N-1-i}y^i) = \xi^{-2N+1+2i}(x^{2N-1-i}y^i) \quad \text{with } A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

on the standard basis $\{x^{2N-1}, x^{2N-2}y, \dots, xy^{2N-2}, y^{2N-1}\}$ of V_{2N}.

Definition 2.1 Let W be a finite CW-complex and let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$. The twisted chain complex $C_*(W;V_n)$ with coefficients in V_n twisted by ρ is defined as a chain complex that consists of

$$C_i(W;V_n) = V_n \otimes_{\sigma_n\circ\rho} C_i(\widetilde{W};\mathbb{Z}),$$

where $\widetilde{W}$ is the universal cover of W and $C_i(\widetilde{W};\mathbb{Z})$ is a left $\mathbb{Z}[\pi_1(W)]$-module.

We assume that each twisted chain module $C_*(W;V_n)$ is equipped with a fixed basis $\boldsymbol{c}^i$ given by $v_j \otimes \widetilde{e}^i_k$ where v_j is a vector in a basis of V_n and $\widetilde{e}^i_k$ is a lift of an i-dimensional cell e^i_k in W.

Definition 2.2 Suppose that the twisted chain complex $C_*(W;V_n)$ is acyclic, *i.e.*, $\operatorname{Im}\partial_i = \ker\partial_{i-1}$ for all i. Each chain module $C_i(W;V_n)$ has the following decomposition:

$$C_i(W;V_n) = \partial_{i+1}\widetilde{B}_{i+1} \oplus \widetilde{B}_i,$$

where $\widetilde{B}_i$ is a lift of $\operatorname{Im}\partial_i$. Then we will denote by $\operatorname{Tor}(W;\sigma_n\circ\rho)$ the n-dimensional Reidemeister torsion for W and ρ, which is given by the following alternating product:

$$\prod_{i\geq 0} \det(\partial_{i+1}\widetilde{\boldsymbol{b}}^{i+1} \cup \widetilde{\boldsymbol{b}}^{i}/\boldsymbol{c}^i)^{(-1)^{i+1}}, \tag{2.1}$$

where $\widetilde{\boldsymbol{b}}^i$ is a basis of $\widetilde{B}_i$, $\boldsymbol{c}^i$ is the fixed basis of $C_i(W;V_n)$ and $(\partial_{i+1}\widetilde{\boldsymbol{b}}^{i+1} \cup \widetilde{\boldsymbol{b}}^{i}/\boldsymbol{c}^i)$ is the base change matrix from $\boldsymbol{c}^i$ to $\partial_{i+1}\widetilde{\boldsymbol{b}}^{i+1} \cup \widetilde{\boldsymbol{b}}^{i}$.

There are several choices in the definition of the n-dimensional Reidemeister torsion. For example, there are many choices of a lift of each cell e^i_k. It is known that the Reidemeister torsion does not depend on the choice of a lift $\widetilde{e}^i_k$ for $\mathrm{SL}_n(\mathbb{C})$-representations. Let us mention the well-definedness of the Reidemeister torsion without proofs. We refer the reader to [Por, Yam] for the details.

Remark 2.3 The alternating product (2.1) is independent of a choice of a lift of $\operatorname{Im}\partial_i$. The acyclicity of the twisted chain complex for W implies that the Euler characteristic of W must be zero. In this case, $\operatorname{Tor}(W;\sigma_n\circ\rho)$ is also independent of a choice of a basis of V_n. It is known that $\operatorname{Tor}(W;\sigma_n\circ\rho)$ does not depend on the ordering and orientation of cells in $\boldsymbol{c}^i$ when n is even. If n is odd, then the sign of $\operatorname{Tor}(W;\sigma_n\circ\rho)$ depends on the ordering and orientation of cells in $\boldsymbol{c}^i$ in general. This is one reason why we restrict our attention to $2N$-dimensional ones.

We give an example of $2N$-dimensional Reidemeister torsion that will be needed in this paper.

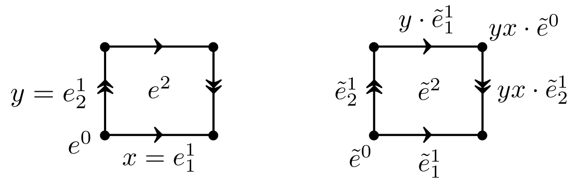

Figure 1: a cell decomposition of Kb (left) and a lift to $\widetilde{Kb}$ (right)

Example 2.4 Suppose that the Klein bottle Kb is decomposed as in Fig. 1 and ρ is an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(Kb)$. The fundamental group has the presentation $\pi_1(Kb) = \langle x, y \mid yx = xy^{-1} \rangle$. The twisted chain complex $C_*(Kb; V_n)$ is expressed as

$$0 \to C_2(Kb; V_n) = V_n \xrightarrow{\partial_2} C_1(Kb; V_n) = V_n \oplus V_n \xrightarrow{\partial_1} C_0(Kb; V_n) = V_n \to 0$$

$$\partial_2 = \begin{pmatrix} \mathbf{1} - Y \\ -XY - \mathbf{1} \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} X - \mathbf{1} & Y - \mathbf{1} \end{pmatrix},$$

where $X = \sigma_n \circ \rho(x)$ and $Y = \sigma_n \circ \rho(y)$. By the relation $x^{-1}yx = y^{-1}$, the $\mathrm{SL}_2(\mathbb{C})$-representation ρ is classified into the following three cases, up to conjugation:

(a) $\rho(y) = \pm\mathbf{1}$ and $\rho(x)$ is arbitrary,

(b) $\rho(y) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ $(\eta \neq \pm 1)$ and $\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

(c) $\rho(y) = \begin{pmatrix} \pm 1 & \omega \\ 0 & \pm 1 \end{pmatrix}$ $(\omega \neq 0)$ and $\rho(x) = \begin{pmatrix} \pm\sqrt{-1} & \omega' \\ 0 & \mp\sqrt{-1} \end{pmatrix}$.

We can express the $2N$-dimensional Reidemeister torsion $\mathrm{Tor}(Kb; \sigma_{2N} \circ \rho)$ as

$$\mathrm{Tor}(Kb; \sigma_{2N} \circ \rho) = \begin{cases} \dfrac{\det(\mathbf{1} - Y)}{\det(Y - \mathbf{1})} & \det(Y - \mathbf{1}) \neq 0, \\ \dfrac{\det(-XY - \mathbf{1})}{\det(X - \mathbf{1})} & \det(Y - \mathbf{1}) = 0. \end{cases} \tag{2.2}$$

Note that the left edge in Figure 1 is moved to the right one by the covering transformation of yx, since the starting point of the left edge is moved to that of the right edge by $yx \in \pi_1(Kb)$.

We will use the following gluing formula of the $2N$-dimensional Reidemeister torsion. This is an application of the Multiplicativity property of the Reidemeister torsion to a torus decomposition of a 3-manifold. In the case of the $2N$-dimensional Reidemeister torsion, we can determine the sign in the gluing formula easily. For the details on applying the Multiplicativity property to a decomposition along a torus, we refer to [Yam, Subsection 2.3 and Section 3] and the references given there.

Lemma 2.5 (Consequence of the Multiplicativity property for a decomposition along a torus) *Suppose that a compact* 3*-manifold M is the union $M_1 \cup_{T^2} M_2$ and each M_i is given a CW-structure such that both of them induce the same CW-structure of T^2. If an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(M)$ induces the acyclic complexes $C_*(M_1; V_{2N})$, $C_*(M_2; V_{2N})$, and $C_*(T^2; V_{2N})$, then the twisted chain complex $C_*(M; V_{2N})$ defined*

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by ρ is also acyclic, and the $2N$-dimensional Reidemeister torsion $\mathrm{Tor}(M;\sigma_{2N}\circ\rho)$ *is expressed as*

$$\mathrm{Tor}(M;\sigma_{2N}\circ\rho) = \mathrm{Tor}(M_1;\sigma_{2N}\circ\rho)\,\mathrm{Tor}(M_2;\sigma_{2N}\circ\rho).$$

Remark 2.6 Usually we have the equality that

$$\mathrm{Tor}(M;\sigma_{2N}\circ\rho)\,\mathrm{Tor}(T^2;\sigma_{2N}\circ\rho) = \mathrm{Tor}(M_1;\sigma_{2N}\circ\rho)\,\mathrm{Tor}(M_2;\sigma_{2N}\circ\rho)$$

as a consequence of the Multiplicativity property. It is known that $\mathrm{Tor}(T^2;\sigma_{2N}\circ\rho) = 1$ if it is defined.

2.2 $\mathrm{SL}_2(\mathbb{C})$-representations of Twist Knot Groups

We review several results concerning $\mathrm{SL}_2(\mathbb{C})$-representations of the fundamental groups of our graph manifolds. We write E_K for the knot exterior of a knot K, which is obtained by removing an open tubular neighbourhood of K from S^3. We mainly consider the n-twist knot K_n, illustrated in Figure 2. The horizontal twists are right-handed if n is positive and left-handed if n is negative. Under our convention, the

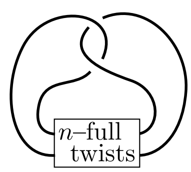

Figure 2: a diagram of K_n

1-twist knot K_1 is the figure-eight knot.

It is known that K_n is a hyperbolic knot and that 4-surgery along K_n yields a graph manifold M when $n \neq 0, -1$. The fundamental group $\pi_1(M)$ has the following presentation.

Proposition 2.7 ([Ter13, Proposition 2.2]) *The graph manifold M consists of a torus knot exterior $E_{T(2,2n+1)}$ and the twisted I-bundle over the Klein bottle. The fundamental group has presentation*

$$\pi_1(M) = \langle a, b, x, y \mid a^2 = b^{2n+1}, x^{-1}yx = y^{-1}, \mu = y^{-1}, h = y^{-1}x^2\rangle, \tag{2.3}$$

where $\mu = b^{-n}a$ and h correspond to a meridian and a regular fiber of the torus knot exterior (with the Seifert fibration), respectively.

Since $\pi_1(M)$ is isomorphic to the quotient group $\pi_1(E_{K_n})/\langle\!\langle m^4\ell\rangle\!\rangle$, where m and ℓ are a meridian and a preferred longitude on ∂E_{K_n}, Proposition 2.7 shows that the quotient $\pi_1(E_{K_n})/\langle\!\langle m^4\ell\rangle\!\rangle$ may be expressed as (2.3). We denote by $\overline{\rho}$ the induced

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homomorphism from $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$:

$$\begin{array}{ccc} \pi_1(E_{K_n}) & \xrightarrow{\ \rho\ } & \mathrm{SL}_2(\mathbb{C}) \\ \big\downarrow & \nearrow_{\overline{\rho}} & \\ \pi_1(M). & & \end{array}$$

Definition 2.8 An $\mathrm{SL}_2(\mathbb{C})$-representation ρ of a group H is referred to as *irreducible* if the invariant subspaces of $\mathbb{C}^2$ under the action of $\rho(H)$ are only $\{\mathbf{0}\}$ and $\mathbb{C}^2$. An $\mathrm{SL}_2(\mathbb{C})$-representation ρ is called *reducible* if it is not irreducible. We also call ρ *abelian* if the image $\rho(H)$ is an abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$.

Remark 2.9 The image of $\pi_1(E_{K_n})$ under ρ coincides with that of $\pi_1(M)$ under $\overline{\rho}$. Hence, $\overline{\rho}$ is irreducible if and only if ρ is irreducible.

Remark 2.10 We have seen the classification of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(Kb)$ in Example 2.4. Case (a) consists of abelian representations; Case (b) consists of irreducible ones, and Case (c) consists of reducible and non–abelian ones.

Definition 2.11 We write $R(X)$ for the set of homomorphisms from $\pi_1(X)$ into $\mathrm{SL}_2(\mathbb{C})$. We call $R(X)$ the $\mathrm{SL}_2(\mathbb{C})$-representation space of $\pi_1(X)$. The symbol $R^{\mathrm{irr}}(X)$ denotes the subset of irreducible representations in $R(X)$.

The pull-back by the quotient induces an inclusion from $R(M)$ into $R(E_{K_n})$. We can regard the representation space $R(M)$ as a subset in $R(E_{K_n})$. From this viewpoint, $R(M)$ is expressed as

$$R(M) = \left\{ \rho \in R(E_{K_n}) \mid \rho(m^4 \ell) = \mathbf{1} \right\}.$$

Definition 2.12 An $\mathrm{SL}_2(\mathbb{C})$-representation of a group H is called *metabelian* if the image $\rho([H,H])$ of the commutator subgroup is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Note that all second commutators of H are sent to the identity matrix under every metabelian $\mathrm{SL}_2(\mathbb{C})$-representation ρ.

Lemma 2.13 *Every irreducible metabelian representation of $\pi_1(E_{K_n})$ is contained in $R(M)$.*

Proof Since a preferred longitude is contained in the second commutator subgroup of a knot group, all metabelian representations send a preferred longitude to $\mathbf{1}$. It was shown in [Nag07, Proposition 1.1] that any irreducible metabelian representation of a knot group sends a meridian to a trace-free matrix in $\mathrm{SL}_2(\mathbb{C})$, which has eigenvalues $\pm\sqrt{-1}$. Hence the matrix corresponding to a meridian has order 4. ■

For any knot K, we can express the set of irreducible metabelian representations as the union of $(|\Delta_K(-1)| - 1)/2$ conjugacy classes, where $\Delta_K(t)$ is the Alexander polynomial of K. If K is a twist knot K_n, then we have the following representatives of conjugacy classes. Here we suppose that $\pi_1(E_{K_n})$ has a presentation $\pi_1(E_{K_n}) =$

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$\langle \alpha, \beta \mid \omega^n \alpha = \beta \omega^n \rangle$, where α, β are meridians and $\omega = \beta\alpha^{-1}\beta^{-1}\alpha$. A twist knot K_n has $(|4n+1|-1)/2$ conjugacy classes, since its Alexander polynomial is given by $-nt^2 + (2n+1)t - n$.

Proposition 2.14 ([NY12, Theorem 3] for K_n) *The set of irreducible metabelian representations of $\pi_1(E_{K_n})$ consists of $(|4n+1|-1)/2$ conjugacy classes. The representatives are given by the following ρ_k ($k = 1, \dots, (|4n+1|-1)/2$):*

$$\rho_k(\alpha) = \begin{pmatrix} \sqrt{-1} & -\sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \rho_k(\beta) = \begin{pmatrix} \sqrt{-1} & 0 \\ -u_k\sqrt{-1} & -\sqrt{-1} \end{pmatrix}, \quad u_k = -4\sin^2 \frac{k\pi}{4n+1}.$$

3 Representation Spaces for Resulting Graph Manifolds

Let M be the graph manifold obtained by 4-surgery along a hyperbolic twist knot K_n.

3.1 $R(M)$ as a Subspace of $R(E_{K_n})$

We determine the $\mathrm{SL}_2(\mathbb{C})$-representation space $R(M)$ as a subset in $R(K_n)$.

Proposition 3.1 *Every irreducible representation of $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$ is induced by an irreducible metabelian one of $\pi_1(E_{K_n})$*, i.e.,

$$R^{\mathrm{irr}}(M) = \left\{ \rho \in R(E_{K_n}) \mid \rho \text{ is irreducible metabelian} \right\}.$$

Proof By Lemma 2.13, it is sufficient to show that if any irreducible representation ρ of $\pi_1(E_{K_n})$ factors through the quotient group $\pi_1(E_{K_n})/\langle\langle m^4\ell\rangle\rangle$, then ρ is metabelian. When $\mathcal{M}^{\pm 1}$ denote the eigenvalues of $\rho(m)$, the trace $\mathcal{M} + \mathcal{M}^{-1}$ of $\rho(m)$ must be zero by Lemmas 3.2 and 3.3. Since K_n is a two–bridge knot, it follows from [NY12, Lemma 23] that ρ must be a metabelian representation. ∎

Lemma 3.2 *If an irreducible representation $\rho \in R(E_{K_n})$ factors through $\pi_1(M)$, then the eigenvalue $\mathcal{M}$ satisfies that $A_{K_n}(\mathcal{M}^{-4}, \mathcal{M}) = 0$, where $A_{K_n}(\mathcal{L}, \mathcal{M})$ is the A-polynomial of K_n.*

Proof The A-polynomial $A_{K_n}(\mathcal{L}, \mathcal{M})$ gives the defining equation of $R(\partial E_{K_n})$. Since the peripheral group $\pi_1(\partial E_{K_n})$ is an abelian group, we can assume that the images of $\rho(m)$ and $\rho(\ell)$ are upper triangular matrices whose diagonal entries are $\mathcal{M}^{\pm 1}$ and $\mathcal{L}^{\pm 1}$ respectively. Then we can rewrite the constraint that $\rho(m^4\ell) = \mathbf{1}$ as $\mathcal{L} = \mathcal{M}^{-4}$. The lemma follows. ∎

Lemma 3.3 *The A-polynomial of K_n for $\mathcal{L} = \mathcal{M}^{-4}$ is expressed as*

$$A_{K_n}(\mathcal{M}^{-4}, \mathcal{M}) = \begin{cases} \mathcal{M}^{-8n}(\mathcal{M} + \mathcal{M}^{-1})^{2n} & n > 0, \\ \mathcal{M}^{-8|n|+3}(\mathcal{M} + \mathcal{M}^{-1})^{2|n|-1} & n < 0. \end{cases}$$

Proof Since the knot K_n is the mirror image of $J(2, -2n)$ in [HS04], the A-polynomial $A_{K_n}(\mathcal{L}, \mathcal{M})$ coincides with $A_{J(2,-2n)}(\mathcal{L}, \mathcal{M}^{-1})$. Hence, we have that

$$A_{K_n}(\mathcal{M}^{-4}, \mathcal{M}) = A_{J(2,-2n)}(\mathcal{M}^{-4}, \mathcal{M}^{-1}).$$

By induction and the recursive formula in [HS04, Theorem 1], one can show that

$$A_{J(2,2n)}(\mathcal{M}^{-4}, \mathcal{M}^{-1}) = \begin{cases} \mathcal{M}^{-8n+3}(\mathcal{M}+\mathcal{M}^{-1})^{2n-1} & n > 0, \\ \mathcal{M}^{-8|n|}(\mathcal{M}+\mathcal{M}^{-1})^{2|n|} & n < 0. \end{cases}$$

The lemma then follows. ∎

3.2 The Restrictions to Seifert Pieces

We will see the restriction of $\overline{\rho} \in R^{\mathrm{irr}}(M)$ to the fundamental group of each Seifert piece. Recall that the graph manifold M is the union the torus knot exterior $E_{T(2,2n+1)}$ and the twisted I-bundle $N(Kb)$ over the Klein bottle Kb. The fundamental group $\pi_1(M)$ contains the twist knot group $\pi_1(E_{K_n})$ and the torus knot group $\pi_1(E_{T(2,2n+1)})$. We will distinguish the pairs of meridian and longitude for these knots by using (m, ℓ) for the twist knot K_n and (μ, λ) for the torus knot $T(2, 2n+1)$.

Proposition 3.4 *For every $\overline{\rho} \in R^{\mathrm{irr}}(M)$, the restriction of $\overline{\rho}$ to $\pi_1(E_{T(2,2n+1)})$ is abelian.*

Proof It was shown by [Ter03, Theorem 1.2] that a twist knot K_n bounds a once-punctured Klein bottle whose boundary slope is 4. We can think of loops in $E_{T(2,2n+1)}$ as loops outside a non-orientable spanning surface of K_n in E_{K_n}. A loop γ outside a non–orientable spanning surface of K_n has an even linking number with K_n. Write $\gamma \in \pi_1(E_{K_n})$ as

$$\gamma = m^{\ell\mathrm{k}(\gamma,K_n)}(m^{-\ell\mathrm{k}(\gamma,K_n)}\gamma).$$

Note that $\ell\mathrm{k}(\gamma, K_n)$ is an even integer and $m^{-\ell\mathrm{k}(\gamma,K_n)}\gamma$ is a commutator. By Proposition 3.1, one can see that $\overline{\rho}$ is induced by an irreducible metabelian representation ρ of $\pi_1(E_{K_n})$. Since ρ sends m^2 and the commutator subgroup to $-\mathbf{1}$ and an abelian subgroup, respectively, the image of $\pi_1(E_{T(2,2n+1)})$ by ρ is contained in the abelian subgroup. ∎

In general, any abelian representation of a knot group $\pi_1(E_K)$ is determined, up to conjugation, by the eigenvalues of the matrix corresponding to a meridian. This follows from the fact that any abelian representation factors through the abelianization $\pi_1(E_K) \to H_1(E_K; \mathbb{Z})$, and $H_1(E_K; \mathbb{Z})$ is generated by the homology class of a meridian of K.

Lemma 3.5 *For every $\overline{\rho} \in R^{\mathrm{irr}}(M)$, the restriction to $\pi_1(E_{T(2,2n+1)})$ is determined by the eigenvalues of $\overline{\rho}(\mu)$ up to conjugation. Here, μ is a meridian of $T(2, 2n+1)$.*

Remark 3.6 By conjugation, we can assume that the $\overline{\rho}(a)$, $\overline{\rho}(b)$, and $\overline{\rho}(y)$ are diagonal matrices and $\overline{\rho}(x)$ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ for any $\overline{\rho} \in R^{\mathrm{irr}}(M)$. This is due to the fact that $\overline{\rho}(a)$, $\overline{\rho}(b)$, and $\overline{\rho}(y) = \overline{\rho}(\mu)^{-1}$ are contained in the same maximal abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$. Thus, every $\overline{\rho} \in R^{\mathrm{irr}}(M)$ itself is determined by the eigenvalues of $\overline{\rho}(\mu)$ up to conjugation.

Furthermore, the set of eigenvalues is determined as follows.

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Proposition 3.7 *Suppose that $\rho_k \in R(K_n)$ is an irreducible metabelian representation in Proposition 2.14 and μ is a meridian of the torus knot in presentation (2.3). Let $\xi_k^{\pm 1}$ be the eigenvalues of $\overline{\rho}_k(\mu)$. Then the set $\{\xi_k^{\pm 1} \mid k = 1, \dots, (|4n+1|-1)/2\}$ is given by*

$$\left\{ e^{\pm\theta\sqrt{-1}} \mid \theta = \pi(2j-1)/|4n+1|, j = 1, \dots, (|4n+1|-1)/2 \right\}.$$

Proof Let p be $|4n+1|$. We regard elements of $\pi_1(E_{T(2,2n+1)})$ as the products $m^{2r}\gamma$ where $r \in \mathbb{Z}$ and γ is a commutator of $\pi_1(E_{K_n})$, as in the proof of Proposition 3.4. It follows from [Yam13, Proposition 2.8] that the eigenvalues of $\rho_k(\gamma)$ are p-th roots of unity. Since $\rho_k(m^2) = -\mathbf{1}$ and p is odd, one can see that for the generators a and $b \in \pi_1(E_{T(2,2n+1)})$,

$$\overline{\rho}_k(a)^p = \pm\mathbf{1} \quad \text{and} \quad \overline{\rho}_k(b)^p = \pm\mathbf{1}.$$

By the relation $a^2 = b^{2n+1}$, we can conclude that $\overline{\rho}_k(b)^p = \mathbf{1}$. On the other hand, we can see that $\overline{\rho}_k(a)^p = -\mathbf{1}$, since the image of $\pi_1(E_{T(2,2n+1)})$ by $\overline{\rho}_k$ contains $-\mathbf{1}$ and p is odd. Hence the eigenvalues $\xi_k^{\pm 1}$ of $\overline{\rho}_k(\mu) = \overline{\rho}_k(b^{-n}a)$ satisfy that $\xi_k^{\pm p} = -1$. We can exclude the case that $\overline{\rho}_k(\mu) = -\mathbf{1}$ by the irreducibility of $\overline{\rho}_k$.

There exist at least $(|4n+1|-1)/2$ distinct pairs of eigenvalues by Proposition 3.1 and Remark 3.6. On the other hand, there exist at most $(|4n+1|-1)/2$ distinct pairs in the set of $2p$-th roots of unity to be the eigenvalues $\xi_k^{\pm 1}$ of $\rho_k(\mu)$ $(k = 1, \dots, (|4n+1|-1)/2)$. This proves Proposition 3.7. ∎

Corollary 3.8 *The order of $\overline{\rho}_k(\mu)$ is given by $2p_k$ where p_k divides $|\Delta_{K_n}(-1)| = |4n+1|$.*

We next turn to the restriction to $\pi_1(N(Kb))$.

Proposition 3.9 *For every $\overline{\rho} \in R^{\mathrm{irr}}(M)$, the restriction of $\overline{\rho}$ to $\pi_1(N(Kb))$ is irreducible.*

Proof Following the notation of Proposition 2.7, we denote by x and y the generators of $\pi_1(N(Kb))$. Note that $\operatorname{tr}\overline{\rho}(y) = \operatorname{tr}\overline{\rho}(\mu)^{-1}$, since $\mu = y^{-1}$ in the presentation (2.3). Proposition 3.7 shows that $\operatorname{tr}\overline{\rho}(y) \neq \pm 2$. Note that $\pi_1(N(Kb))$ is isomorphic to $\pi_1(Kb)$. Under the identification between $\pi_1(N(Kb))$ and $\pi_1(Kb)$, the restriction of $\overline{\rho}$ to $\pi_1(N(Kb))$ is an $\mathrm{SL}_2(\mathbb{C})$-representation of the type given in Example 2.4(b), and hence is irreducible. ∎

4 Asymptotic Behavior of Reidemeister Torsion for Graph Manifolds

We will consider the limit of the leading coefficient in the asymptotic behavior of Reidemeister torsion. We use the symbols $\xi_k^{\pm 1}$ to denote the eigenvalues of $\overline{\rho}_k(\mu)$. We will compute the higher dimensional Reidemeister torsion and its asymptotic behavior for M from the decomposition of a graph manifold.

Proposition 4.1 *Let ρ_k be an irreducible metabelian representation. Then the Reidemeister torsion* $\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)$ *is expressed as*

$$\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)=\prod_{i=1}^{N}\frac{\Delta_{T(2,2n+1)}(\xi_k^{2i-1})\Delta_{T(2,2n+1)}(\xi_k^{-2i+1})}{(\xi_k^{2i-1}-1)(\xi_k^{-2i+1}-1)}.$$

Proof Applying Lemma 2.5 to the decomposition $M=E_{T(2,2n+1)}\cup N(Kb)$, we have that

$$\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)=\mathrm{Tor}(E_{T(2,2n+1)};\sigma_{2N}\circ\overline{\rho}_k)\,\mathrm{Tor}(N(Kb);\sigma_{2N}\circ\overline{\rho}_k).$$

By Proposition 3.4 and Corollary 3.8, the restriction $\overline{\rho}_k$ to $\pi_1(E_{T(2,2n+1)})$ is an abelian representation such that the matrix $\overline{\rho}_k(\mu)$ corresponding to a meridian has an even order. Our claim follows from Lemmas 4.2 and 4.3. ∎

Lemma 4.2 *The Reidemeister torsion* $\mathrm{Tor}(N(Kb);\sigma_{2N}\circ\overline{\rho}_k)$ *is equal to 1 for all N.*

Proof By the homotopy equivalence between $N(Kb)$ and Kb, $\mathrm{Tor}(N(Kb);\sigma_{2N}\circ\overline{\rho}_k)$ coincides with $\mathrm{Tor}(Kb;\sigma_{2N}\circ\overline{\rho}_k)$. The Reidemeister torsion $\mathrm{Tor}(Kb;\sigma_{2N}\circ\overline{\rho}_k)$ is given by equation (2.2). The eigenvalues of $\sigma_{2N}\circ\overline{\rho}_k(y)=\sigma_{2N}\circ\overline{\rho}_k(\mu)^{-1}$ are given by $\xi_k^{\mp(2i-1)}$ $(i=1,\dots,N)$. Proposition 3.7 shows that the orders of $\xi_k^{\pm1}$ are even. Hence, $\sigma_{2N}\circ\overline{\rho}_k(y)$ does not have the eigenvalue 1 for any N. Hence, Example 2.4 shows that

$$\mathrm{Tor}(N(Kb);\sigma_{2N}\circ\overline{\rho}_k)=\mathrm{Tor}(Kb;\sigma_{2N}\circ\overline{\rho}_k)=\frac{\det(\mathbf{1}-Y)}{\det(Y-\mathbf{1})}=1$$

for any N. ∎

Lemma 4.3 *Let φ be an abelian representation of a knot group $\pi_1(E_K)$ that sends a meridian to a matrix with eigenvalues $\xi^{\pm1}$. If ξ is not a $(2r-1)$-root of unity for any $r\in\mathbb{N}$, then the Reidemeister torsion* $\mathrm{Tor}(E_K;\sigma_{2N}\circ\varphi)$ *is expressed as*

$$\mathrm{Tor}(E_K;\sigma_{2N}\circ\varphi)=\prod_{i=1}^{N}\frac{\Delta_K(\xi^{2i-1})}{\xi^{2i-1}-1}\frac{\Delta_K(\xi^{-2i+1})}{\xi^{-2i+1}-1}$$

for all N.

Proof Since φ is abelian, φ factors through $H_1(E_K;\mathbb{Z})$, and we can assume that φ sends all meridians to the matrix $\begin{pmatrix}\xi & *\\ 0 & \xi^{-1}\end{pmatrix}$ up to conjugation. Then $\mathrm{Tor}(E_K;\varphi)$ is given by

$$\frac{\Delta_K(\xi)\Delta_K(\xi^{-1})}{(\xi-1)(\xi^{-1}-1)}.$$

This formula follows from a computation similar to that in [Yam07, proof of Proposition 3.8]. which shows how to compute the Reidemeister torsion of E_K for a representation sending all meridians to upper triangular matrices with diagonal entries ξ and ξ^{-1}. A computation similar to that in [Yam07, proof of Proposition 3.8] shows that $\mathrm{Tor}(E_K;\varphi)$ is a fraction whose numerator is given by the product of $\epsilon\xi^{n'}\Delta_K(\xi)$

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and $\epsilon\xi^{-n'}\Delta_K(\xi^{-1})$, where $\epsilon \in \{\pm 1\}$ and $n' \in \mathbb{Z}$. The denominator of $\mathrm{Tor}(E_K;\varphi)$ is expressed as

$$\det\Big(\begin{pmatrix}\xi & * \\ 0 & \xi^{-1}\end{pmatrix} - \mathbf{1}\Big) = (\xi - 1)(\xi^{-1} - 1).$$

Note that the sign term τ_0 of [Yam07, Proposition 3.8] is dropped from our definition of $\mathrm{Tor}(E_K;\sigma_{2N}\circ\varphi)$ since the Reidemeister torsion has no sign ambiguity for $\mathrm{SL}_{2N}(\mathbb{C})$-representations.

The $\mathrm{SL}_{2N}(\mathbb{C})$-representation $\sigma_{2N}\circ\varphi$ is decomposed into the direct sum $\oplus_{i=1}^{N}\varphi_i$, where φ_i is an abelian representation sending a meridian to an $\mathrm{SL}_2(\mathbb{C})$-matrix with eigenvalues $\xi^{\pm(2i-1)}$. For the direct sum of representations, the Reidemeister torsion is given by the product of those for each direct summand φ_i. This implies our claim. ■

Theorem 4.4 *Let $\overline{\rho}_k$ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$, which sends μ to a matrix of order $2p_k$ where p_k is a divisor of $p = |\Delta_{K_n}(-1)|$. Then the growth order of $\log|\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)|$ is equal to $2N$. Moreover, the convergence of the leading coefficient is expressed as*

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)|}{2N} = \frac{1}{p_k}\Big(\log|\Delta_{T(2,2n+1)}(-1)| - \log 2\Big). \tag{4.1}$$

Proof It is sufficient to show that the left-hand side of (4.1) converges to the right-hand side. By Proposition 4.1, the left-hand side of (4.1) turns out to be

$$\begin{aligned}\lim_{N\to\infty}&\frac{\log|\mathrm{Tor}(M;\sigma_{2N}\circ\overline{\rho}_k)|}{2N}\\ &= \lim_{N\to\infty}\frac{1}{2N}\sum_{i=1}^{N}\log|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})\Delta_{T(2,2n+1)}(\xi_k^{-(2i-1)})|\\ &\quad + \lim_{N\to\infty}\frac{1}{2N}\sum_{i=1}^{N}\log|(\xi_k^{2i-1}-1)(\xi_k^{-2i+1}-1)|^{-1}.\end{aligned}$$

The eigenvalues $\xi_k^{\pm 1}$ are primitive $2p_k$-th roots of unity as in Propositions 3.7 and 3.8. It follows from [Yam, Proposition 3.9] that the second term in the right-hand side converges to $-(\log 2)/p_k$. Note that we can ignore the indeterminacy of a factor t^j ($j \in \mathbb{Z}$) in the Alexander polynomial in the computation of the first term. The first term is rewritten as

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{2N}\sum_{i=1}^{N}\log|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})\Delta_{T(2,2n+1)}(\xi_k^{-(2i-1)})| \\ &\quad= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\log|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})|\\ &\quad= \frac{1}{p_k}\sum_{i=1}^{p_k}\log|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})|\\ &\quad= \frac{1}{p_k}\log\prod_{i=1}^{p_k}|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})|\end{aligned} \tag{4.2}$$

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by the symmetry that $\Delta_K(t) = t^j \Delta_K(t^{-1})$ $(j \in \mathbb{Z})$ and [Yam, Lemma 3.11]. The Alexander polynomial $\Delta_{T(2,2n+1)}(t)$ is given by $(t^{2n+1}+1)/(t+1)$. We have seen that p_k is a divisor of p in Corollary 3.8. Since $\gcd(p, 2n+1) = 1$, we can see that $\gcd(2p_k, 2n+1) = 1$. From this, we see that the denominator coincides with the numerator in the product of $|\Delta_{T(2,2n+1)}(\xi_k^{2i-1})|$ except for $i = (p_k+1)/2$; *i.e.,* we have that

$$\prod_{\substack{1 \le i \le p_k, \\ i \ne (p_k+1)/2}} |\Delta_{T(2,2n+1)}(\xi_k^{2i-1})| = \prod_{\substack{1 \le i \le p_k, \\ i \ne (p_k+1)/2}} \frac{|\xi_k^{(2i-1)(2n+1)} + 1|}{|\xi_k^{2i-1} + 1|} = 1.$$

The right-hand side of (4.2) is thus $(\log |\Delta_{T(2,2n+1)}(-1)|)/p_k$. Hence, the left-hand side of (4.1) is $(\log |\Delta_{T(2,2n+1)}(-1)| - \log 2)/p_k$. ∎

It follows from Proposition 3.7 that the integer p_k, which gives the order of $\overline{\rho}(\mu)$ by $2p_k$, runs over all divisors of $|\Delta_{K_n}(-1)|$ except for 1.

Corollary 4.5 *The set of the limits of the leading coefficients is given by*

$$\left\{ \frac{1}{p_k} \Big(\log |\Delta_{T(2,2n+1)}(-1)| - \log 2 \Big) \;\Big|\; p_k > 1, p_k \textit{ is a divisor of } |\Delta_{K_n}(-1)| \right\}. \tag{4.3}$$

In particular, the minimum in the set (4.3) *is given by*

$$\Big(\log |\Delta_{T(2,2n+1)}(-1)| - \log 2 \Big) / \big| \Delta_{K_n}(-1) \big| = \Big(\log |2n+1| - \log 2 \Big) / |4n+1|.$$

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