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BOOK REVIEWS

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Peter Kronheimer and Tomasz Mrowka *Monopoles and Three-Manifolds* (Cambridge University Press, 2007), 808 pp, 9 780 521 88022 0 (hardback), £80.00, 9 780 521 18476 2 (paperback), £40.00.

Morse theory computes the homology groups of a manifold by analysing the critical points and gradient flow of a given function f on the manifold. Floer theory is an analogue of Morse theory for certain infinite-dimensional spaces: specifically, the infinite-dimensional spaces should be associated somehow with 3-manifolds Y. This book constructs Floer theory for the Chern–Simons–Dirac functional $\mathcal{L}$ on the 'configuration space' $\mathcal{C}$ of sections of and connections on a complex spinor bundle over Y, or, more precisely, the quotient of this space by the action of the gauge group $\mathcal{G}$. This particular version of Floer theory has its roots in a celebrated paper of Witten [**1**], which introduced the Seiberg–Witten equations (the gradient flow equations for $\mathcal{L}$) to the mathematical world. In this book the Floer theory associated with the Seiberg–Witten equations is developed from scratch, assuming only a basic grounding in functional analysis and geometry. It represents an important development, being the first textbook on Seiberg–Witten Floer theory, and only the second on Floer theory (it is preceded by Donaldson's book [**2**] on Floer theory for the instanton equations).

The book has two major goals: to construct the Floer homology groups for the Seiberg–Witten equations; and to show their functorality (essentially that cobordisms induce maps between Floer homology groups). Roughly two-thirds of the book is devoted to developing the machinery necessary to achieve these two goals; the remaining third outlines some interesting developments of the theory. The functorality is particularly important, because it implies that the Floer homology groups are topological invariants of the underlying 3-manifold Y.

The first chapter of the book introduces the main ideas of Floer theory and summarizes the results that will be proved. A significant novelty of the Seiberg–Witten Floer homology compared with standard Morse homology is that one is dealing with the quotient $\mathcal{C}/\mathcal{G}$ rather than the manifold $\mathcal{C}$: since the action of $\mathcal{G}$ is not free, the quotient may not be smooth. The solution is to use a blow-up of $\mathcal{C}$, whose quotient is a smooth manifold with boundary: the authors helpfully give the finite-dimensional analogue of this construction. One ends up with three kinds of homology group, computing the homology of the manifold, the homology of its boundary and the homology of the pair.

As is usually the case in Floer homology, the analytical underpinnings of the subject are significant, dealing with infinite-dimensional manifolds; Chapters 2–5 are devoted to developing this theory in detail. One needs to show that the moduli spaces of solutions to the gradient flow (Seiberg–Witten) equations are manifolds (or something similar) and compute their dimension (in the present case, using index theorems and spectral flow). One needs to show that the moduli spaces (or suitable completions of them) are compact. One also needs to show that the Chern–Simons–Dirac functional can be perturbed so that it has no degenerate critical points (and other nice properties).

Chapters 6 and 7 are devoted to the construction of the Floer homology groups and to the proof of functorality. Apart from analytic ingredients, these objectives require a systematic way of orienting moduli spaces. The authors also describe in these chapters the natural grading set for the Floer homology groups and discuss the relationship between Floer homology and the monopole invariants of 4-manifolds.

Chapter 8 discusses a more general Floer theory based on perturbations of the Seiberg–Witten equations by 2-forms. Chapter 9 is devoted to calculating Floer homology groups in a number of situations. Finally, Chapter 10 describes some links between Floer theory and other areas of mathematics: an integer invariant for homology 3-spheres is constructed; genus bounds on the Chern numbers of complex spinor bundles are discussed; non-vanishing theorems are proved for the Floer groups of manifolds with taut foliations; and the relationship with the Alexander polynomial of a knot is discussed.

The book is presented systematically, with the aim of giving as complete a picture of the Floer homology groups as possible. The style nicely complements Donaldson's book [**2**], which offers a more rapid, but less detailed, introduction. Many of the arguments used in this book are drawn from the literature, and the authors have carefully pointed out the original sources. Newcomers to the subject will no doubt find the references to the literature helpful in orientating themselves.

In short, there is little doubt that this book will become a standard reference for mathematicians working in the areas of gauge theory and Floer theory.

References

1. E. Witten, Monopoles and 4-manifolds, *J. Diff. Geom.* **17** (1982), 661–692.
2. S. K. Donaldson, *Floer homology groups in Yang–Mills theory*, Cambridge Tracts in Mathematics, Volume 147 (Cambridge University Press, 2002).

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