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On the theory of soluble factorizable groups

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Suppose that a finite soluble group G is the product AB of subgroups A and B. Our question is the following: what conclusions can be made about G if A and B are suitably restricted? First we shall prove that the p-length of G is restricted by the derived lengths of the Sylow p-subgroups of A and B, if A and B are p-closed and p'-closed. Moreover, if in such a group the Sylow p-subgroups of A and B are modular, the p-length of G is at most 1. Next we obtain a general estimate for the derived length of the group $G = AB$ of odd order in terms of the derived lengths of A and B. Furthermore it will be possible to bound the nilpotent length of G and also the p-length of G in terms of other invariants of special subgroups of G.

Suppose that a finite group G is the product AB of subgroups A and B. Our question is the following: what conclusions can be made about G if A and B are suitably restricted? For example, there is the well-known fact, first proved by Wielandt and Kegel [3, VI, 4.3], that G is soluble if A and B are nilpotent. But it is possible to get some more information about the structure of G. So, for example, Gross proved in [1] that the derived length of G modulo its Frattini subgroup is at most the sum of the (nilpotent) classes of A and B.

In the first section we shall show that there are similar conclusions if the subgroups A and B are only p-decomposed for a prime p; that

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is p-closed and p'-closed. (The group G is π-closed for a set π of primes if the π-elements of G generate a (normal) π-group.) The basis of this section is a result on primitive soluble groups. It says that in such a group G which is the product of the p-decomposed subgroups A and B, where p is the characteristic of the only minimal normal subgroup M of G, one may assume that one of A and B is a Sylow p-subgroup while the other is a Hall p'-subgroup of G. Then we shall generalize results of Gross [1], Maier [5], Ward [7], and Kegel [4]. It may be pointed out that the arguments presented here shorten the analogous arguments in the papers mentioned.

In the second section there are no special restrictions on the subgroups A and B. We obtain a general estimate for the derived length of the group $G = AB$ of odd order in terms of the derived lengths of A and B. As a corollary of this theorem we get a generalization of a result of Inagaki [3, VI, 9.10]. Furthermore it will be possible to bound the nilpotent length of G and also the p-length of G in terms of other invariants of special subgroups of G.

Wielandt proved that a group G is soluble if and only if it has three subgroups with pairwise relatively prime indices. At the end of this note we shall prove a result that restricts the structure of such a group G in terms of the derived lengths of the three subgroups with relatively prime indices. Throughout this paper all groups considered will be finite and soluble unless otherwise stated. The notation agrees with that in Huppert's book [3].

1.

For the following considerations we first need a lemma, which is a generalization of a theorem of Kegel [4].

LEMMA 1. *Let π be a set of primes and $G = AB$ with $A \neq B$, where A is a nilpotent π-group and B a π-closed subgroup with a nilpotent Hall π-subgroup. Then there is a proper normal subgroup K of G containing A or B.*

Proof. If $A = 1$, then any normal subgroup of G contains A. So we may assume without loss of generality that $A \neq 1$. Let G be a minimal counter-example to the lemma and M be a minimal normal subgroup

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of G. Let q be the characteristic of M. If $AM < G$ then, since $G/M = (AM/M)(BM/M)$, due to the minimality of G there exists a proper normal subgroup K/M of G/M containing AM/M or BM/M. But then K is a proper normal subgroup of G containing A or B, a contradiction. Hence suppose $AM = G = BM$. If $q \in \pi$, then G is a π-group, and in this case G is the product of two nilpotent subgroups by the hypothesis. Now the theorem of Kegel [4] mentioned above gives the desired contradiction. So let $q \notin \pi$. Then $G/M = AM/M \cong A/(M\cap A) = A$, and therefore G/M is a nilpotent π-group. So we have $B_{\pi'} = G_{\pi'} = M$ and $B \geq M$.

If $B < G$, then any maximal proper subgroup $K \geq B$ is normal by the nilpotency of G/M. So we may assume $B = G$. Since B is π-closed we get $B_\pi = G_\pi = A \trianglelefteq G$. As A is a proper subgroup of G, $K = A$ is a proper normal subgroup of G which contains A, a final contradiction. //

DEFINITION. A group G is called p-decomposed for a prime p, if G is p-closed and p'-closed; that is, if the Sylow p-subgroup is a direct factor of G.

A basic result for our considerations in this section is the following lemma.

LEMMA 2. *Let* $G = AB$ *with* $A \neq B$ *be a primitive group and* p *the characteristic of the only minimal normal subgroup* M *of* G. *Assume that* A *and* B *are* p*-decomposed subgroups. Then one of* A *and* B *is a Sylow* p*-subgroup of* G *while the other is a Hall* p'*-subgroup of* G.

Proof. Since A and B are p'-closed subgroups of G, it follows from [6, Lemma I(2)] that $[A_{p'}, B_{p'}] \leq O_{p'}(G) = 1$. Let $g = ab \in G$ with $a \in A$ and $b \in B$. Then

$$\left[A_{p'}^{g}, B_{p'}\right] = \left[A_{p'}^{ab}, B_{p'}\right] = \left[A_{p'}^{b}, B_{p'}\right] = [A_{p'}, B_{p'}]^{b} = 1 .$$

So we get $\left[A_{p'}^{G}, B_{p'}\right] = 1$. If $A_{p'}$ is nontrivial, then $A_{p'}^{G} \geq M$ since M is the only minimal normal subgroup of G. Therefore $B_{p'}$ centralize the (selfcentralizing) normal p-subgroup M. So $B_{p'} = 1$. In any case $A_{p'}$ or $B_{p'}$ is trivial. So, without loss of generality, let A be a

p-group. It follows that $B_{p'} \in \mathrm{Hall}_{p'}(G)$. Now choose a complement W of M in G containing $B_{p'}$. (This is possible since all complements of M are conjugate.) Since B is p-closed and p'-closed, the Sylow p-subgroup B_p of B centralizes the Hall p'-subgroup $B_{p'}$ of B . Let $T = F(W)$ be the Fitting subgroup of W . Then we have $C_W(T) \le T$, and T is a p'-group since $O_p(W) = 1$ by the primitivity of G . Hence T is contained in $B_{p'}$. Therefore we get $B_p \le C_G(B_{p'}) \le C_G(T) \le N_G(T) = W$, since W is a maximal subgroup of G and G contains only one minimal normal subgroup. So we have $C_G(T) = C_W(T) \le T$. Therefore we get $B_p \le T$. As T is a p'-subgroup and B_p a p-subgroup we have $B_p = 1$. Therefore $B \in \mathrm{Hall}_{p'}(G)$ and $A \in \mathrm{Syl}_p(G)$. //

A group is called modular if its subgroup lattice is modular in the lattice-theoretical sense. A nilpotent group is modular if and only if any two of its subgroups are permutable.

THEOREM 1. *Let the group* G *be the product of the* p*-decomposed subgroups* A *and* B *. Assume that the Sylow* p*-subgroups of* A *and* B *are modular. Then the* p*-length of* G *is at most* 1 .

Proof. Since the class of all groups with p-length at most 1 is a saturated formation (see, for example, [3, p. 689]), a minimal counter-example G to Theorem 1 is primitive. Therefore G has exactly one minimal normal subgroup, M, and we may assume that the characteristic of M is p . Further, suppose $A \ne B$, so that the hypothesis of Lemma 2 is satisfied. Then Lemma 2 implies that, without loss of generality, $A \in \mathrm{Syl}_p(G)$ and $B \in \mathrm{Hall}_{p'}(G)$. Further, Lemma 1 implies (with $\pi = \{p\}$) that there exists a normal subgroup K of G which contains A or B . Suppose K is a maximal normal subgroup with this property.

First, let $A \le K$. Then $K = AB_1$ with $B_1 = K \cap B$. Therefore the maximal normal subgroup K satisfies the hypothesis of the theorem. Hence due to the minimality of G it follows that the p-length of K is at most 1 . In particular p does not divide the order of K/M . Therefore $A \le M$ and we get that G/M is a p'-group. Hence the p-length of G is as most 1 , a contradiction.

So we may assume $B \leq K$. Again K satisfies the hypothesis of the theorem $(K = A_1B$ with $A_1 = A \cap K)$ and therefore due to the minimality of G it follows that the p-length of K is at most 1. We get $p \nmid |K/M|$. Therefore M is a maximal normal subgroup of A. Let W be a complement of M in G. Then $L = A \cap W$ is a complement of M in A. It follows from [5, Lemma 3] that A is an abelian subgroup and hence $A = M$. Therefore we get that G/M is a p'-group and we conclude that the p-length of G is at most 1, a final contradiction. //

The symmetric group S_4 on four elements is the product of a symmetric group S_3 on three elements and an elementary abelian subgroup of order 4, the V_4. S_3 and V_4 are p'-closed (for $p = 2$) and have modular Sylow subgroups. But S_4 has p-length 2.

Further, S_4 is the split extension of the alternating group A_4 on four elements by a cyclic group C_2 of order 2. Here A_4 and C_2 are 2-closed and have modular Sylow subgroups. Thus we see that in Theorem 1 we can omit neither the assumption that the subgroups A and B are p-closed nor the assumption that they are p'-closed.

THEOREM 2. *If the group G is the product of two nilpotent subgroups A and B which have modular Sylow p-subgroups, then the nilpotent residual G^N is p'-closed.*

Proof. A minimal counter-example G is primitive and the minimal normal subgroup M has characteristic p again. By Theorem 1, the p-length of G is at most 1, in particular G/M is a p'-group. By Lemma 2 we may assume that A is a Sylow p-subgroup and B is a Hall p'-subgroup of G. Since B is nilpotent, also G/M is nilpotent, and so the nilpotent residual of G is a p-group; in particular it is p'-closed, a contradiction. //

The quaternion group $B = Q_8$ of order 8 has a faithful irreducible G-module A (of degree 2) over $GF(5)$. Let G be the split extension of A by B. Then G is the product of A and B and these are nilpotent subgroups of G with modular p-subgroups for $p = 5$. But G' is not p'-closed. The symmetric group $S_3 = B$ on three elements has a

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faithful irreducible G-module A over $\mathrm{GF}(p)$ (with $p \geq 5$). Let G be the split extension of A by B . Then A and B are p-decomposed $(p = 5)$, but the nilpotent residual G^N is not p'-closed.

In the following theorem we shall prove that the p-length of G is restricted by the derived lengths of the Sylow p-subgroups of A and B , if G is the product of the p-decomposed subgroups A and B . (Denote by $l_p(G)$ the p-length of G and by $d_p(G)$ the derived length of the Sylow p-subgroups of G .)

THEOREM 3. *Let* $G = AB$ *be the product of the* p*-decomposed subgroups* A *and* B *. If the nilpotent class of the Sylow* 2*-subgroups of both* A *and* B *is at most* 3 *; then we have* $l_p(G) \leq \max\left(d_p(A), d_p(B)\right)$.

Proof. A minimal counter-example G is primitive and we may again assume that the characteristic of the only minimal normal subgroup M of G is p . Then the hypothesis of Lemma 2 is satisfied. So we may assume, without loss of generality, A to be a Sylow-p-subgroup and B to be a Hall p'-subgroup. Therefore we have $l_p(G) = l_p(G/M) + 1 \leq d_p(A/M) + 1$. Since G is a counter-example we must have $d_p(A/M) = d_p(A)$. Then it follows from Hall and Higman [2, Theorem 3.2.1] that $p = 2$ and $\mathrm{cl}(A) = 3$. Lemma 3 in [1] then implies that $l_2(G) \leq 2$. Since A is not abelian, we get $l_2(G) \leq \max\left(d_2(A), d_2(B)\right)$, a final contradiction. //

In this context we refer to the following theorem.

THEOREM 4. *Let* $2 \nmid |G|$ *and assume that* A_i *are subgroups of* G *with* $\gcd_i \left(|G : A_i|\right) = 1$ $(i = 1, \ldots, r)$ *. If* $\exp_p\left(A_i\right)$ *denotes the exponent of the Sylow* p*-subgroups of* A_i *, the* p*-length of* G *is restricted by* $l_p(G) \leq \max_i \left(\exp_p\left(A_i\right)\right)$.

Proof. Since $\gcd_i \left(|G : A_i|\right) = 1$, there exists a $j \in \{1, \ldots, r\}$ with $p \nmid |G : A_j|$. A Sylow p-subgroup P of A_j therefore is a Sylow p-subgroup of G . So we have $\exp_p(G) = \exp_p\left(A_j\right)$. By Hall and Higman [2] we get $l_p(G) \leq \exp_p(G)$, and therefore

$$l_p(G) \le \exp_p(A_j) = \max_j \left(\exp_p(A_i)\right) . \qquad //$$

The proofs of Theorems 3 and 4 show how to get analogous estimates for the p-length of G from the corresponding theorems of Hall and Higman [2]. So, for example, the following is true.

Let 2 and 3 not divide the order of G and let A_i be subgroups of G with $\gcd_i \left(|G : A_i|\right) = 1$ $(i = 1, \ldots, r)$. If the Sylow subgroups of A_i have nilpotent classes not greater than 2, the nilpotent length of G is restricted by $l(G) \le \sum_{i=1}^{r} l\left(A_i\right)$. If 2 divides the order of G, this result becomes false as one can see by the counter-example S_4. S_4 is the product of the quaternion group of order 8, the Q_8, and a cyclic subgroup C_3. We have $\mathrm{cl}\left(Q_8\right) = 2$ and $\mathrm{cl}\left(C_3\right) = 1$, but $l(G) = 3 > 2 = l\left(Q_8\right) + l\left(C_3\right)$.

Finally we want to generalize a theorem of Ward [7]. But before we do this we prove the following result.

THEOREM 5. *Let G be the product of the p-decomposed subgroups A and B. Assume that A is abelian and that the derived length of B is at most r. Then $G^{\left(\mathsf{S}_p\mathsf{A}\right)^r}$ is p'-closed. (Here $\mathsf{S}_p = \{G \mid G$ is a p-group$\}$ and $\mathsf{A} = \{G \mid G$ is abelian$\}$ and $G^{\left(\mathsf{S}_p\mathsf{A}\right)^r}$ denotes the residual of the formation $\left(\mathsf{S}_p\mathsf{A}\right)^r$.)*

Proof. The class

$$\mathsf{X} = \mathsf{S}_{p'}\mathsf{S}_p\left(\mathsf{S}_p\mathsf{A}\right)^r = \left\{G \mid G^{\left(\mathsf{S}_p\mathsf{A}\right)^r} \text{ is } p'\text{-closed}\right\}$$

is a saturated formation. Therefore a minimal counter-example is primitive and we again may assume that the minimal normal subgroup M has characteristic p. Lemma 2 implies that either $A \in \mathrm{Syl}_p(G)$ and $B \in \mathrm{Hall}_{p'}(G)$ or $B \in \mathrm{Syl}_p(G)$ and $A \in \mathrm{Hall}_{p'}(G)$. So we have to distinguish two cases.

First, let $A \in \mathrm{Syl}_p(G)$ and $B \in \mathrm{Hall}_{p'}(G)$. Since A is abelian and $A \geq M$ we get $A = M$. Therefore $G/M \cong B$, and so we have $G^r \leq M$; in particular $G^{(S_pA)^r} \leq M$. Since M is p'-closed, we get a contradiction.

So we may assume that $B \in \mathrm{Syl}_p(G)$ and $A \in \mathrm{Hall}_{p'}(G)$. In this case we have $B \geq M$. By the lemma of Ward [7] we have that $B^{(r-1)}$ centralizes any abelian normal subgroup of G/M. Since by the primitivity of G the Fitting subgroup of G/M is a p'-group, and since A is an abelian Hall p'-subgroup, $F_2(G)/M = F(G/M)$ is abelian. Therefore we get $B^{(r-1)}M/M \leq C_{G/M}(F_2(G)/M) \leq F_2(G)/M$. Since $B^{(r-1)}M$ is a p-group and $F_2(G)M/M$ is a p'-group, we have $B^{(r-1)} \leq M$. (This follows also from a theorem of Hall and Higman if $2 \nmid |G|$ without the use of the lemma of Ward.) Due to the minimality of G the theorem is true for G/M. Hence it follows that $G^{(S_pA)^{r-1}}/M$ is p'-closed, a final contradiction. //

In just the same way we can prove:

THEOREM 6. *Let $G = AB$ be the product of the abelian subgroup A and the nilpotent subgroup B. Assume that the derived length of B is at most r. Then G^r is nilpotent.*

At the end of this section we prove an extended version of a result of Kegel [4] similar to Lemma 1.

THEOREM 7. *Let G be the product of two different π-closed subgroups A and B with nilpotent Hall π-subgroups. Then there exists a proper normal subgroup K of G containing A_π or B_π.*

Proof. Let G be a minimal counter-example and M be a minimal normal subgroup of G. If $AM < G$, then since $G/M = (AM/M)(BM/M)$, due to the minimality of G, there is a proper normal subgroup K/M of G/M containing $A_\pi M/M$ or $B_\pi M/M$. In any case $K \geq A_\pi M$ or $K \geq B_\pi M$ and so

$K \geq A_\pi$ or $K \geq B_\pi$, a contradiction. So we may assume $AM = G = BM$. By the Isomorphism Theorem and as A and B are π-closed, it follows that MA_π and MB_π are normal subgroups of G. It is obvious that we may assume $MA_\pi = G = MB_\pi$. If M is a π-group, then also G is a π-group, and in this case G is the product of the two nilpotent subgroups A and B. By a theorem of Kegel [4] we get the desired contradiction. So M is a π'-group. Then A_π and B_π are Hall π-subgroups of G. By [3, VI, 4.6] we have that also $A_\pi B_\pi \in \mathrm{Hall}_\pi(G)$. Therefore $A_\pi = B_\pi$. Now, by the hypothesis of the theorem, $A_\pi \trianglelefteq A$ and $B_\pi \trianglelefteq B$. Therefore $N_G(A_\pi) \geq AB = G$. So A is normal in G, and since G is not a π-group, it follows that A_π is a proper normal subgroup of G, a final contradiction. //

The direct product $S_3 \times S_3$ is the product of $S_3 \times C_2$ with $S_3 \times C_2$. $S_3 \times C_2$ is π-closed for $\pi = \{3\}$. But there does not exist a proper normal subgroup of $S_3 \times S_3$ containing one of the factors $S_3 \times C_2$.

2.

In this second part of the paper we want to remove the special restrictions on the structure of the single factors, as for example that they are p-decomposed. We want to get an upper bound for the derived length of G in terms of the derived lengths of the single factors in general.

First we state such an estimate for the case that the single subgroups are A-groups; that is, that they have only abelian Sylow subgroups.

THEOREM 8. *Let A_i be subgroups of G with $\gcd_i(|G : A_i|) = 1$ and $i \in \{1, \ldots, r\}$. Assume that the A_i are A-groups. Then we have*

$$d(G) \leq \sum_{i=1}^{r} d(A_i) .$$

Proof. Since homomorphic images of G satisfy the same conditions, there exists exactly one minimal normal subgroup in a minimal counter-example G. Therefore the Fitting subgroup G_N is a p-group. Since $\gcd_i(|G:A_i|) = 1$, there exists a $j \in \{1, \ldots, r\}$ with $p \nmid |G:A_j|$. A Sylow p-subgroup P of A_j therefore is a Sylow p-subgroup of G. So we have $A_j \geq G_N$. Since G_N contains its own centralizer, the Fitting subgroup of A_j is a p-group. As a Fitting subgroup of an A-group $(A_j)_N$ is abelian. Since $C_G(G_N) \leq G_N \leq (A_j)_N$, it follows that $G_N = (A_j)_N$. Therefore we have $d(A_j/G_N) \leq d(A_j) - 1$. By induction we get $d(G/G_N) \leq \sum_{i \neq j} d(A_i) + d(A_j) - 1$. Therefore it follows that

$$d(G) \leq \sum_{i \neq j} d(A_i) + d(A_j) - 1 + 1 = \sum_{i=1}^{r} d(A_i) . \qquad //$$

Now we handle the general case and prove:

THEOREM 9. *Let* $G = AB$ *with* $(|G:A|, |G:B|) = 1$ *and* G *of odd order. Then* $d(G) \leq \big((d(A)+d(B))(d(A)+d(B)-1)\big)/2 + 1$.

Proof. A minimal counter-example G has exactly one minimal normal subgroup. This implies that the Fitting subgroup of G is a p-group for a prime p. Let $n = d(A)$ be the derived length of A and $m = d(B)$ that of B. Since $(|G:A|, |G:B|) = 1$, without loss of generality, p does not divide $|G:A|$. Therefore $A \geq G_N$ and a Sylow p-subgroup P of A is one of G. As $[O_{p'}(A), G_N] \leq O_{p'}(A) \cap G_N = 1$, it follows that $O_{p'}(A) \leq C_G(G_N) \leq G_N$ and so $O_{p'}(A) = 1$. Therefore we have that A_N is a p-group and $A_N \leq P \in \mathrm{Syl}_p(G)$. If A is abelian, we get $A = G_N$, and so $d(G/G_N) \leq d(B) = m$ and $d(G) \leq m + 1$. But $m + 1 \leq (m+1)m/2 + 1$, since $m \geq 1$. Therefore we get a contradiction in this case. Otherwise $A^{d(A)-1}$ is an abelian normal subgroup of P. By Hall and Higman [2, Theorem 3.2.1] we get $A^{d(A)-1} \leq O_{p'p}(G) = G_N$. So we have $d(A/G_N) \leq n - 1$. By induction it follows that

$$d(G/G_N) \leq (n-1+m)(n-1+m-1)/2 + 1 .$$

Since $A \geq G_N$, the derived length of G_N is not greater than that of A. Therefore $d(G_N) \leq d(A) = n$. Furthermore we have $d(G) \leq d(G/G_N) + d(G_N)$. Summarizing we have

$$d(G) \leq (n+m-1)(n+m-2)/2 + 1 + n = (n+m)(n+m-1)/2 - (n+m-1) + 1 + n \leq$$
$$\leq (n+m)(n+m-1)/2 + 1 ,$$

contradicting the choice of G. //

In the case $n = m = 1$ our result is a theorem of Itô's [3, VI, 4.4].

COROLLARY 1. *Let G be of odd order and $G = AB$ with $(|G : A|, |G : B|) = 1$. Put $t = (d(A)+d(B)-1)(d(A)+d(B)-2)/2 + 1$. Then $G \in \mathsf{NA}^t$; that is, G^t is nilpotent. In particular, if G is the product of two metabelian groups with relatively prime indices and $2 \nmid |G|$, then G^4 is nilpotent.*

The proof is similar to that of Theorem 9.

COROLLARY 2. *Let $H^n \trianglelefteq G$ for all Hall subgroups H of G and suppose $2 \nmid |G|$. Then $G^{(2n^2-3n+2)}$ is nilpotent.*

Proof. Since any homomorphic image of G satisfies the same conditions, a minimal counter-example G is primitive. Let p be the characteristic of the only minimal normal subgroup M and let $P \in \mathrm{Syl}_p(G)$. Since $P^n \trianglelefteq G$ and $P^n \leq \Phi(P)$ we get $P^n \leq \Phi(G) = 1$. Therefore the derived length of P is at most n. Let Q be a p-complement of G, a Hall p'-subgroup of G. Since $Q^n \trianglelefteq G$ and $Q^n \cap M = 1$, we get $Q^n = 1$. Therefore G is the product of the two subgroups P and Q, which both have derived length at most n. By Corollary 1 of Theorem 9 we get that $G^{(2n-1)(2n-2)/2+1}$ is nilpotent; that is, $G \in \mathsf{NA}^{2n^2-3n+2}$, a contradiction to the choice of G. //

In the case $n = 1$ we have the well-known result of Inagaki's that G' is nilpotent, if H' is a normal subgroup of G for all Hall subgroups H of G. For $n = 2$ we get: if $2 \nmid |G|$ and $H'' \trianglelefteq G$ for all Hall subgroups H of G, then G^4 is nilpotent.

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By a theorem of Wielandt [3, VI, 1.9] a group G is soluble if it possesses three soluble subgroups whose indices are pairwise relatively prime. In the last theorem we want to find an upper bound for the nilpotent length of such a group in terms of the derived lengths of the subgroups in question. We prove:

THEOREM 10. *Let G be a group of odd order. Assume that G has three subgroups U_i $(i = 1, 2, 3)$, whose indices are pairwise relatively prime. Let $n = d(U_1)$, $m = d(U_2)$, and $l = d(U_3)$. Then $G \in \mathrm{N}^{(n+m+l)/2-2}\mathrm{A}^2$, if $n + m + l$ is even and $G \in \mathrm{N}^{(n+m+l+1)/2-2}\mathrm{A}$, if $n + m + l$ is odd.*

Proof. If $n = m = l = 1$, G has three abelian subgroups with pairwise relatively prime indices and thus it is abelian. So the theorem is true in this case. For $n = m = 1$ and $l = 2$ we have that G is metabelian by a theorem of Itô's. For $n + m + l \geq 5$ the classes $\mathrm{N}^{(n+m+l+1)/2-2}\mathrm{A}$ and $\mathrm{N}^{(n+m+l)/2-2}\mathrm{A}^2$ are saturated formations. Therefore a minimal counter-example to the theorem is primitive. Let M be the only minimal normal subgroup of G and $|M| = p^k$. As the index of any subgroup, which does not contain M is divisible by p, we may assume, without loss of generality, that $U_1, U_2 \geq M$. Since $[O_{p'}(U_i), M] \leq O_{p'}(U_i) \cap M = 1$ for $i = 1, 2$, it follows that $O_{p'}(U_i) \leq C_G(M) = M$. Therefore $O_{p'}(U_i) = 1$ and so $(U_i)_N$ is a p-group. If, without loss of generality, U_i is abelian, we have $U_1 = M$ and so U_1 is a p-group. Therefore any prime divisor $q \neq p$ of $|G|$ is a divisor of the index $|G : U_1|$. So in this case one of the three subgroups U_1, U_2, and U_3 is equal to G; that is, $U_2 = G$, say. Then we have $d(G/M) \leq d(U_3)$, so $d(G) \leq d(U_3) + 1 = l + 1$. So G is contained in $\mathrm{A}^m \cap \mathrm{A}^{l+1}$. We distinguish two cases.

If $m = l + 1$, then $1 + l + m$ is even and G belongs to $\mathrm{A}^{l+1} = \mathrm{A}^{(1+l+l+1)/2-2}\mathrm{A}^2$. As in this case $n = 1$ and $l = m + 1$ we get the desired result.

If $m < l + 1$ and $1 + l + m$ is odd, we have $(1+l+m+1)/2 - 1 \geq m$, whence G belongs to $A^{(n+m+l+1)/2-2}A$, again a contradiction. The other cases are obvious.

So suppose that none of the U_i $(i = 1, 2)$ is abelian. Therefore $U_i^{d(U_i)-1} \leq (U_i)_N \leq P$ is an abelian normal subgroup of $P \in \mathrm{Syl}_p(G)$. By Hall and Higman [2, Theorem 3.2.1], we now have

$$U_i^{d(U_i)-1} \leq O_{p'p}(G) = M .$$

Therefore $d(U_i/M) \leq d(U_i) - 1$. If $n + m + l$ is even, then also $n - 1 + m - 1 + l$ is even. So we get $G/M \in N^{(n+m+l-2)/2-2}A^2$ by induction, whence G'' belongs to $N^{(n+m+l)/2-2}$.

If $n + m + l$ is odd, then also $n - 1 + m - 1 + l$ is odd. In this case we get by induction that $G/M \in N^{(n+m+l-1)/2-2}A$, whence G' belongs to $N^{(n+m+l+1)/2-2}$, which proves the theorem. //

COROLLARY. *If G has three metabelian subgroups with pairwise relatively prime indices, then G'' is nilpotent. If G has two metabelian subgroups and an abelian subgroup with pairwise relatively prime indices, G' is nilpotent. Finally, if G has one abelian, one metabelian and a third subgroup, which has derived length at most 3 and if these subgroups have pairwise relatively prime indices, then G'' is nilpotent.*

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