

# Generalized Eigenfunctions and a Borel Theorem on the Sierpinski Gasket

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*Abstract.* We prove there exist exponentially decaying generalized eigenfunctions on a blow-up of the Sierpinski gasket with boundary. These are used to show a Borel-type theorem, specifically that for a prescribed jet at the boundary point there is a smooth function having that jet.

## 1 Introduction

There is a well developed theory of analysis on certain types of fractal sets, of which the Sierpinski gasket (SG) is the simplest non-trivial example (see [5, 9]). In this theory the fractals are viewed as limits of graphs, and notions analogous to the Dirichlet energy and the Laplacian are constructed as renormalized limits of the corresponding objects on the approximating graphs. The nature of this construction has naturally led to extensive study of the eigenfunctions of this Laplacian, and to functional-analytic notions based on the eigenfunctions. However, more recent work has examined other elementary functions on SG, including analogues of polynomials, analytic functions and certain exponentials [2, 7]. A forthcoming paper will extend this investigation to study smooth bump functions and a method for partitioning smooth functions subordinate to an open cover [8].

In the present work we prove there are exponentially decaying generalized eigenfunctions on a blow-up of SG with boundary (which we denote  $SG_\infty$ ), proving:

**Theorem 1.1** *For each  $\lambda < 0$  and  $j \in \mathbb{N}$  there is a smooth function  $E_\lambda^j$  on  $SG_\infty$  such that for each  $j$  we have  $(\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1}$ . Moreover  $E_\lambda^j$  decays exponentially away from the boundary point of  $SG_\infty$  and satisfies  $|E_\lambda^j| \leq j!|\lambda|^{-j}$ .*

There are sufficiently many of these generalized eigenfunctions that they can be used to prove a Borel-type theorem on  $SG_\infty$ , thereby answering a question asked in [2] and [7]. Using the term *jet* for the sequence of values of the natural derivatives at a junction point of SG, our result may be summarized as follows.

**Theorem 1.2** *Given an arbitrary jet there is a smooth function on  $SG_\infty$  with that jet at the boundary point.*

Our motivation for studying generalized eigenfunctions and for proving Theorem 1.1 was to prove Theorem 1.2. The structure of the paper reflects this motivation. Apart from some background in Section 2, our first results (in Section 3) are those

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showing that Theorem 1.2 follows from Theorem 1.1 and some known results about localized eigenfunctions on  $SG$ . Section 4 is then devoted to the construction of the generalized eigenfunctions and the proof of Theorem 1.1.

## 2 Setting

We give a brief description of some parts of the theory of analysis on the Sierpinski gasket, more details of which are in [9]. For the general theory of analysis on fractals the standard reference is [5].

### $SG$ and $SG_\infty$

The Sierpinski gasket  $SG$  is the unique non-empty compact set in  $\mathbb{R}^2$  that is invariant under the iterated function system  $f_i = \frac{1}{2}(x + q_i)$ ,  $i = 0, 1, 2$ , in the sense that  $SG = \bigcup_{i=0}^2 f_i(SG)$ , where the points  $q_i$  are the vertices of an equilateral triangle. For  $m \in \mathbb{N}$  and  $(i_1, i_2, \dots, i_m) \in \{0, 1, 2\}^m$  we call  $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_m}(SG)$  a *cell of level  $m$* . The points  $V_0 = \{q_i : i = 0, 1, 2\}$  are the boundary of  $SG$ , and we view  $SG$  as the limit of graphs  $\Gamma_m$  with vertices defined inductively by  $V_m = \bigcup_{i=0}^2 f_i(V_{m-1})$  and edge relation  $x \sim_m y$  if  $x$  and  $y$  are in the same  $m$ -cell. The set of all vertices is  $V_\infty = \bigcup_m V_m$ , and the *junction points* are  $V_\infty \setminus V_0$ . We let  $\mu$  be the usual self-similar probability measure on  $SG$  with  $\mu(f_w(SG)) = 3^{-|w|}$ , and we also use  $\mu$  to denote the obvious extension to  $SG_\infty$ .

The infinite blow-up of  $SG$  with boundary point  $q_0$  is denoted  $SG_\infty$  and defined by

$$SG_\infty = \bigcup_{m=0}^\infty f_0^{o(-m)}(SG)$$

This is the simplest blow-up of  $SG$ ; we could also consider arbitrary sequences of blow-ups  $\bigcup_{m=1}^\infty f_{i_1}^{-1} f_{i_2}^{-1} \dots f_{i_m}^{-1}(SG)$ , but all that have a boundary point necessarily have  $\{i_m\}$  to be constant after some  $m_0$  and are isometric, so among those with a boundary it suffices to consider  $SG_\infty$  (see Lemma 2.3 in [11]). For results in this paper it is crucial that we are on a blow-up with boundary. We refer to [10, 11] for more information about blow-ups and the Laplacian on  $SG_\infty$ .

### Laplacian, Derivatives, and Jets

Each graph approximation  $\Gamma_m$  of  $SG$  supports a graph Laplacian  $\Delta_m$  defined at non-boundary points  $x \in V_m \setminus V_0$ , and we define a Laplacian  $\Delta$  at junction points of  $SG$  as a renormalized limit of the graph Laplacians

$$\begin{aligned} \Delta_m u(x) &= \sum_{y \sim_m x} (u(y) - u(x)), \\ (2.1) \quad \Delta u(x) &= \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \Delta_m u(x). \end{aligned}$$

A continuous function  $u$  is in the domain of the Laplacian,  $u \in \text{dom}(\Delta)$ , if there is a continuous  $f$  such that the right side of (2.1) converges uniformly to  $f$  on  $V_\infty \setminus V_0$ . Then we write  $\Delta u = f$ , extending  $\Delta u$  to all points of SG by continuity. The factor  $3/2$  in (2.1) is for consistency with an alternative definition of the Laplacian using a renormalized Dirichlet energy (see [5, 9]). We make the obvious definition of  $\Delta^k u$  and  $\text{dom}(\Delta^k)$ , and call a function smooth if it is in  $\text{dom}(\Delta^\infty) = \bigcap_k \text{dom}(\Delta^k)$ . One additional property of the Laplacian that we will use extensively is its scaling; it is immediate from (2.1) that for  $u \in \text{dom}(\Delta)$

$$(2.2) \quad \Delta(u \circ f_0^{-1}) = 5(\Delta u) \circ f_0^{-1} \quad \text{on } f_0(SG)$$

In addition to the Laplacian there are two derivatives at boundary points, the normal derivative  $\partial_n$  and tangential derivative  $\partial_T$ , defined by

$$(2.3) \quad \partial_n u(q_i) = \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m (2u(q_i) - u(f_j^{\circ m}(q_{i+1})) - u(f_j^{\circ m}(q_{i+2})))$$

$$(2.4) \quad \partial_T u(q_i) = \lim_{m \rightarrow \infty} 5^m (u(f_j^{\circ m}(q_{j+1})) - u(f_j^{\circ m}(q_{j+2})))$$

(with  $q_{i+3} = q_i$ ). The former exists for any  $u \in \text{dom}(\Delta)$ , and the latter exists under the additional assumption that  $\Delta u$  is Hölder continuous. Both may be localized to boundary points of cells. The normal derivatives are much better understood than the tangential derivatives, and have considerable application. For this paper their important feature is the *matching condition* for the normal derivatives: if  $u \in \text{dom}(\Delta)$  then at any junction point of two cells the normal derivatives corresponding to these cells sum to zero. Conversely, if  $u$  is continuous and  $\Delta u = f$  on each  $m$ -cell then  $\Delta u = f$  on SG if and only if  $f$  is continuous, and the matching condition holds at each point of  $V_m \setminus V_0$ .

At a boundary point  $q$  we will call the values of  $\Delta^k u(q)$ ,  $\partial_n \Delta^k u(q)$  and  $\partial_T \Delta^k u(q)$  the Laplacian powers, normal derivatives and tangential derivatives, respectively. The  $j$ -jet of  $u$  at  $q$  is  $(u(q), \partial_n u(q), \partial_T u(q), \dots, \Delta^j u(q), \partial_n \Delta^j u(q), \partial_T \Delta^j u(q))$  and the infinite jet is the corresponding sequence of Laplacian powers and derivatives.

### Spectral decimation

A useful feature of the Laplacian on SG and  $SG_\infty$  is that restricting a Laplacian eigenfunction  $E_\lambda$  to the graph approximation  $\Gamma_m$  produces an eigenfunction of the graph Laplacian, with a shift in the eigenvalue. This phenomenon is known as spectral decimation [3, 4, 11]. Specifically, if  $(\Delta + \lambda)E_\lambda = 0$  on SG or  $SG_\infty$ , then  $(\Delta_m + \lambda_m)E_\lambda = 0$  on  $\Gamma_m$ , where

$$(2.5) \quad \lambda_{m-1} = \lambda_m(5 - \lambda_m)$$

$$(2.6) \quad \lambda = \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \lambda_m$$

$$(2.7) \quad E_\lambda(y_i) = \frac{2E_\lambda(x_i) + (4 - \lambda_m)(E_\lambda(x_{i+1}) + E_\lambda(x_{i+2}))}{(2 - \lambda_m)(5 - \lambda_m)}$$

in which the points  $x_i$  are the vertices of an  $(m - 1)$ -cell, and each  $y_i$  is the point from  $V_m$  opposite to  $x_i$  as shown in Figure 1. For proofs we refer to [3] or [9]; an explanation of spectral decimation in a more general context is found in [6].

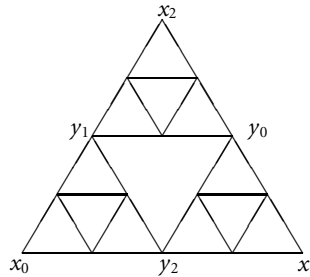


Figure 1: Points  $x_i$  in  $V_{m-1}$  and  $y_i$  in  $V_m$ .

We will need one technical result about spectral decimation on  $SG$  that is well known but for which there does not appear to be a proof in the literature.

**Lemma 2.1** For  $\lambda < 0$  there is an entire function  $\Psi$  such that  $\lambda_m = \Psi(5^{-m}\lambda)$ .

**Proof** Considering (2.5) we define functions by  $2\phi_{\pm}(\zeta) = 5 \pm \sqrt{25 - 4\zeta}$ , so that  $\lambda_m$  is one of  $\phi_{\pm}(\lambda_{m-1})$ . Observe that if  $\lambda_{m-1} \geq 0$  then  $\lambda_m \geq 0$ , so from (2.6) and  $\lambda < 0$  we must have  $\lambda_m < 0$  and  $\lambda_m = \phi_{-}(\lambda_{m-1})$  for all  $m$ . The renormalized limit  $\Phi(\zeta) = \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \phi_{-}^m(\zeta)$  is analytic in a neighborhood of the origin and has  $\Phi'(0) = 3/2$  by virtue of the fact that  $\phi_{-}(\zeta) = \zeta/5 + O(\zeta^2)$  for sufficiently small  $\zeta$ . It follows that  $\Phi$  has an analytic inverse  $\Psi(\zeta) = \Phi^{-1}(\zeta) = \sum_{k=0}^{\infty} \alpha_k \zeta^k$  in a neighborhood of 0.

Using (2.5) and (2.6) we find  $\Psi(5^{-m}\lambda) = \lambda_m$  for all sufficiently small  $5^{-m}\lambda$ . This gives a recursion for the coefficients  $\alpha_k$ , beginning with  $\alpha_0 = 0$ ,  $\alpha_1 = 2/3$  and continuing according to

$$(2.8) \quad (5^k - 5)\alpha_k = - \sum_{l=1}^{k-1} \alpha_l \alpha_{k-l}$$

for  $k \geq 2$ . An almost identical recursion appears for a different purpose in [7] (as Equation 2.9 and in Theorem 2.7), and their argument shows that

$$|\alpha_k| \leq C(k!)^{-\log 5 / \log 2}.$$

It follows immediately that  $\Psi$  is entire, while  $\lambda_m = \Psi(5^{-m}\lambda)$  is true by construction. ■

### 3 The Borel Theorem

In this section we prove Theorem 1.2 under the assumption of Theorem 1.1. First we construct smooth functions with finitely many prescribed values of the Laplacian powers and tangential derivatives at  $q_0$  using known results about the existence of localized eigenfunctions. Then we use Theorem 1.1 and linear algebra to prove that there are smooth functions with finitely many prescribed normal derivatives. Finally we state a precise version of Theorem 1.2 and show that its validity for finite jets gives the full result by a scaling and convergence argument.

#### Localized Eigenfunctions

A curious feature of many highly symmetric fractals is that their Laplacians have localized eigenfunctions. We will not need the details of the theory, for which we refer to [1, 5], but only the existence of two specific eigenfunctions  $u_1$  and  $u_2$  on SG. The values of  $u_1$  on  $V_1$  and  $u_2$  on  $V_2$  are shown in Figure 2. From the values shown

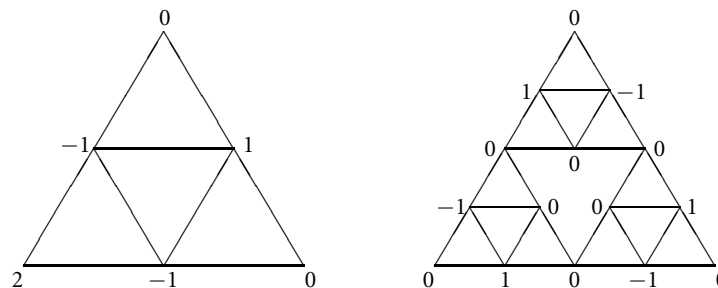


Figure 2: The functions  $u_1$  and  $u_2$ .

we can compute  $u_1$  and  $u_2$  at any scale by the method of spectral decimation given in (2.5)–(2.7) (with the caveat that for  $u_1$  the positive root must be taken at the first step of the recursion (2.5)). What is important here is that the normal derivatives of both  $u_i$  vanish at the points  $q_1$  and  $q_2$ , which we see from (2.3) and the antisymmetry of the  $u_i$  on the cells  $f_1(SG)$  and  $f_2(SG)$ . Since the  $u_i$  are eigenfunctions, we then find that all of the values  $\Delta^k u_i$  and  $\partial_n \Delta^k u_i$  vanish at  $q_1$  and  $q_2$ , and (2.2) shows the same is true for  $u_i \circ f_0^{-m}$  at  $f_0^m(q_1)$  and  $f_0^m(q_2)$  for any  $m$ . It follows from the matching condition that

$$u_{i,m} = \begin{cases} u_i \circ f_0^{-m} & \text{on } f_0^m(SG), \\ 0 & \text{elsewhere,} \end{cases}$$

are smooth functions, and therefore are Laplacian eigenfunctions with eigenvalues  $-5^m \lambda_i$ . For obvious reasons they are called *localized eigenfunctions*.

The jets of the  $u_{i,m}$  at  $q_0$  are easily computed. The eigenfunction equations  $\Delta u_i = -\lambda_i u_i$  give the higher order terms from the initial ones, so by simple algebra from

(2.7) and some symmetry arguments,

$$\begin{aligned} \Delta^k u_{i,m}(q_0) &= \begin{cases} 2(-\lambda_1)^k 5^{km} & i = 1, \\ 0 & i = 2, \end{cases} \\ \partial_T \Delta^k u_{i,m}(q_0) &= \begin{cases} 0 & i = 1, \\ 2(-\lambda_2)^k 5^{km} & i = 2, \end{cases} \\ \partial_n \Delta^k u_{i,m}(q_0) &= 0 \quad i = 1, 2. \end{aligned}$$

With these functions as building blocks we show that there is a smooth function with finitely prescribed values of the Laplacian powers and tangential derivatives at  $q_0$ , and whose normal derivatives are all zero.

**Lemma 3.1** For  $n \in \mathbb{N}$  and values  $\zeta_0, \dots, \zeta_n$  and  $\theta_0, \dots, \theta_n$  there is  $u \in \text{dom}(\Delta^\infty)$  such that  $\Delta^k u(q_0) = \zeta_k$ ,  $\partial_n \Delta^k u(q_0) = 0$ , and  $\partial_T \Delta^k u(q_0) = \theta_k$  for all  $0 \leq k \leq n$ .

**Proof** We observe that the vectors

$$(u_{1,m}(q_0), \Delta u_{1,m}(q_0), \dots, \Delta^n u_{1,m}(q_0)) = 2(1, (-\lambda_1)5^m, \dots, (-\lambda_1)^n 5^{nm})$$

are linearly independent with respect to  $m$ . A similar result is true for the vector of tangential derivatives of  $u_{2,m}$ . We may then obtain the desired  $u$  as a linear combination of the functions  $u_{i,m}$  for  $0 \leq m \leq n$  by linear algebra. ■

We remark that this method cannot be applied to prescribe values of the normal derivatives at  $q_0$  using localized eigenfunctions. The structure of the localized eigenfunctions is well understood (see [9]), and non-zero normal derivatives can occur only in “closed loops” circling the holes in the gasket. As each junction point corresponds to a hole of precisely one size, our scaling arguments are not applicable. Similar arguments are needed in Lemma 3.4 below, so any proof of Theorem 1.2 using only localized eigenfunctions would need to be quite different from ours.

### Generalized Eigenfunctions and Normal Derivatives

The generalized eigenfunctions produced in Theorem 1.1 are a sufficiently rich class that we can use a finite linear combination of them to match finitely many prescribed normal derivatives at  $q_0$ .

**Lemma 3.2** For  $n \in \mathbb{N}$ ,  $\lambda < 0$ , and values  $\eta_0, \dots, \eta_n$ , there is  $u \in \text{dom}(\Delta^\infty)$  which is a finite linear combination of the  $E_\lambda^j$ ,  $0 \leq j \leq n$ , and has  $\partial_n \Delta^k u(q_0) = \eta_k$  for all  $0 \leq k \leq n$ .

**Proof** Let  $a_{j,k} = \partial_n \Delta^k E_\lambda^j(q_0)$ . Clearly, it suffices to show that the matrix  $[a_{j,k}]_{j,k=0}^n$  is invertible, so we examine its determinant. Writing the generalized eigenfunction equation  $(\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1}$  in terms of  $a_{j,k}$  we have  $a_{j,k} + \lambda a_{j,k-1} = -j a_{j-1,k-1}$ , which suggests a column operation on  $[a_{j,k}]$ . For all columns  $k \geq 1$  we replace  $a_{j,k}$  with  $-j a_{j-1,k-1}$ , which makes the first row zero except in the first place simply

because  $j = 0$  on this row. For concreteness, the result of this computation for the determinant in the case  $n = 2$  is given below.

$$\begin{vmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{vmatrix} = \begin{vmatrix} a_{0,0} & 0 & 0 \\ a_{1,0} & -a_{0,0} & -a_{0,1} \\ a_{2,0} & -2a_{1,0} & -2a_{1,1} \end{vmatrix} = a_{0,0} \begin{vmatrix} -a_{0,0} & -a_{0,1} \\ -2a_{1,0} & -2a_{1,1} \end{vmatrix}.$$

This operation can be repeated inductively, because it shows

$$\det [a_{j,k}]_0^n = a_{0,0} \det [-ja_{j-1,k-1}]_1^n = a_{0,0} \det [-(j+1)a_{j,k}]_0^{n-1},$$

and the only change to the matrix at each stage is to multiply each row by a constant and reduce the degree, so the same column operations apply each time. We conclude that  $\det [a_{j,k}]_0^n = (-1)^n n! a_{0,0}^{n+1}$  and is non-zero by (4.2) below. ■

**Corollary 3.3** *Given values  $(\zeta_k, \eta_k, \theta_k)$  for  $0 \leq k \leq n$ , there is a finite linear combination  $u$  of localized eigenfunctions and generalized eigenfunctions such that  $\Delta^k u(q_0) = \zeta_k$ ,  $\partial_n \Delta^k u(q_0) = \eta_k$  and  $\partial_T \Delta^k u(q_0) = \theta_k$  for all  $0 \leq k \leq n$ .*

**Proof** Apply Lemma 3.2 to match the normal derivatives and then Lemma 3.1 to correct the Laplacian powers and tangential derivatives without affecting the values of the normal derivatives. ■

**Proof of the Borel Theorem**

Corollary 3.3 supplies the natural building blocks for obtaining a smooth function with any given jet at  $q_0$ . We define functions  $F_j$  from which we will determine the Laplacian powers,  $G_j$  for the normal derivatives and  $H_j$  for the tangential derivatives, by requiring that for all  $0 \leq k \leq j$ ,

$$\begin{aligned} \Delta^k F_j(q_0) &= \delta_{j,k}, & \partial_n \Delta^k F_j(q_0) &= 0, & \partial_T \Delta^k F_j(q_0) &= 0, \\ \Delta^k G_j(q_0) &= 0, & \partial_n \Delta^k G_j(q_0) &= \delta_{j,k}, & \partial_T \Delta^k G_j(q_0) &= 0, \\ \Delta^k H_j(q_0) &= 0, & \partial_n \Delta^k H_j(q_0) &= 0, & \partial_T \Delta^k H_j(q_0) &= \delta_{j,k}, \end{aligned}$$

where  $\delta_{j,k}$  is the Kronecker delta. The natural goal is to construct a smooth function with prescribed (infinite) jet by using the terms of the jet as coefficients in a series with functions like the  $F_j$ ,  $G_j$  and  $H_j$ . To make the series converge to a smooth function we will need some estimates on these functions and their Laplacian powers. What we know so far is that they are finite linear combinations of the localized eigenfunctions  $u_{i,m}$  with  $0 \leq m \leq j$  and  $i = 1, 2$ , and the generalized eigenfunctions  $E_\lambda^k$  for a fixed  $\lambda$  and  $0 \leq k \leq j$ . All of these functions and their Laplacian powers of order at most  $j$  are bounded: for the localized eigenfunctions this is obvious, while for the generalized eigenfunctions it follows from the bound in Theorem 1.1 and the recursion (4.3). We conclude that for each  $j$  there is a constant  $C(j)$  such that for all  $0 \leq k \leq j$ ,

$$|\Delta^k F_j| \leq C(j), \quad |\Delta^k G_j| \leq C(j), \quad |\Delta^k H_j| \leq C(j),$$

and turn now to a scaling argument that allows us to make these as small as desired.

**Lemma 3.4** *If  $m \in \mathbb{N}$  then the functions*

$$F_{j,m} = 5^{-jm} F_j \circ f_0^{-m}, \quad G_{j,m} = 5^{-jm} G_j \circ f_0^{-m}, \quad H_{j,m} = 5^{-jm} H_j \circ f_0^{-m},$$

*have the same  $j$ -jets at  $q_0$  as  $F_j$ ,  $G_j$  and  $H_j$  respectively, but for  $0 \leq k \leq j$  they satisfy the following estimates on  $SG_\infty$*

$$|\Delta^k F_{j,m}| \leq C(j)5^{(k-j)m}, \quad |\Delta^k G_{j,m}| \leq C(j)5^{(k-j)m}, \quad |\Delta^k H_{j,m}| \leq C(j)5^{(k-j)m}.$$

**Proof** The result is an elementary consequence of the scaling property of the Laplacian. By induction from (2.2) we see that  $\Delta^k(u \circ f_0^{-m}) = 5^{km}(\Delta^k u) \circ f_0^{-m}$ . Both statements of the lemma are immediate consequences of this and the definitions (2.3) and (2.4) of the normal and tangential derivatives. ■

**Proof of Theorem 1.2** We are supplied with values  $(\zeta_k, \eta_k, \theta_k)$  for  $k \in \mathbb{N}$  and seek a smooth function  $u$  such that  $\Delta^k u(q_0) = \zeta_k$ ,  $\partial_n \Delta^k u(q_0) = \eta_k$  and  $\partial_T \Delta^k u(q_0) = \theta_k$  for all  $k$ . This will be certainly be the case for the function

$$(3.1) \quad u = \sum_{j=0}^{\infty} (\zeta_j F_{j,m_j} + \eta_j G_{j,m_j} + \theta_j H_{j,m_j})$$

provided that applying any power of the Laplacian yields a uniformly convergent series. However by Lemma 3.4 we may choose the sequence  $m_j$  such that terms after the  $j$ -th have only a small effect on the Laplacian powers of order at most  $j$ . Specifically, given any  $\epsilon > 0$  we may make  $m_j$  so large that for  $0 \leq k \leq j - 1$ ,

$$\begin{aligned} & \left| \Delta^k (\zeta_j 5^{-jm_j} F_{j,m_j} + \eta_j 5^{-jm_j} G_{j,m_j} + \theta_j 5^{-jm_j} H_{j,m_j}) \right| \\ & \leq C(j)5^{(k-j)m_j} \max\{|\zeta_k|, |\eta_k|, |\theta_k| : 0 \leq k \leq j - 1\} \\ & \leq \epsilon 2^{k-j}, \end{aligned}$$

providing a bound on the tail of the series obtained by applying  $\Delta^k$  to (3.1). We conclude that  $u$  is smooth, that it has the desired jet at  $q_0$ , and moreover that most of the contribution to the  $k$ -th jet is from the first  $k$  terms in the sum

$$|\Delta^k u| \leq \epsilon + C(k) \sum_{j=0}^k (|\zeta_j| + |\eta_j| + |\theta_j|). \quad \blacksquare$$

### 4 Generalized Eigenfunctions with Decay

In this section we prove Theorem 1.1, showing that there are exponentially decaying generalized eigenfunctions  $E_\lambda^j$  of the Laplacian on  $SG_\infty$ . Our results depend on work in [7], where the negative-eigenvalue eigenfunctions of  $-\Delta$  were studied using spectral decimation. Using notation from (2.5), the results we need may be summarized as follows.



**Proposition 4.1** ([7], Section 6) For each  $\lambda < 0$  there is an eigenfunction  $E_\lambda$  on  $SG_\infty$  which is symmetrical under the reflection that fixes  $q_0$  and exchanges  $q_1$  with  $q_2$ , and which satisfies  $(\Delta + \lambda)E_\lambda = 0$ . There is an explicit formula for  $E_\lambda$  at the points  $z_m = f_0^{-m}(q_1)$

$$(4.1) \quad E_\lambda(z_m) = 1 - \frac{\lambda_m}{4} + \frac{\lambda_m}{4} \prod_{j=0}^{\infty} \left( 1 + \frac{4}{2 - \lambda_{m-j}} \right)$$

which is uniformly continuous on compacta in  $V_m$ , and  $E_\lambda$  is the limit of this on  $SG_\infty$ . These functions have exponential decay  $|E_\lambda(z_m)| = O(|\lambda_m|^{-1}) = O(2^{-2^{-m}})$  as  $m \rightarrow -\infty$ , and the normal derivative at  $q_0$  is given by

$$(4.2) \quad \partial_n E_\lambda(q_0) = \lambda \prod_{m=0}^{\infty} \left( 1 + \frac{4}{2 - \lambda_{-m}} \right) \prod_{n=1}^{\infty} \left( \frac{6 - \lambda_n}{6 - 3\lambda_n} \right) > 0.$$

Our construction is motivated as follows. Formally setting  $E_\lambda^j = \left(\frac{d}{d\lambda}\right)^j E_\lambda$  we find that the  $E_\lambda^j$  satisfy the generalized eigenfunction equation

$$(4.3) \quad (\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1},$$

and we hope that the decay of  $E_\lambda$  will ensure exponential decay for  $E_\lambda^j$ . This argument is made rigorous by Lemma 4.2, but it will initially be simpler to construct the  $E_\lambda^j$  from (4.3) than by proving  $E_\lambda$  can be differentiated with respect to  $\lambda$ .

Observe that on  $SG$  we can inductively obtain solutions  $E_\lambda^j$  of (4.3) for  $j \in \mathbb{N}$  starting with  $E_\lambda^0 = E_\lambda$ , merely because  $\lambda < 0$  and the spectrum of  $\Delta$  is positive. The resulting functions are clearly in  $\text{dom}(\Delta^\infty)$  and depend on the boundary data we assign. Guided by the formal idea that  $E_\lambda^j$  should be  $\left(\frac{d}{d\lambda}\right)^j E_\lambda$ , we set

$$(4.4) \quad E_\lambda^j(z) = \left(\frac{d}{d\lambda}\right)^j E_\lambda(z)$$

at each of the three boundary points  $z = q_0, q_1, q_2$ . The definition is legitimate because  $\lambda_m = \Psi(5^{-m}\lambda)$  and  $\Psi$  is entire (Lemma 2.1), so the rapid growth of  $\lambda_m$  ensures the expression (4.1) is analytic with respect to  $\lambda$ .

**Lemma 4.2** Using the supremum norm, the functions  $E_\lambda^j$  are differentiable with respect to  $\lambda$  and  $\frac{d}{d\lambda} E_\lambda^{j-1} = E_\lambda^j$ .

**Proof** We require a standard estimate (like Lemma 5.2.8 of [5]). Let  $u$  be in  $\text{dom}(\Delta)$  and subtract the harmonic function  $u_0$  with the same boundary values. It is well known that  $\|u - u_0\|_\infty \leq c\|\Delta u\|_2$ , and by the maximum principle we conclude  $\|u\|_\infty \leq c\|\Delta u\|_2 + \max_{V_0} |u|$ . Let  $\lambda < 0$ , and let  $\kappa_1 > 0$  be the first Dirichlet eigenvalue of  $-\Delta$ . The spectral representation of  $\Delta$  immediately shows

$$(4.5) \quad \|u\|_\infty \leq c \left( 1 + \frac{|\lambda|}{\kappa_1} \right) \|(\Delta + \lambda)u\|_2 + \max_{V_0} |u|.$$

Now suppose inductively that the lemma is true up to  $j - 1$ . For the difference between  $E_\lambda^j$  and the Newton quotient for the derivative of  $E_\lambda^{j-1}$ , we have

$$\begin{aligned} & (\Delta + \lambda) \left( E_\lambda^j - \frac{1}{t} (E_{\lambda+t}^{j-1} - E_\lambda^{j-1}) \right) \\ &= -jE_\lambda^{j-1} - \frac{1}{t} \left( -(j-1)E_{\lambda+t}^{j-2} - tE_{\lambda+t}^{j-1} + (j-1)E_\lambda^{j-2} \right) \\ &= (E_{\lambda+t}^{j-1} - E_\lambda^{j-1}) + (j-1) \frac{1}{t} (E_{\lambda+t}^{j-2} - E_\lambda^{j-2}) - (j-1)E_\lambda^{j-1} \\ &\rightarrow 0 \quad \text{in } L^2(SG) \end{aligned}$$

by induction. From (4.5) and the fact that the boundary data varies analytically with  $\lambda$ , we conclude

$$\left\| E_\lambda^j - \frac{1}{t} (E_{\lambda+t}^{j-1} - E_\lambda^{j-1}) \right\|_\infty \rightarrow 0.$$

The same reasoning reduces the base case of the induction to showing  $\|E_{\lambda+t} - E_\lambda\|_2 \rightarrow 0$ , which is a consequence of the fact (from Proposition 4.1) that  $E_\lambda$  is uniformly approximated by the analytic function of  $\lambda$  in (4.1). ■

With this lemma in hand we can describe the natural scaling behavior of the  $E_\lambda^j$ . From (2.2) we know that  $E_{5\lambda} = E_\lambda \circ f_0^{-1}$ , whence on  $f_0(SG)$

$$E_\lambda^j \circ f_0^{-1} = \left( \frac{d}{d\lambda} \right)^j E_\lambda \circ f_0^{-1} = \left( \frac{d}{d\lambda} \right)^j E_{5\lambda} = 5^j E_{5\lambda}^j,$$

and therefore, the natural definition of  $E_\lambda^j$  on  $f_0^{-1}(SG)$  is to set  $E_\lambda^j = 5^j E_{5\lambda}^j \circ f_0$ . Inductively, for each  $n \in \mathbb{N}$  we let

$$E_\lambda^j = 5^{jn} E_{5^n \lambda} \circ f_0^n \quad \text{on } f_0^{-n}(SG)$$

to extend  $E_\lambda^j$  to all of  $SG_\infty$ . We remark that this gives the same result as solving (4.3) on  $f_0^{-n}(SG)$  with boundary data (4.4) at the points  $z = q_0, f_0^{-n}(q_1), f_0^{-n}(z_2)$ .

Combining the above results we have proven the existence statements of Theorem 1.1. What remains to be proven is the content of the following lemma, which is regrettably but perhaps unavoidably technical.

**Lemma 4.3** *The generalized eigenfunctions satisfy  $|E_\lambda^j| \leq j!|\lambda|^{-j}$  and have decay*

$$|E_\lambda^j(z_m)| = O(2^{-2^{-m}}) \quad \text{as } m \rightarrow -\infty.$$

**Proof** We first prove the decay. Recall that the recursion (2.5) guarantees  $|\lambda_m| \geq C2^{2^{-m}}$  as  $m \rightarrow -\infty$ . Set  $\beta_m^j = (d/d\lambda)^j \lambda_m$ . It is elementary to verify that  $|\beta_m^j| = O(|\lambda_m|)$  as  $m \rightarrow -\infty$  from the definition of  $\Psi$  and the recursion (2.8). Using the explicit formula (4.1) we write  $E_\lambda(z_m) = 1 + (P_m - \lambda_m)/4$ , where

$$P_m = \lambda_m \prod_{n=0}^\infty \left( 1 + \frac{4}{2 - \lambda_{m-n}} \right),$$

and define  $S_m^j$  by  $(d/d\lambda)^j P_m = P_m S_m^j$ . Examining  $S_m^1$  we have

$$\begin{aligned} S_m^1 &= \frac{\beta_m^1}{\lambda_m} + \sum_{n=0}^{\infty} \frac{4\beta_{m-n}^1}{(2 - \lambda_{m-n})(6 - \lambda_{m-n})} \\ &= \frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + O(\lambda_m^{-2}) + \sum_{n=1}^{\infty} \frac{4\beta_{m-n}^1}{(2 - \lambda_{m-n})(6 - \lambda_{m-n})} \\ &= \frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + O(\lambda_m^{-2}), \end{aligned}$$

where the penultimate estimate uses the series expansion for  $1/(2 - \lambda_m)(6 - \lambda_m)$ , and the final one uses the structure of the series. This series consists of terms which are rational functions of  $\lambda_{m-n}$  and  $\beta_{m-n}^1$ , and in which the degree of the denominator strictly exceeds that of the numerator. The rapid growth of  $|\lambda_m|$  ensures this is bounded by a multiple of the first term, which is  $O(|\lambda_{m-1}|^{-1}) = O(|\lambda_m|^{-2})$ .

We will call a series in which the terms are rational functions that depend on  $\lambda_{m-n}, \beta_{m-n}^1, \dots, \beta_m^j$  for  $n \geq 1$ , but must always have the degree of the denominator to be strictly greater than that of the numerator, a good rational series or GRS. Notice that the derivative of a GRS is a GRS, and that the product of a GRS with a GRS or a rational function in which the numerator has the same or lesser degree than the denominator is also a GRS. Our estimates on  $\lambda_{m-n}$  and  $\beta_{m-n}^1$  guarantee that any GRS sums to a value which is  $O(|\lambda_m|^{-2})$ .

Using induction over  $j$  we see that the function  $S_m^j - \beta_m^j/\lambda_m - 4\beta_m^j/\lambda_m^2$  is always a GRS, and so is  $O(|\lambda_m|^{-2})$ . Indeed, this is true for  $j = 1$ , and if we assume it to be true for  $j - 1$  and apply the recursion  $S_m^j = S_m^1 S_m^{j-1} + (dS_m^{j-1}/d\lambda)$ , then both

$$\begin{aligned} S_m^1 S^{j-1,m} &= \left( \frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + \text{GRS} \right) \left( \frac{\beta_m^{j-1}}{\lambda_m} + \frac{4\beta_m^{j-1}}{\lambda_m^2} + \text{GRS} \right) \\ &= \frac{\beta_m^1 \beta_m^{j-1}}{\lambda_m^2} + \frac{8\beta_m^1 \beta_m^{j-1}}{\lambda_m^3} + \text{GRS} \end{aligned}$$

and

$$\frac{dS_{j-1,m}}{d\lambda} = \frac{\beta_m^j}{\lambda_m} - \frac{\beta_m^{j-1} \beta_m^1}{\lambda_m^2} + \frac{4\beta_m^j}{\lambda_m^2} - \frac{8\beta_m^{j-1} \beta_m^1}{\lambda_m^3} + \text{GRS}$$

hold, so we may sum them to complete the induction. In particular we conclude that

$$\begin{aligned} 4E_\lambda^j(z_m) &= \frac{d^j P_m}{d\lambda^j} - \beta_m^j \\ &= P_m S_m^j - \beta_m^j \\ &= P_m \left( S_m^j - \frac{\beta_m^j}{\lambda_m} - \frac{4\beta_m^j}{\lambda_m^2} \right) + \frac{\beta_m^j}{\lambda_m} (P_m - \lambda_m + 4) + \frac{4\beta_m^j}{\lambda_m^2} (P_m - \lambda_m) \\ &= O(|\lambda_m|^{-1}) = O(2^{-2^{-m}}), \end{aligned}$$

where in the last step we used that the first bracketed term is  $O(|\lambda_m|^{-2})$ , and that  $P_m = O(|\lambda_m|)$ ,  $(P_m - \lambda_m + 4) = O(|\lambda_m|^{-1})$ , and  $(P_m - \lambda_m) = O(1)$ , all of which are from the fact that  $|E_\lambda(z_m)| = O(|\lambda_m|^{-1})$  (see Proposition 4.1) and the definition of  $P_m$ .

Now that we know  $|E_\lambda^j|$  has exponential decay it must be the case that its maximum occurs at an interior point of some  $f_0^{-m}(SG)$ . It is well known (see [7] Proposition 2.11) that  $E_\lambda^j$  and  $\Delta E_\lambda^j$  must have opposite signs at any local extreme point of  $|E_\lambda^j|$ . Since  $\Delta E_\lambda^j = -\lambda E_\lambda^j - jE_\lambda^{j-1}$  and  $\lambda < 0$ , we find that  $\text{sgn}(\Delta E_\lambda^j) = -\text{sgn}(E_\lambda^j)$  implies  $|\lambda E_\lambda^j| \leq |jE_\lambda^{j-1}|$ . The bound on  $E_\lambda^j$  follows by induction and the fact that  $|E_\lambda| \leq |E_\lambda(q_0)| = 1$ . ■

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