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A q-SUPERCONGRUENCE ARISING FROM ANDREWS' $_4\phi_3$ IDENTITY

JI-CAI LIU and JING LIU

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Abstract

We establish a q-analogue of a supercongruence related to a supercongruence of Rodriguez-Villegas, which extends a q-congruence of Guo and Zeng ['Some q-analogues of supercongruences of Rodriguez-Villegas', *J. Number Theory* **145** (2014), 301–316]. The important ingredients in the proof include Andrews' $_4\phi_3$ terminating identity.

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1. Introduction

In 2003, Rodriguez-Villegas [5] investigated hypergeometric families of Calabi–Yau manifolds. He observed numerically some remarkable supercongruences between the values of the truncated hypergeometric series and expressions derived from the number of $\mathbb{F}_p$-points of the associated Calabi–Yau manifolds. For manifolds of dimension $d = 1$, he conjectured four interesting supercongruences associated to certain elliptic curves, one of which is

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{1.1}$$

where $p \geq 5$ is a prime. The conjectural supercongruence (1.1) was first proved by Mortenson [4].

For polynomials $A_1(q), A_2(q), P(q) \in \mathbb{Z}[q]$, the q-congruence

$$A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}$$

is understood as $A_1(q)$ is divisible by $P(q)$, and $A_2(q)$ is coprime with $P(q)$. In general, for rational functions $A(q), B(q) \in \mathbb{Q}(q)$ and polynomial $P(q) \in \mathbb{Z}[q]$,

$$A(q) \equiv B(q) \pmod{P(q)} \Longleftrightarrow A(q) - B(q) \equiv 0 \pmod{P(q)}.$$

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Guo and Zeng [3, Corollary 2.2] established a q-analogue of (1.1) as follows:

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(p-1)/2} q^{(1-p^2)/4} \pmod{[p]^2}.$$

Here and in what follows, the q-analogue of the natural number n is defined by $[n] = (1-q^n)/(1-q)$, and for $n \geq 1$, the q-shifted factorial is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ with $(a;q)_0 = 1$.

In 2011, Sun [7, Conjecture 5.5] conjectured a supercongruence related to (1.1): modulo p^2,

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{32^k} \equiv \begin{cases} 2x - \dfrac{p}{2x} & \text{if } p \equiv 1 \pmod 4 \text{ and } p = x^2 + y^2 \text{ with } 4 \mid (x-1), \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

which was proved by Tauraso [8] and Sun [6, Theorem 2.2].

Guo and Zeng [3, Corollary 2.7] established a partial q-analogue of (1.2):

$$\sum_{k=0}^{(p-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv 0 \pmod{[p]^2} \tag{1.3}$$

for all primes $p \equiv 3 \pmod 4$.

To continue the q-story of (1.2), we recall some q-series notation. The basic hypergeometric series is defined by

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k,$$

where $(a_1, a_2, \ldots, a_m; q)_k = (a_1;q)_k (a_2;q)_k \cdots (a_m;q)_k$. The nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ (n,k)=1}} (q - \zeta^k),$$

where ζ denotes a primitive nth root of unity.

The motivation for this paper is to extend the q-congruence (1.3) of Guo and Zeng, and establish a complete q-analogue of (1.2).

THEOREM 1.1. *Let n be an odd positive integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \begin{cases} (-1)^{(n-1)/4} q^{(n-1)(n+3)/8} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{1.4}$$

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The important ingredients in the proof of (1.4) include Andrews' ${}_4\phi_3$ terminating identity [2, (II.17), page 355]:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a^2q^{n+1}, c, -c \\ aq, -aq, c^2 \end{matrix}; q, q\right] = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2, \\ \dfrac{c^n(q, a^2q^2/c^2; q^2)_{n/2}}{(a^2q^2, c^2q; q^2)_{n/2}} & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{1.5}
$$

The rest of the paper is organised as follows. In the next section, we shall explain why (1.4) is a q-analogue of (1.2). The proof of Theorem 1.1 will be presented in Section 3.

2. Why (1.4) is a q-analogue of (1.2)

Let p be an odd prime. It is clear that

$$
\Phi_p(q) = 1 + q + \cdots + q^{p-1}.
$$

Setting $n \to p$ and $q \to 1$ on both sides of (1.4) gives, modulo p^2,

$$
\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{32^k} \equiv \begin{cases} \dfrac{(-1)^{(p-1)/4}}{2^{(p-1)/2}}\dbinom{(p-1)/2}{(p-1)/4} & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.1}
$$

By a result due to Chowla *et al.* [1],

$$
\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2},
$$

where $p \equiv 1 \pmod 4$ and $p = x^2 + y^2$ with $4 \mid (x-1)$. It follows that

$$
\begin{aligned}
\frac{(-1)^{(p-1)/4}}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4} &\equiv (-1)^{(p-1)/4}\frac{2^{p-1}+1}{2^{(p+1)/2}}\left(2x - \frac{p}{2x}\right) \\
&\equiv 2x - \frac{p}{2x} \pmod{p^2},
\end{aligned} \tag{2.2}
$$

where we have used the fact [6, page 1918]:

$$
\frac{2^{p-1}+1}{2^{(p+1)/2}} \equiv (-1)^{(p-1)/4} \pmod{p^2}.
$$

Combining (2.1) and (2.2), we arrive at (1.2). Thus, (1.4) is indeed a q-analogue of (1.2).

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3. Proof of Theorem 1.1

Let n be an odd positive integer. Setting $n \to (n-1)/2, q \to q^2, a \to 1$ on both sides of (1.5) gives

$$\sum_{k=0}^{(n-1)/2} \frac{(q^{1-n}, q^{1+n}, c, -c; q^2)_k}{(q^2, q^2, -q^2, c^2; q^2)_k} q^{2k} = \begin{cases} \dfrac{c^{(n-1)/2}(q^2, q^4/c^2; q^4)_{(n-1)/4}}{(q^4, c^2q^2; q^4)_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.1}$$

Letting $c \to 0$ on both sides of (3.1) and noting that for $n \equiv 1 \pmod 4$,

$$\lim_{c\to 0}(c, -c; q^2)_k = \lim_{c\to 0}(c^2; q^2)_k = \lim_{c\to 0}(c^2q^2; q^4)_{(n-1)/4} = 1$$

and

$$\lim_{c\to 0} c^{(n-1)/2}(q^4/c^2; q^4)_{(n-1)/4} = (-1)^{(n-1)/4} q^{(n-1)(n+3)/8},$$

we obtain

$$\sum_{k=0}^{(n-1)/2} \frac{(q^{1-n}, q^{1+n}; q^2)_k}{(q^2; q^2)_k (q^4; q^4)_k} q^{2k} = \begin{cases} (-1)^{(n-1)/4} q^{(n-1)(n+3)/8} \dfrac{(q^2; q^4)_{(n-1)/4}}{(q^4; q^4)_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.2}$$

Since

$$(1 - q^{n-2j+1})(1 - q^{n+2j-1}) + (1 - q^{2j-1})^2 q^{n-2j+1} = (1 - q^n)^2$$

and $1 - q^n \equiv 0 \pmod{\Phi_n(q)}$,

$$(1 - q^{n-2j+1})(1 - q^{n+2j-1}) \equiv -(1 - q^{2j-1})^2 q^{n-2j+1} \pmod{\Phi_n(q)^2}.$$

It follows that

$$(1 - q^{-n+2j-1})(1 - q^{n+2j-1}) \equiv (1 - q^{2j-1})^2 \pmod{\Phi_n(q)^2}.$$

Thus,

$$\begin{aligned}(q^{1-n}, q^{1+n}; q^2)_k &= \prod_{j=1}^{k} (1 - q^{-n+2j-1})(1 - q^{n+2j-1}) \\ &\equiv \prod_{j=1}^{k} (1 - q^{2j-1})^2 = (q; q^2)_k^2 \pmod{\Phi_n(q)^2}. \end{aligned} \tag{3.3}$$

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Finally, substituting (3.3) into the left-hand side of (3.2) gives, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k(q^4;q^4)_k} q^{2k} \equiv \begin{cases} (-1)^{(n-1)/4} q^{(n-1)(n+3)/8} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

as desired.

References

[1] S. Chowla, B. Dwork and R. J. Evans, 'On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$', *J. Number Theory* **24** (1986), 188–196.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd edn, Encyclopedia of Mathematics and Its Applications, 96 (Cambridge University Press, Cambridge, 2004).

[3] V. J. W. Guo and J. Zeng, 'Some q-analogues of supercongruences of Rodriguez-Villegas', *J. Number Theory* **145** (2014), 301–316.

[4] E. Mortenson, 'A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function', *J. Number Theory* **99** (2003), 139–147.

[5] F. Rodriguez-Villegas, 'Hypergeometric families of Calabi–Yau manifolds', in: *Calabi–Yau Varieties and Mirror Symmetry (Toronto, ON, 2001)*, Fields Institute Communications, 38 (eds. N. Yui and J. D. Lewis) (American Mathematical Society, Providence, RI, 2003), 223–231.

[6] Z.-H. Sun, 'Congruences concerning Legendre polynomials', *Proc. Amer. Math. Soc.* **139** (2011), 1915–1929.

[7] Z.-W. Sun, 'On congruences related to central binomial coefficients', *J. Number Theory* **131** (2011), 2219–2238.

[8] R. Tauraso, 'An elementary proof of a Rodriguez-Villegas supercongruence', Preprint, 2009, arXiv:0911.4261v1.

JI-CAI LIU, Department of Mathematics,
Wenzhou University, Wenzhou 325035, PR China
e-mail: jcliu2016@gmail.com

JING LIU, Department of Mathematics,
Wenzhou University, Wenzhou 325035, PR China
e-mail: 22451025009@stu.wzu.edu.cn