ON THE NUMBER OF PRIMITIVE LATTICE POINTS IN A PARALLELOGRAM

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1. Let α be any irrational real number, and let $F(u)$ denote the number of those positive integers $n \leqslant u$ for which $(n, [n\alpha]) = 1$. Watson proved in the preceding paper that

(1) $$\lim_{u\to\infty} \{u^{-1} F(u)\} = 6\pi^{-2}.$$

The object of this paper is to give a different proof of a slight generalization of this result.

In what follows, a lattice point is a point in the plane whose cartesian coordinates are integers. It is said to be primitive if its coordinates are relatively prime. For any positive numbers u and a, let $g(u, a)$ denote the number of lattice points, $f(u, a)$ the number of primitive lattice points in the set of points given by

$$0 < x \leqslant u, \qquad \alpha x - a < y \leqslant \alpha x$$

(a parallelogram with two of its sides included). Then $F(u) = f(u, 1)$, and thus the formula

(2) $$\lim_{u\to\infty} \{u^{-1} f(u, a)\} = 6\pi^{-2} a$$

is a generalization of (1).

2. My proof is based on the formula

(3) $$\lim_{u\to\infty} \{u^{-1} g(u, a)\} = a.$$

This is equivalent to a well-known theorem of Bohl, Sierpinski, and Weyl [**2**, Satz 2]. The following simple elementary proof of (3) is reconstructed from what I remember of a lecture given by Hecke about thirty years ago. I cannot trace it in the literature, but its main idea, at any rate, is due to Hecke.

Since the addition of an integer to α does not alter $g(u, a)$, we may assume that $\alpha > 0$. Then, by a theorem of Kronecker [**1**, Theorem 438], the numbers of the form $m\alpha - n$, where m and n are positive integers, are everywhere dense. It is therefore sufficient to consider the case when a is of this form. Then

(4) $$g(u, a) = g(u, m\alpha - n) = g(u, m\alpha) - n[u],$$

and $g(u, m\alpha) = A + B - C$, where A, B, and C are, respectively, the numbers

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of lattice points in the parallelogram

$$0 < y \leqslant \alpha u, \quad y/\alpha \leqslant x < y/\alpha + m,$$

in the triangle

$$0 < x, \quad \alpha x - m\alpha < y \leqslant 0$$

and in the triangle

$$u < x, \quad \alpha x - m\alpha < y \leqslant \alpha u.$$

Now $A = m[\alpha u]$, B is independent of u, and C is a bounded function of u. Hence

$$\lim_{u\to\infty} \{u^{-1} g(u, m\alpha)\} = m\alpha,$$

and, by (4),

$$\lim_{u\to\infty} \{u^{-1} g(u, a)\} = m\alpha - n = a.$$

3. Throughout the remainder of this paper, let the letters d, k, and n denote positive integers, m an integer, a and u positive numbers, and μ Möbius' function. Then

$$(5) \qquad f(u, a) = \sum_{n\leqslant u} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n \\ (n,m)=1}} 1 = \sum_{n\leqslant u} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n}} \sum_{d\,|\,(n,m)} \mu(d)$$

$$= \sum_{d} \mu(d) \sum_{\substack{n\leqslant u \\ d\,|\,n}} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n \\ d\,|\,m}} 1 = \sum_{d} \mu(d) \sum_{n'\leqslant u/d} \sum_{\substack{m' \\ \alpha n' - a/d < m' \leqslant \alpha n'}} 1$$

$$= \sum_{d} \mu(d)\, g(u/d, a/d),$$

so that

$$(6) \qquad u^{-1} f(u, a) = \sum_{d} \mu(d)\, d^{-1} (u/d)^{-1} g(u/d, a/d).$$

Also, by (3),

$$(7) \qquad \lim_{u\to\infty} \{(u/d)^{-1} g(u/d, a/d)\} = a/d.$$

Thus, if it were permissible to proceed to the limit term by term, it would follow from (6) that

$$\lim_{u\to\infty} \{u^{-1} f(u, a)\} = a \sum_{d} \mu(d)\, d^{-2} = 6\pi^{-2} a;$$

but I do not know any direct method of justifying this process.

4. A slight modification of the preceding argument, however, will lead to

$$(8) \qquad \overline{\lim_{u\to\infty}} \{u^{-1} f(u, a)\} \leqslant 6\pi^{-2} a.$$

Let

$$(9) \qquad h(u, a, k) = \sum_{n\leqslant u} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n \\ (n,m,k!)=1}} 1.$$

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Then, by the first equation in (5),

$$f(u, a) \leqslant h(u, a, k) \tag{10}$$

and, by the argument that led to (6),

$$u^{-1} h(u, a, k) = \sum_{d \mid k!} \mu(d)\, d^{-1} (u/d)^{-1} g(u/d, a/d).$$

Here it is obviously permissible to proceed to the limit term by term. From this and (7) it follows that

$$\lim_{u\to\infty} \{u^{-1} h(u, a, k)\} = a \sum_{d \mid k!} \mu(d)\, d^{-2}. \tag{11}$$

By (10) and (11),

$$\overline{\lim_{u\to\infty}} \{u^{-1} f(u, a)\} \leqslant a \sum_{d \mid k!} \mu(d)\, d^{-2}.$$

Since this holds for every k, and

$$\lim_{k\to\infty} \sum_{d \mid k!} \mu(d)\, d^{-2} = \sum_{d} \mu(d)\, d^{-2} = 6\pi^{-2},$$

we deduce (8).

5. To complete the proof of (2), we note that, by (9),

$$h(u, a, k) = \sum_{\substack{d \\ (d, k!)=1}} \sum_{n \leqslant u} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n \\ (n,m)=d}} 1$$

$$= \sum_{\substack{d \\ (d, k!)=1}} \sum_{n' \leqslant u/d} \sum_{\substack{m' \\ \alpha n' - a/d < m' \leqslant \alpha n' \\ (n',m')=1}} 1 = \sum_{\substack{d \\ (d, k!)=1}} f(u/d, a/d).$$

Hence

$$f(u, a) = h(u, a, k) - \sum_{\substack{d>1 \\ (d, k!)=1}} f(u/d, a/d). \tag{12}$$

LEMMA 1. $f(u, a) - f(\tfrac{1}{2}u, a) \leqslant 2au + 1$.

Proof.

$$f(u, a) - f(\tfrac{1}{2}u, a) = \sum_{\substack{n \\ \frac{1}{2}u < n \leqslant u}} \sum_{\substack{m \\ \alpha n - a < m \leqslant \alpha n \\ (n,m)=1}} 1$$

$$\leqslant \sum_{n \leqslant u} \sum_{\substack{m \\ \alpha - 2a/u \leqslant m/n \leqslant \alpha \\ (n,m)=1}} 1.$$

This is the number of fractions, in their lowest terms, with positive denominators less than or equal to u, in an interval of length $2a/u$. Since any two such fractions differ by at least u^{-2}, the result follows.

LEMMA 2. *Let* $u \geqslant 1$. *Then* $f(u, a) \leqslant 4au + \log(2u)/\log 2$.

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Proof. Let $b = [\log u/\log 2]$. Then, by Lemma 1,

$$f(u, a) = f(u, a) - f(2^{-b-1}u, a) = \sum_{m=0}^{b} \{f(2^{-m}u, a) - f(2^{-m-1}u, a)\}$$

$$\leqslant \sum_{m=0}^{b} (2^{1-m}au + 1) < 4au + b + 1,$$

and the result follows.

LEMMA 3. *Let $au \leqslant 1$. Then $f(u, a) \leqslant 1$.*

Proof. Otherwise there would be two distinct fractions m_1/n_1 and m_2/n_2, such that

$$n_1 \leqslant n_2 \leqslant u, \quad \alpha - a/n_1 < m_1/n_1 \leqslant \alpha,$$
$$\alpha - a/n_1 \leqslant \alpha - a/n_2 < m_2/n_2 \leqslant \alpha,$$

which implies that $|m_1/n_1 - m_2/n_2| < a/n_1$; but

$$|m_1/n_1 - m_2/n_2| \geqslant 1/(n_1 n_2) \geqslant 1/(n_1 u) \geqslant a/n_1.$$

6. Let $u \geqslant 1$. Then, since the conditions $d > 1$ and $(d, k!) = 1$ imply that $d > k$, and since $f(u/d, a/d) = 0$ if $d > u$, it follows from Lemmas 2 and 3 that

$$\sum_{\substack{d \\ 1<d\leqslant\sqrt{(au)} \\ (d,k!)=1}} f(u/d, a/d) \leqslant \sum_{\substack{d \\ k<d\leqslant\sqrt{(au)}}} (4aud^{-2} + 2\log u) \leqslant 4auk^{-1} + 2\sqrt{(au)}\log u$$

and

$$\sum_{\substack{d>\sqrt{(au)} \\ (d,k!)=1}} f(u/d, a/d) \leqslant \sum_{\substack{d\leqslant u \\ (d,k!)=1}} 1 .$$

Hence, by (12),

$$u^{-1}f(u, a) \geqslant u^{-1}h(u, a, k) - \frac{4a}{k} - 2\sqrt{(a/u)}\log u - \frac{1}{u}\sum_{\substack{d\leqslant u \\ (d,k!)=1}} 1 ,$$

and hence, by (11),

$$\varliminf_{u\to\infty} \{u^{-1}f(u, a)\} \geqslant a\sum_{d|k!} \mu(d)\, d^{-2} - \frac{4a}{k} - \frac{\phi(k!)}{k!},$$

where ϕ denotes Euler's function. Since this holds for every k, and the right-hand side tends to $6\pi^{-2}a$ as $k \to \infty$, it follows that

$$\varliminf_{u\to\infty} \{u^{-1}f(u, a)\} \geqslant 6\pi^{-2}a,$$

which, together with (8), proves (2).

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