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COMMUTING DILATIONS AND UNIFORM ALGEBRAS

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1. Introduction. Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Fix a nonzero complex homomorphism τ on A and a representing measure m for τ on X. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak*-closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$.

Let M be an invariant subspace of H^2 under the multiplications of functions in A and N the orthogonal complement of M in H^2, that is, $N = H^2 \ominus M$. The orthogonal projection in L^2 with range N will be denoted by P. For f a function in H^∞ let S_f denote the projection onto N of the operator M_f on L^2 of multiplication by f, that is, $S_f = PM_f|N$. If A is a disc algebra and $\tau(f) = \tilde{f}(0)$ where $\tilde{f}$ denotes the holomomorphic extension of f in A, then τ is a complex homomorphism on A. Let m be a normalized Lebesgue measure on the unit circle ∂U; then m is a representing measure for τ. Then H^2 is the classical Hardy space $H^2(U)$.

SARASON THEOREM. *Let H^2 be the classical Hardy space $H^2(U)$. If T is a bounded linear operator on N that commutes with S_f ($f \in A$), then there is a function ϕ in H^∞ such that*

$$\|\phi\|_\infty = \|T\| \quad \text{and} \quad T = S_\phi.$$

The Sarason Theorem implies that $\|S_\phi\| = \|\phi + M \cap L^\infty\|$ for any ϕ in H^∞, and hence it is close to Nehari's theorem. The author ([**10**], [**11**]) generalized Nehari's theorem to general uniform algebras. In this paper generalizations of the Sarason Theorem to general uniform algebras will be proved using the method in the author's previous papers ([**10**], [**11**]). The proofs are different from Sarason's proof and simpler than his in the classical Hardy space $H^2(U)$. In Section 2, we will consider the relation between $\|S_\phi\|$ and $\|\phi + M \cap L^\infty\|$. In Section 3, we will apply the result in Section 2 to get Pick's theorem. In the special case, this gives a theorem of Abrahamse [**1**, Theorem 1] that implies Pick's theorem in a multiply

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connected domain. In Section 4, we will study generalizations of the Sarason Theorem. This gives a dilation of the commutant of some representation of a uniform algebra, extending partially the dilation theorem of Sz-Nagy and Foiaş, ([**9**]), in case A is a disc algebra. In Section 5, we will give dilations of the commutants of other representations of a uniform algebra relating with Hankel operators and Toeplitz operators. In Section 6, we will give concrete examples for which we can apply theorems in previous sections. That is, a uniform algebra which consists of rational functions on a multiply connected domain, a subalgebra of a disc algebra which contains the constants and which has finite codimension, and a polydisc algebra. However the Sarason Theorem is not true in an exact meaning. It is interesting to compare a recent paper of R. G. Douglas and V. I. Paulsen [**6**] or an example of S. Parrott [**13**] with this.

Throughout this paper, we use the following definition and assume that $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$. This assumption is satisfied in many examples (see Section 6).

DEFINITION. For an invariant subspace M in H^2, $M^\perp$ denotes the orthogonal complement of M in L^2. Moreover set

$$M^\perp \cap L^\infty = \left\{ f \in L^\infty : \int_X f\bar{g}\, dm = 0 \text{ for all } g \text{ in } M \right\}$$

and

$$M^\perp \cap L^1 = \left\{ f \in L^1 : \int_X f\bar{g}\, dm = 0 \text{ for all } g \text{ in } M \cap L^\infty \right\}.$$

2. Generalized Interpolation. Put $\mathcal{L} = \{ v \in L^\infty;\ v^{-1} \in L^\infty \text{ and } v \geq 0 \}$. Let M be an invariant subspace of H^2 and N be an orthogonal complement of M in H^2, that is, $N = H^2 \ominus M$, as in the Introduction. For each v in $\mathcal{L}$, let $N^v = vH^2 \ominus vM$ and P^v the orthogonal projection from L^2 onto N^v. For ϕ in H^∞ and g in N^v, S_ϕ^v is the operator defined by

$$S_\phi^v g = P^v M_\phi g.$$

If v is a constant function, then $N^v = N$, $P^v = P$ and $S_\phi^v = S_\phi$.

Denoting by (f) the coset in $(L^\infty)^{-1}/(H^\infty)^{-1}$ of an f in $(L^\infty)^{-1}$, define

$$\|(f)\| = \inf\{ \|g\|_\infty \|g^{-1}\|_\infty;\ g \in (f)\}$$

and

$$\gamma_0 = \sup\{ \|(f)\|;\ (f) \in (L^\infty)^{-1}/(H^\infty)^{-1}\}.$$

This constant γ_0 was introduced in [**11**], and used in [**11**] and [**12**]. In the definition above, we can use $\mathcal{L}/|(H^\infty)^{-1}|$ instead of $(L^\infty)^{-1}/(H^\infty)^{-1}$.

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LEMMA 1. *If h is in L^∞ then there exists a sequence $\{v_n\}$ in $\mathcal{L}$ such that*

$$\lim_{n\to\infty}\int_X v_n^2\, dm = \int_X |h|\, dm$$

and

$$\lim_{n\to\infty}\int_X |h|^2 v_n^{-2}\, dm = \int_X |h|\, dm.$$

PROOF. This is in the proof of Theorem 1 in [**10**]. In fact set $E_n = \{x \in X;\ 0 < |h(x)| < 1/n\}$, $F_0 = \{x \in X;\ h(x) = 0\}$ and $F_n = \{x \in X;\ |h(x)| \geq 1/n\}$. Define v_n by the formula

$$v_n(x) = \begin{cases} 1 & x \in E_n \\ 1/n & x \in F_0 \\ |h(x)|^{1/2} & x \in F_n \end{cases}$$

LEMMA 2. *If ϕ is in H^∞, then for any v in $\mathcal{L}$*

$$\|S_\phi^v\| = \sup\left\{\left|\int \phi h\bar{k}\, dm\right|;\ h \in vH^2,\ k \in v^{-1}M^\perp,\ \|h\|_2 \leq 1 \textit{ and } \|k\|_2 \leq 1\right\}.$$

PROOF. Since $N^v = (vH^2) \cap v^{-1}M^\perp$, it is sufficient to show that

$$\|S_\phi^v\| \geq \sup\left\{\left|\int \phi h\bar{k}\, dm\right|;\ h \in vH^2,\ k \in v^{-1}M^\perp,\ \|h\|_2 \leq 1 \text{ and } \|k\|_2 \leq 1\right\}.$$

For $h \in vH^2$ and $k \in v^{-1}M^\perp$

$$\begin{aligned}\left|\int \phi h\bar{k}\, dm\right| &= |(\phi h, k)| \\ &= |(\phi P^v h, k)| \\ &= |(P^v \phi P^v h, k)| \\ &\leq \|S_\phi^v\|\ \|h\|_2\ \|k\|_2\end{aligned}$$

where (,) denotes the usual inner product with respect to dm. Hence the lemma follows.

THEOREM 1. *Suppose M is an invariant subspace of H^2 and $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$. Let ϕ be a function in H^∞; then the following are valid.*
(1) $\sup\{\|S_\phi^v\|;\ v \in \mathcal{L}\} = \|\phi + M \cap L^\infty\|$.
(2) $\|S_\phi^v\| = \|S_\phi^u\|$ *if* $(v) = (u)$.
(3) $\|(v^{-1})\| \leq \|S_\phi^v\| / \|S_\phi\| \leq \|(v)\|$ *for any v in $\mathcal{L}$.*

PROOF.
(1) If $g \in M \cap L^\infty$ and $h \in N^v$ then $gh \in vM$. Hence

$$S_{\phi+g}^v h = P^v\big((\phi + g)h\big) = P^v(\phi h).$$

Thus for any $v \in \mathcal{L}$ $\|S_\phi^v\| \leq \|\phi + M \cap L^\infty\|$. If $h \in M^\perp \cap L^\infty$ then $h = v_n \times v_n^{-1} h$, $v_n \in v_n H^2$ and $v_n^{-1} h \in (v_n M)^\perp = v_n^{-1} M^\perp$. By Lemma 2

$$\left|\int_X \phi \bar{h}\, dm\right| = \left|\int_X \phi\, v_n(v_n^{-1}\bar{h})\, dm\right| \\ \leq \|S_\phi^{v_n}\|\ \|v_n\|_2\ \|v_n^{-1} h\|_2.$$

As $n \longrightarrow \infty$, by Lemma 1

$$\left|\int_X \phi \bar{h}\, dm\right| \leq \sup_v \|S_\phi^v\| \int_X |\bar{h}|\, dm.$$

Since $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$

$$\|\phi + M \cap L^\infty\| \leq \sup_v \|S_\phi^v\|.$$

(2) If $f \in (H^\infty)^{-1}$ then $v|f|H^2 = q(vH^2)$ and $v^{-1}|f|^{-1}M^\perp = q(v^{-1}M^\perp)$ with $q = |f|/f = \bar{f}/|f|$. Hence

$$\sup\left\{\left|\int_X a\bar{b}\phi\, dm\right|;\ a \in vH^2,\ b \in v^{-1}M^\perp,\ \|a\|_2 \leq 1 \text{ and } \|b\|_2 \leq 1\right\} \\ = \sup\left\{\left|\int_X c\bar{d}\phi\, dm\right|;\ c \in v|f|H^2,\ d \in v^{-1}|f|^{-1}M,\ \|c\|_2 \leq 1 \text{ and } \|d\|_2 \leq 1\right\}.$$

By Lemma 2 $\|S_\phi^v\| = \|S_\phi^u\|$ if $(v) = (u)$.

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(3) Let $v \in \mathcal{L}$. If $\int_X |k|^2 v^2\, dm \leq 1$ and $\int_X |h|^2 v^{-2}\, dm \leq 1$, then $\int_X |k|^2\, dm \leq \|v^{-2}\|_\infty$ and $\int_X |h|^2\, dm \leq \|v^2\|_\infty$. Hence

$$\begin{aligned}
\|S_\phi^v\| &= \sup\left\{\left|\int_X a\bar{b}\phi\, dm\right|;\ a \in N^v,\ b \in N^v,\ \|a\|_2 \leq 1 \text{ and } \|b\|_2 \leq 1\right\} \\
&\leq \sup\left\{\left|\int_X vk \times v^{-1}\bar{h}\phi\, dm\right|;\ vk \in vH^2, v^{-1}k \in (vM)^\perp,\ \|vk\|_2 \leq 1 \text{ and } \|v^{-1}h\|_2 \leq 1\right\} \\
&\leq \sup\left\{\left|\int_X k\bar{h}\phi\, dm\right|;\ k \in H^2,\ h \in M^\perp,\ \|k\|_2 \leq \|v^{-2}\|_\infty^{1/2} \text{ and } \|h\|_2 \leq \|v^2\|_\infty^{1/2}\right\} \\
&\leq \left(\|v^{-2}\|_\infty\, \|v^2\|_\infty\right)^{1/2} \sup\left\{\left|\int_X k\bar{h}\phi\, dm\right|;\ k \in H^2,\ h \in M^\perp,\ \|k\|_2 \leq 1 \text{ and } \|h\|_2 \leq 1\right\}.
\end{aligned}$$

By Lemma 2

$$\|S_\phi^v\| \leq \|v^{-1}\|_\infty\, \|v\|_\infty\, \|S_\phi\|$$

and by (2) in this theorem

$$\|S_\phi^v\| \leq \|(v)\|\, \|S_\phi\|.$$

This implies (3).

The proof of (1) of Theorem 1 is similar to that of Theorem 1 in [**10**]. The proofs of (2) and (3) are similar to that of Theorem 3 in [**11**].

COROLLARY 1. *Suppose M is an invariant subspace of H^2 such that $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$. If γ_0 is finite, then*

$$\|S_\phi\| \leq \|\phi + M \cap L^\infty\| \leq \gamma_0 \|S_\phi\|$$

for any ϕ in H^∞.

If A is a disc algebra then $\gamma_0 = 1$ and hence Corollary 1 is a part of the Sarason Theorem. Let N_τ denote the set of representing measures on the Shilov boundary of A for τ. Suppose N_τ is finite dimensional and m is a core point of N_τ. Let N^∞ be the real annihilator of A in L_R^∞; then N^∞ is also finite dimensional. Set $\mathcal{E} = \exp N^\infty$; then $\mathcal{E}$ is a subgroup of $\mathcal{L}$. If $n = 0$ then $\mathcal{E} = \{1\}$.

COROLLARY 2. *Suppose N_T is finite dimensional and m is a core point of N_T. Let M be an invariant subspace of H^2 and ϕ a function in H^∞.*

(1) $\sup\{\|S_\phi^v\|;\ v \in \mathcal{E}\} = \|\phi + M \cap L^\infty\|$.

(2) If m is a unique logmodular measure then there exists v_0 in $\mathcal{E}$ such that

$$\|S_\phi^{v_0}\| = \|\phi + M \cap L^\infty\|.$$

Moreover γ_0 is finite and so

$$\|S_\phi\| \leq \|\phi + M \cap L^\infty\| \leq \gamma_0 \|S_\phi\|.$$

PROOF. (1) By the proof of Theorem 2 in [**10**] and (2) of Theorem 1, we can choose $\mathcal{E}$ instead of $\mathcal{L}$ in (1) of Theorem 1. By Theorem 6.1 in [**7**, Chapter V], $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$ and hence we need not assume it. (2) By (1) and the proof of Theorem 3 in [**10**] there exists v_0 in $\mathcal{E}$ such that $\|S_\phi^{v_0}\| = \|\phi + M \cap L^\infty\|$. γ_0 is finite by Theorem in [**11**] and hence Corollary 1 completes the proof.

3. Pick Interpolation Theorem. The proofs in this section are modeled after the Pick intepolation theorem for a bounded domain in the plane whose boundary consists of finite disjoint analytic Jordan curves due to M. B. Abrahamse [**1**].

Let $E = \{s_1, s_2, \ldots, s_n\}$ be the finite set of independent continuous linear functionals on H^2. Suppose if $s \in E$ then for any ϕ in H^∞ and h in H^2 $s(\phi h) = s(\phi)s(h)$, and we will write $s(h) = h(s)$. Put $M = \{f \in H^2 : f(s) = 0 \text{ for all } s \in E\}$. Then M is an invariant subspace and $N = H^2 \ominus M$ is an n-dimensional subspace. For each v in $\mathcal{L}$, $N^v = vH^2 \ominus vM$ is also an n-dimensional subspace. Let $(\ ,\ )_v$ denote the usual inner product with respect to $v^2\, dm$. For each v in $\mathcal{L}$ and $s \in E$, there exists k_s^v in H^2 such that for any h in H^2

$$h(s) = (h, k_s^v)_v = \int_X h\overline{k_s^v}\, v^2\, dm.$$

If v is constant we will write $k_s^v = k_s$. Put $k^v(s,t) = (k_s^v, k_t^v)_v$; then $k^v(s,t)$ is a kernel function on $E \times E$. If f is in M then for any $s \in E$ we have $(f, k_s^v)_v = 0$ and hence

$$\int_X \bar{f} k_s^v v^2\, dm = \int_X \overline{vf}\, vk_s^v\, dm = 0.$$

Therefore vk_s^v belongs to N^v and $\{vk_{s_1}^v, \ldots, vk_{s_n}^v\}$ is a basis in N^v.

LEMMA 3. *For ϕ in H^∞, $P^v(\bar{\phi} k_s^v) = \overline{\phi(s)} k_s^v$.*

THEOREM 2. *Let $E = \{s_1, s_2, \ldots, s_n\}$ be the finite set of independent continuous linear functionals on H^2 which if $s \in E$ then for any ϕ in H^∞ and h in H^2 $s(\phi h) =$*

$s(\phi)s(h)$, and let $w_1, w_2, \ldots, w_n$ be complex numbers. Suppose $M^\perp \cap L^\infty$ is dense in $M^\perp \cap L^1$ where $M = \{h \in H^2;\ h(s) = 0 \text{ for all } s \in E\}$.

(1) There is an analytic function ϕ in H^∞ satisfying $\|\phi\|_\infty \leq 1$ and $\phi(s_i) = w_i$ for $i = 1, \ldots, n$ if and only if the matrix

$$[(1 - w_i \bar{w}_j)k^v(s_i, s_j)]$$

is nonnegative for each v in $\mathcal{L}$.

(2) When $(v) = (u)$, the matrix $[(1 - w_i\bar{w}_j)k^v(s_i, s_j)]$ is nonnegative if and only if $[(1 - w_i\bar{w}_j)k^v(s_i, s_j)]$ is nonnegative.

(3) When γ_0 is finite, if the matix

$$[(1 - w_i\bar{w}_j)k(s_i, s_j)]$$

is nonnegative then there is an analytic function ϕ in H^∞ satisfying $\|\phi\|_\infty \leq \gamma_0$ and $\phi(s_i) = w_i$ for $i = 1, \ldots, n$.

PROOF. For $s \in E$, let α_s be a complex number and set

$$k = \sum_s \bar{\alpha}_s v k_s^v.$$

Then

$$\|k\|_2^2 = \sum_{s,t} \alpha_s \bar{\alpha}_t k^v(s, t)$$

and

$$\|P^v(\bar{\phi} k)\|_2^2 = \sum_{s,t} \alpha_s \bar{\alpha}_t \phi(s) \bar{\phi}(t) k^v(s, t).$$

Hence the assertion

$$\|P^v(\bar{\phi} k)\|_2^2 \leq \|k\|_2^2$$

for all k in N^v is equivalent to the assertion

$$[(1 - w_i\bar{w}_j)k^v(s_i, s_j)] \geq 0.$$

Since $\|P^v(\bar{\phi} k)\|_2 = \|(S_\phi^v)^* k\|_2$, the matrix above is nonnegative for each v if and only if $\sup\{\|S_\phi^v\|;\ v \in \mathcal{L}\} \leq 1$.

(1) If $\|\phi\|_\infty \leq 1$ and $\phi(s_i) = w_i$ for $i = 1, \ldots, n$ then $\sup\{\|S_\phi^v\|;\ v \in \mathcal{L}\} \leq 1$ and hence from the above remark the part of 'only if' follows. Conversely if the matrix is positive for each $v \in \mathcal{L}$, by what was shown above $\sup\{\|S_\phi^v\|;\ v \in \mathcal{L}\} \leq 1$ and by (1) of Theorem 1 $\|\phi + M \cap L^\infty\| \leq 1$.

(2) follows from (2) of Theorem 1 and what was shown above.

(3) If $[(1 - w_i\bar{w}_j)k^v(s_i, s_j)]$ is nonnegative then $\|S_\phi\| \leq 1$. Since γ_0 is finite, by Corollary 1 $\|\phi + M \cap L^\infty\| \leq \gamma_0$ and this implies (2).

COROLLARY 3. *Suppose N_τ is finite dimensional and m is a core point of N_τ. Let $E = \{s_1, s_2, \ldots, s_n\}$ be the finite set of independent continuous linear functionals on H^2 and $w_1, w_2, \ldots, w_n$ complex numbers.*

(1) There is an analytic function ϕ in H^∞ satisfying $\|\phi\|_\infty \leq 1$ and $\phi(s_i) = w_i$ for $i = 1, \ldots, n$ if and only if the matrix

$$[(1 - w_i\bar{w}_j)k^v(s_i, s_j)]$$

is nonnegative for each v in $\mathcal{E}$

(2) When m is a unique logmodular measure, if the matrix

$$[(1 - w_i\bar{w}_j)k(s_i, s_j)]$$

is nonnegative then there is an analytic function ϕ in H^∞ satisfying $\|\phi\|_\infty \leq \gamma_0$ and $\phi(s_i) = w_i$ for $i = 1, \ldots, n$.

In this section, we used a well known result, that is, when $E = \{s_1, s_2, \ldots, s_n\}$ is a finite set there exists at least one function f in H^∞ such that $f(s_i) = w_i$ for $i = 1, \ldots, n$.

4. Dilations of Commutants. Let L be a complex Hilbert space and $\mathcal{B}(L)$ the algebra of all bounded linear operators on L. I denotes the identity operator in L. An algebra homomorphism $f \longrightarrow \mathcal{M}_f$ of H^∞ in $\mathcal{B}(L)$ which satisfies

$$\mathcal{M}_1 = I \quad \text{and} \quad \|\mathcal{M}_f\| \leq \|f\|_\infty$$

is called a representation of H^∞ on L. If $\mathcal{N}$ is a closed subspace of L and $\mathcal{P}$ is the orthogonal projection onto $\mathcal{N}$, then $\mathcal{N}$ is called semi-invariant under H^∞ provided $\mathcal{P}\mathcal{M}_f\mathcal{P}\mathcal{M}_g\mathcal{P} = \mathcal{P}\mathcal{M}_f\mathcal{M}_g$ for all f and g in H^∞. For ϕ in H^∞ and h in $\mathcal{N}$, S_ϕ is the operator defined by

$$S_\phi h = \mathcal{P}\mathcal{M}_\phi h.$$

D. Sarason [**14**] showed that every semi-invariant subspace of H^∞ is equal to the orthogonal complement of one invariant subspace of H^∞ with respect to a larger one, and every subspace of the latter form is semi-invariant under H^∞. By the Sarason Theorem it is natural to assume that for any f, g in H^∞ $\mathcal{M}_f^*\mathcal{M}_g = \mathcal{M}_g\mathcal{M}_f^*$. A question is that if T is a bounded operator on $\mathcal{N}$ that commutes with $\mathcal{S}_f$ for any f in H^∞ then $T = \mathcal{S}_\phi$ for some ϕ in H^∞ and $\|T\| = \|\phi\|_\infty$. However the conjecture can be answered negatively even if $\mathcal{N}$ is two dimensional because the Pick interpolation theorem for two points is not true in the original form for the annulus algebra [**1**, p. 202]. If the question can be answered positively for the disc algebra, then it contains the part of a theorem of B. Sz-Nagy and C. Foiaş [**9**] and hence a theorem of T. Ando [**3**]. The question is not true for the polydisc algebra [**8**]. This is not so surprizing. For when $\mathcal{N} = N = H^2 \ominus M$, if the question is true then for any ϕ in H^∞ $\|S_\phi\| = \|\phi + M \cap L^\infty\|$. This negative answer for the polydisc algebra is related with examples of S. Parrott [**13**] and N. J. Varopoulos [**15**].

In this section we concentrate on a special case. We assume that $H^\infty = H^2 \cap L^\infty$. As in Section 2 let $\mathcal{N} = N = H^2 \ominus M$, $L = L^2$ and $S_\phi = S_\phi(\phi \in H^\infty)$. Suppose $N \cap L^\infty$ is dense in N, then $N^v \cap L^\infty$ is dense in N^v for any v in $\mathcal{L}$. For ϕ in H^2 and g in $N^v \cap L^\infty$, $\mathring{S}_\phi^v$ is the operator defined by

$$\mathring{S}_\phi^v\, g = P^v M_\phi g.$$

If ϕ is in H^∞ then $\mathring{S}_\phi^v = S_\phi^v$.

THEOREM 3. *Let M be an invariant subspace of H^2 and let $M^\perp \cap L^\infty$ be dense in $M^\perp$ and $M^\perp \cap L^1$, and $N \cap L^\infty$ dense in N. Suppose T is a bounded operator on N which commutes with S_f for any f in H^∞.*

(1) There exists a function ϕ in H^2 such that $T = \mathring{S}_\phi$.

(2) If $TP1$ is in H^∞ then there exists a function ϕ in H^∞ such that

$$\|T\| \leq \|\phi\|_\infty, \ T = S_\phi \ \textit{and} \ \|\phi\|_\infty = \sup\{\|\mathring{S}_\phi^v\|;\ v \in \mathcal{L}\}.$$

(3) If γ_0 is finite then there exists a function ϕ in H^∞ such that

$$\|T\| \leq \|\phi\|_\infty \leq \gamma_0 \|T\| \ \textit{and} \ T = S_\phi.$$

PROOF.

(1) Put $\phi = TP1$ then $\phi \in H^2$. For any $h, k \in N \cap L^\infty$

$$\begin{aligned}(Th, k) &= (hP1, T^* k)\\ &= (TS_h P1, k)\\ &= (S_h TP1, k)\\ &= (\phi h, k)\\ &= (\mathring{S}_\phi h, k)\end{aligned}$$

because T commutes with S_h. Thus $T = \mathring{S}_\phi$ because $N \cap L^\infty$ is dense in N.

(2) $\phi_1 = TP1$ is in H^∞ and hence by the proof of (1) $T = S_{\phi_1}$. By (1) of Theorem 1 we can choose ϕ in H^∞ such that $\|T\| \leq \|\phi\|_\infty$, $T = S_\phi$ and $\|\phi\|_\infty = \sup\{\|S_\phi^v\|;\ v \in \mathcal{L}\}$.

(3) Put $\phi_1 = TP1$ then $\phi_1 \in H^2$. As in the proof of (1) of Theorem 1 we can show that

$$\left|\int_X \phi_1 \bar{h}\, dm\right| \le \sup_v \|\mathring{S}^v_{\phi_1}\| \int_X |\bar{h}|\, dm$$

for $h \in M^\perp \cap L^\infty$. Moreover as in the proof of (3) of Theorem 1 we can show that

$$\|S^v_{\phi_1}\| \le \left(\|v^{-2}\|_\infty \|v^2\|_\infty\right)^{\frac{1}{2}} \sup\left\{\left|\int_X k\bar{h}\phi_1\, dm\right|;\right.$$
$$\left. k \in H^\infty, h \in M^\perp \cap L^\infty, \|k\|_2 \le 1 \text{ and } \|h\|_2 \le 1\right\}.$$

For any $h, k \in N \cap L^\infty$, $(Th, k) = (\mathring{S}^v_{\phi_1} h, k)$ and hence as in the proof of Lemma 2 we can show that

$$\|T\| = \sup\{\left|\int_X k\bar{h}\phi_1\, dm\right|; k \in H^\infty,\ h \in M^\perp \cap L^\infty,\ \|k\|_2 \le 1$$
$$\text{and } \|h\|_2 \le 1\}$$

because $M^\perp \cap L^\infty$ is dense in M. Since γ_0 is finite, $\sup_v \|\mathring{S}^v_{\phi_1}\| \le \gamma_0 \|T\|$. Therefore $\sup_v \|\mathring{S}^v_{\phi_1}\| < \infty$ and hence by the Hahn-Banach theorem there exists a function $\phi \in L^\infty$ such that $\phi - \phi_1$ is orthogonal to $M^\perp \cap L^\infty$. Since $M^\perp \cap L^\infty$ is dense in $M^\perp$, $\phi - \phi_1$ belongs to M. Thus $\phi \in H^2 \cap L^\infty = H^\infty$ and $S_\phi = \mathring{S}^v_{\phi_1}$.

5. Hankel operators and Toeplitz operators. Let L be a complex Hilbert space and $\mathcal{M}_f (f \in L^\infty)$ a representation of L^∞ on L. If H is a closed subspace of L and Q is the orthogonal projection onto $H^\perp$, then H is called invariant under H^∞ provided $(1 - Q)\mathcal{M}_f(1 - Q) = \mathcal{M}_f(1 - Q)$ for all f in H^∞. For ϕ in L^∞ and h in H, H_ϕ is the operator defined by

$$H_\phi h = Q\mathcal{M}_\phi h$$

and it is called a Hankel operator. For ϕ in L^∞ and h in H, T^+_ϕ is the operator defined by

$$T^+_\phi h = (1 - Q)\mathcal{M}_\phi h$$

and it is called a Toeplitz operator. For ϕ in H^∞ put $T_\phi^- = Q\mathcal{M}_\phi | H$. Two natural questions are following:

(1) If T is a bounded operator from H into $H^\perp$ and $TT_f^+ = T_f^- T$ for any f in H^∞ then $T = H_\phi$ for some ϕ in L^∞ and $\|T\| = \|\mathcal{M}_\phi\|$?

(2) If T is a bounded operator on H that commutes with T_f^+ for any f in H^∞ then $T = T_\phi^+$ for some ϕ in H^∞ and $\|T\| = \|\mathcal{M}_\phi\|$? As in Section 4 if the questions can be answered positively for the disc algebra, then these contain the part of a theorem of B. Sz-Nagy and C. Foiaş.

In this section we concentrate on a special case. Let $L = L^2$ and $H = H^2$.

PROPOSITION 4. *Suppose if h is a function in H^2 with $hH^2 \subset H^2$ then h belongs to H^∞. If T is a bounded linear operator on H that commutes with T_f^+ for all f in H^∞ then $T = T_\phi^+$ for some ϕ in H^∞ and $\|T\| = \|K_\phi\|$.*

PROOF. Put $\phi = T1$ then $\phi \in H^2$. Fix $h \in H^2$. There exists a sequence $\{h_n\}$ in H^∞ such that $\|h_n - h\|_2 \longrightarrow 0$ and $h_n \longrightarrow h$ a. e. as $n \longrightarrow \infty$. Since T commutes with T_f^+ for all f in H^∞,

$$\begin{aligned} Th_n &= T(T_{h_n}^+ 1) \\ &= T_{h_n}^+ T1 \\ &= T_{h_n}^+ \phi \\ &= h_n \phi. \end{aligned}$$

Since T is bounded and $\|Th_n - Th\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$, $\|h_n\phi - Th\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. There exists a subsequence $\{h_{n_j}\}$ in H^∞ such that $h_{n_j}\phi \longrightarrow h\phi$ a. e. as $j \longrightarrow \infty$, and hence $\phi h = Th$. Thus $\phi H^2 \subset H^2$ and by the hypothesis $\phi \in H^\infty$ and $T = T_\phi^+$.

By the proofs of Theorem 1 in [**10**], Theorem 3 in [**11**] and Theorem 3 in this paper, we can prove the following proposition. Let Q^v be the orthogonal projection from L^2 onto $(vH^2)^\perp$ for each v in $\mathcal{L}$. For ϕ in H^∞ and g in vH^2, H_ϕ^v is the operator defined by

$$H_\phi^v g = Q^v M_\phi g.$$

When ϕ in H^2 and g in $vH^2 \cap L^\infty$, $\mathring{H}_\phi^v$ is the operator denfined by $\mathring{H}_\phi^v g = Q^v M_\phi g$.

PROPOSITION 5. *Let $(H^2)^\perp \cap L^\infty$ be dense in $(H^\infty)^\perp \cap L^1$ and $H^\infty = H^2 \cap L^\infty$. Suppose T is a bounded operator from H^2 into $(H^2)^\perp$ and $TT_f^+ = T_f^- T$ for any f in H^∞.*

(1) There exists a function ϕ in H^2 such that $T = \mathring{H}_\phi$.

(2) If $T1$ is in H^∞ then there exists a function in L^∞ such that

$$\|T\| \leq \|\phi\|_\infty,\ T = H_\phi \quad \text{and}$$
$$\|\phi\|_\infty = \sup\{\|H_\phi^\nu\|;\ \nu \in \mathcal{L}\}.$$

(3) If γ_0 is finite then there exists a function ϕ in L^∞ such that

$$\|T\| \leq \|\phi\|_\infty \leq \gamma_0\|T\| \text{ and } T = H_\phi.$$

6. Concrete Examples. All results in this paper were known in the disc algebra. We shall now apply them to some other concrete examples.

(I) Let Γ be a subgroup of reals, endowed with the discrete topology, and X the dual group. Let m be a Haar measure on X and $A = \{f \in C(X);\ \int_X f(x)(-a,x)\ dm(x) = 0$ for any $a \in \Gamma$ with $a > 0\}$, where (a,x) denotes the continuous character of X for $a \in \Gamma$. Then $\dim N_\tau = 0$ and $N_\tau = \{m\}$, and hence $\gamma_0 = 1$. If M is an invariant subspace of H^2, then $M^\perp \cap L^\infty$ is dense in $M^\perp$ and $M^\perp \cap L^1$ but $N = H^2 \ominus M$ is always infinite dimensional. Hence we can not apply Theorem 2 or Corollary 2 to this example. We do not know whether $N \cap L^\infty$ is dense in N or not.

(II) Let Y be a compact subset of the plane, and let $R(Y)$ be the uniform closure of the rational functions in $C(Y)$. We regard $R(Y)$ as a uniform algebra on its Shilov boundary, the topological boundary X of Y. Suppose the complement Y^c of Y has a finite number n of components and the interior Y^0 of Y is a nonempty connected set. Let $A = R(Y)|X$ and $\tau(f) = f(s)$ for some s in Y^0. If m is a harmonic measure on X for s then m is a unique logmodular measure of N_τ and $\dim N_\tau = n < \infty$. Then $\mathcal{E} \subset C(X)$ and γ_0 is finite (see [**11**]). (1) of Corollary 2 is essentially a theorem of M. B. Abrahamse [**1**, Theorem 1]. We can show that $N \cap L^\infty$ is dense in N, hence Theorem 3 gives a generalization of the Sarason Theorem but an example of M. B. Abrahamse [**1**] shows that the Sarason Theorem is not true explicitly.

(III) Let $\mathcal{A}$ be the disc algebra and A be a subalgebra of $\mathcal{A}$ which contains the constants and which has finite codimension in $\mathcal{A}$. If $\tau(f) = \tilde{f}(0)$ for f in A and m is the normalized Lebesgue measure on the unit circle ∂U, then it is easy to check that m is a core point of N_τ and $\dim N_\tau = \dim N^\infty = 2 \dim \mathcal{A}/A$. Hence we can apply (1) of Corollary 3 to this example. But γ_0 is infinte [**11**].

Let $\mathcal{H}^\infty$ be the weak-*closure of $\mathcal{A}$ in L^∞, that is, H^∞ the classical Hardy space. Let $s_1, \ldots, s_n$ be distinct points in the open unit disc U, and let $w_1, \ldots, w_n$ be complex numbers. We wish to know a necessary and sufficient condition for that there is a function f in H^∞ satisfying $\|f\|_\infty \leq 1, f(s_i) = w_i$ for $i = 1, \ldots, n, f'(0) = \cdots = f^{(\ell)}(0) = 0$ and $f(a_1) = \cdots = f(a_k)$. Let $A = \{f \in \mathcal{A};\ f'(0) = \cdots = f^{(\ell)}(0) = 0$ and $f(a_1) = \cdots = f(a_k)\}$, then (1) of Corollary 3 gives a solution, but it is very difficult to check the condition.

(IV) The unit polydisc U^n and the torus $(\partial U)^n$ are cartesian products of n copies of U and of ∂U, respectively. $A(U^n)$ is the class of all continuous complex functions on the closure $\bar{U}^n$ of U^n with holomorphic restrictions to U^n is holomorphic there. Let $A = A(U^n)|X$ and $X = (\partial U)^n$. Let m be normalized Lebesgue measure; then m is a representing measure for τ on X where $\tau(f) = f(0)$ and $0 \in U^n$. We can apply Theorem 1, Theorem 2, (1) and (2) of Theorem 3, Proposition 4 and Proposition 5.

The generalization of the Pick-Nevanlinna interpolation theorem was studied by F. Beatrous and J. Burbea [**5**] when E in Theorem 2 is an infinite uniqueness set in U^n. If $E = \{s_1, s_2, \ldots, s_n\}$ is finite set then nothing was known, where $E \subset U^n$. When $M = \{h \in H^2;\ h(s) = 0$ for all $s \in E\}$, M is an invariant subspace in H^2 which has finite codimension and $N = H^2 \ominus M$ is in H^∞. Hence $M^\perp \cap L^\infty$ is dense in $M^\perp$ and $M^\perp \cap L^1$ and (1) and (2) of Theorem 2 in this paper give a generalization of the Pick interpolation theorem. However we can not apply (3) of Theorem 2. For K. Izuchi noted to me privately that γ_0 is infinite because H^∞ is not a uniform algebra in L^∞. If N is finite dimensional then N is in H^∞ (see [**2**]). Hence by Theorem 3 if T is a bounded operator on N which commutes with S_f for any f in H^∞, then $T = S_\phi$ for some ϕ in H^∞. However, there is an operator T on N such that $\|T\| \lneq \|\phi + M \cap L^\infty\|$. For an example due to Korányi and Pukánski [**8**] shows that a function on a 2 point set in the bi-disk that is not the restriction of any function in the unit ball of H^∞. Thus an exact generalization of the Sarason Theorem (and hence a theorem of Nagy and Foiaş) is not true.

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