INACCESSIBLE VARIETIES OF GROUPS

Dedicated to the memory of Hanna Neumann

L. G. KOVÁCS

(Received 2 January 1974)

Communicated by M. F. Newman

1. Discussion

A weak but still unsolved version of the first half of Problem 7 in Hanna Neumann's book [5] asks whether the product $\mathfrak{U}\mathfrak{V}$ of a nontrivial join-irreducible variety $\mathfrak{U}$ and an arbitrary variety $\mathfrak{V}$ is necessarily also join-irreducible. One of the results of this paper is a positive answer provided either $\mathfrak{U}$ is abelian or the infinite-rank free groups of $\mathfrak{U}$ have no nontrivial abelian verbal subgroups. For a general discussion, see [4]; all unexplained notation is as in [5].

Part of this will be derived from the main result: every product of the form $\mathfrak{A}_p\mathfrak{V}$ is *join-inaccessible* in the sense that $\mathfrak{A}_p\mathfrak{V} \leqq \mathfrak{X} \vee \mathfrak{Y}$ implies that $\mathfrak{A}_p\mathfrak{V} \leqq \mathfrak{X}$ or $\mathfrak{A}_p\mathfrak{V} \leqq \mathfrak{Y}$. (By contrast, an example at the end of this paper shows that $\mathfrak{A}_8\mathfrak{A}_3$ is not join-inaccessible.) The rest follows from the fact that if $\mathfrak{U}$ is a nontrivial *commutator-inaccessible* variety (in the sense that $\mathfrak{U} \leqq [\mathfrak{X}, \mathfrak{Y}]$ implies that $\mathfrak{U} \leqq \mathfrak{X}$ or $\mathfrak{U} \leqq \mathfrak{Y}$), then $\mathfrak{U}\mathfrak{V}$ is always commutator-inaccessible: the proof of Theorem 7.1 in Dunwoody [3] requires only trivial modifications to yield this. The connection is via the elementary fact that a nontrivial variety is commutator-inaccessible if and only if it is join-irreducible and its infinite-rank free groups have no nontrivial abelian verbal subgroups. As $\mathfrak{X} \vee \mathfrak{Y} \leqq [\mathfrak{X}, \mathfrak{Y}]$, a commutator-inaccessible variety is also join-inaccessible; but every variety of the form $\mathfrak{A}_p\mathfrak{V}$ (except for the case $\mathfrak{V} = \mathfrak{O}$) is a counter-example to the converse.

Another inaccessibility condition has been around for some time: call $\mathfrak{U}$ *product-inaccessible* if $\mathfrak{U} \leqq \mathfrak{X}\mathfrak{Y}$ implies that $\mathfrak{U} \leqq \mathfrak{X}$ or $\mathfrak{U} \leqq \mathfrak{Y}$. This is the most restrictive of all, for $\mathfrak{X} \vee \mathfrak{Y} \leqq \mathfrak{X}\mathfrak{Y}$ shows that all product-inaccessible varieties are join-inaccessible, and $[\mathfrak{X}, \mathfrak{Y}] \leqq \mathfrak{A}(\mathfrak{X} \vee \mathfrak{Y}) \leqq \mathfrak{A}\mathfrak{X}\mathfrak{Y}$ shows that the nonabelian product-inaccessibles are also commutator-inaccessible. The variety generated by

---

This is a reconstruction, with one new reference, of a lecture given at the Eleventh Summer Research Institute of the Australian Mathematical Society at the University of Sydney in January, 1971.

https://doi.org/10.1017/S1446788700018103 Published online by Cambridge University Press

any verbally simple group must obviously be product-inaccessible; and it follows from 53.55 in [5] that a Cross variety is product-inaccessible if and only if it can be generated by a single finite simple group. (On the other hand, a nontrivial Cross variety is commutator-inaccessible if and only if it is generated by a critical group with nonabelian monolith.)

Product-inaccessible varieties were called irreducible in Brady [1], where it was established that all insoluble just-non-Cross varieties fall into this class. (The equivalence of Brady's definition with the present one may be seen from the following observation. In testing a variety $\mathfrak{U}$ for product-inaccessibility or commutator-inaccessibility, one may assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are subvarieties of $\mathfrak{U}$; for $\mathfrak{U} \leqq \mathfrak{X}\mathfrak{Y}$ implies $\mathfrak{U} \leqq (\mathfrak{U} \wedge \mathfrak{X})(\mathfrak{U} \wedge \mathfrak{Y})$ and $\mathfrak{U} \leqq [\mathfrak{X}, \mathfrak{Y}]$ implies $\mathfrak{U} \leqq [\mathfrak{U} \wedge \mathfrak{X}, \mathfrak{U} \wedge \mathfrak{Y}]$.) It is possible to show also in some other contexts that all 'least criminals' are product-inaccessible. For instance, a minimal pseudo-abelian variety (see the discussion of Problem 5 of Hanna Neumann [5] in [4]) or a minimal torsionfree nonnilpotent locally nilpotent variety (see Question 4 in [4]) would certainly have to be product-inaccessible.

Generally speaking, one might call a variety inaccessible if it cannot be contained in any variety constructed from others unless it is already contained in at least one of the varieties used in the construction. Specific interpretations of inaccessible (we have just seen three) depend, of course, on what methods of construction are permitted, and on what is meant by 'used'. Finite constructions involving joins, products, and commutators (but no 'constants': see below) yield merely conjunctions of some of the three notions of inaccessibility discussed above. Consider the variety generated by all the groups of order dividing a fixed prime power, p^n say: the join of the ascending chain of these, with p fixed and n growing, is known to be the variety $\mathfrak{O}$ of all groups. Using this fact with two different choices of p, one quickly sees that if infinite joins are permitted in the construction (even if only implicitly), only the trivial variety $\mathfrak{E}$ will be inaccessible. Thus it appears that we have dealt with all interesting interpretations of inaccessibility already. In fact, the condition that the infinite-rank free groups of a variety $\mathfrak{U}$ should have no nontrivial abelian verbal subgroups is equivalent to the requirement that $\mathfrak{U} \leqq \mathfrak{A}\mathfrak{B}$ should imply $\mathfrak{U} \leqq \mathfrak{B}$: this may well be considered yet another kind of inaccessibility condition, involving a 'constant' namely $\mathfrak{A}$. There is unlimited scope for variants of this.

To conclude the discussion, let us return to join-inaccessibles. One of the main difficulties is that in testing a variety $\mathfrak{U}$ for join-inaccessibility, apparently one cannot restrict $\mathfrak{X}$ and $\mathfrak{Y}$ in any way (except in relation to each other: by the modular law, one may assume that $\mathfrak{Y} \leqq \mathfrak{X} \vee \mathfrak{U}$). The only nonabelian varieties which I know to be join-inaccessible are those of the form $\mathfrak{A}_p\mathfrak{B}$ or $\mathfrak{A}\mathfrak{B}$ (the latter simply as a consequence of the former, because $\mathfrak{A}\mathfrak{B} = \bigvee \mathfrak{A}_p\mathfrak{B}$ whenever the join is taken as p runs through an infinite set of primes). In particular, there is the

https://doi.org/10.1017/S1446788700018103 Published online by Cambridge University Press

tantalizing question whether the nilpotent just-nonabelian varieties ($\mathfrak{B}_4 \wedge \mathfrak{N}_2$ and the $\mathfrak{B}_p \wedge \mathfrak{N}_2$ for odd primes p) are join-inaccessible. A positive answer would, o course, follow if one could establish that if a finite group is contained in a join $\mathfrak{X} \vee \mathfrak{Y}$ then it is already contained in the variety generated by one finite group from $\mathfrak{X}$ and one from $\mathfrak{Y}$. However, this seems to be a long-standing and quite intractable problem.

2. Proofs

The main result awaiting proof is that every $\mathfrak{A}_p\mathfrak{B}$ is join-inaccessible. The only other claim not reduced to a routine exercise in the discussion is that, for every prime-power $p^n (\neq 1)$ and every $\mathfrak{B}$, the product $\mathfrak{A}_{p^n}\mathfrak{B}$ is join-irreducible. We start by deriving this from the main result.

One preliminary observation is needed: if $\mathfrak{X} \neq \mathfrak{E}$ and $\mathfrak{B} \neq \mathfrak{O}$ then $\mathfrak{X}\mathfrak{B} = \mathfrak{U} \vee \mathfrak{B}$ implies that $\mathfrak{X}\mathfrak{B} = \mathfrak{U}$. This follows from a lemma of Peter M. Neumann, 24.31 in Hanna Neumann's [5] (in her notation, put $\mathfrak{W} = \mathfrak{X}\mathfrak{B}$ and $\mathfrak{Y} = \mathfrak{B}$). Consequently it is sufficient to prove here that if $\mathfrak{A}_{p^n}\mathfrak{B} = \mathfrak{U} \vee \mathfrak{W}$ then $\mathfrak{A}_{p^n}\mathfrak{B}$ is either $\mathfrak{U} \vee \mathfrak{B}$ or $\mathfrak{W} \vee \mathfrak{B}$. To this end, let F be a free group of infinite rank in the variety $\mathfrak{A}_{p^n}\mathfrak{B}$ and let U, V, W be the verbal subgroups of F corresponding to the varieties $\mathfrak{U}, \mathfrak{B}, \mathfrak{W}$, respectively: given that $U \cap W = 1$, we are required to show that either $U \cap V$ or $W \cap V$ is 1. Write V^0 and V_0 for the verbal subgroups of V corresponding to the varieties $\mathfrak{A}_p$ and $\mathfrak{A}_{p^{n-1}}$, respectively; let U^* be the set of those elements of V which have their p^{n-1}th powers in $U \cap V_0$, and define W^* similarly. Note that V^0 is the verbal subgroup of F corresponding to $\mathfrak{A}_p\mathfrak{B}$, and that U^*, W^* are fully-invariant subgroups of F with $U^* \cap W^* = V^0$. The join-irreducibility of $\mathfrak{A}_p\mathfrak{B}$ (which is part of the main result yet to be proved) yields that one of U^* and W^* is V^0; say, $U^* = V^0$. Every element of $U \cap V_0$ is the p^{n-1}th power of some element of U^* while V^0 has exponent p^{n-1}, so $U \cap V_0 = 1$ follows. On the other hand V_0 contains all elements of order p in V, so this means that $U \cap V = 1$ and the proof is complete.

The proof of the main result will be much easier to express in terms of the following rather convenient if unusual notation. If H and K are normal subgroups of a group G with $H \geqq K$ and C/K denotes the centralizer of H/K in G/K, then G/C has an obvious, faithful action on H/K: for Kh in H/K and Cg in G/C, one takes $(Kh)^{Cg} = K(h^g)$. Write H/K *flit* G, or simply H *flit* G if $K = 1$, for the split extension of H/K by G/C acting in this way. Two simple facts concerning this construction will be used repeatedly, without reference. The first is that if H/K as normal subgroup of G/K is similar to H^*/K^* as normal subgroup of G^*/K^* (in the sense of Definition 53.11 of [5]), then H/K *flit* G is isomorphic to H^*/K^* *flit* G^*. The second is that H/K *flit* G is a homomorphic image of a subgroup of the direct square of G/K (the proof of Lemma 2.2 in Brady [2] shows this), and so H/K *flit* G is contained in the variety generated by G/K.

https://doi.org/10.1017/S1446788700018103 Published online by Cambridge University Press

The heart of the proof is the following argument. Let A be a group of prime order p, and F a relatively free group on an infinite, free generating set $\mathfrak{f}$. It will be shown that if N is a nontrivial normal subgroup of the restricted, standard wreath product P of A and F, then N *flit* P has a subgroup isomorphic to P. The first thing is to observe that since N is assumed nontrivial and the base group $A^{(F)}$ of P is its own centralizer in P, there must be at least one nontrivial element, say ϕ, of $A^{(F)}$ in N. The finite support $\sigma(\phi)$ of ϕ lies in a subgroup of F generated by some finite subset $\mathfrak{f}'$ of $\mathfrak{f}$: write G for the subgroup of F generated by the complement $\mathfrak{f}\backslash\mathfrak{f}'$ of $\mathfrak{f}'$ in $\mathfrak{f}$, and note that $G \cong F$. If g and h are distinct elements of G, then the translates $\sigma(\phi)g$ and $\sigma(\phi)h$ are disjoint: for the endomorphism of F which maps $\mathfrak{f}'$ to $\{1\}$ and leaves each element of $\mathfrak{f}\backslash\mathfrak{f}'$ unchanged, maps $\sigma(\phi)g$ and $\sigma(\phi)h$ to the disjoint sets $\{g\}$ and $\{h\}$, respectively. Therefore each element of the subgroup of $A^{(F)}$ generated by the conjugates $\{\phi^h \mid g \neq h \in G\}$ has its support in

$$\bigcup \{\sigma(\phi)h \mid g \neq h \in G\}$$

which is disjoint from the support $\sigma(\phi)g$ of ϕ^g: so this subgroup meets the cycle generated by ϕ^g in $\{1\}$. It follows that the subgroup B of $A^{(F)}$ generated by $\{\phi^g \mid g \in G\}$ is the (restricted) direct product of the cycles generated by the ϕ^g individually, and hence that the subgroup generated by ϕ and G is the wreath product of the p-cycle generated by ϕ and of G. This subgroup GB is therefore isomorphic to P itself. On the other hand, B is contained in N, while G avoids the centralizer C of N in P (for it even avoids the centralizer of ϕ). It follows that GB is isomorphic to the split extension of B by GC/C (with the obvious action) which in turn is embedded in N *flit* P, the split extension of N by P/C.

The preparations for the proof of the main result are now complete. The case $\mathfrak{V} = \mathfrak{E}$ may, of course, be excluded as obvious, so there is no bar to taking F above as a free group of $\mathfrak{V}$. By 22.32 in [5], P generates $\mathfrak{A}_p\mathfrak{V}$. Suppose $\mathfrak{A}_p\mathfrak{V} \leqq \mathfrak{U} \vee \mathfrak{W}$ and $\mathfrak{A}_p\mathfrak{V} \nleqq \mathfrak{U}$ so $P \notin \mathfrak{U}$: the aim is to show that $P \in \mathfrak{W}$. Let X be the absolutely free group of the same infinite rank as F, and U, W the verbal subgroups of X corresponding to $\mathfrak{U}$, $\mathfrak{W}$, respectively. Take a homomorphism τ of X onto P, with kernel T, say; then $P \in \mathfrak{A}_p\mathfrak{V} \leqq \mathfrak{U} \vee \mathfrak{W}$ gives that $T \geqq U \cap W$, and $P \notin \mathfrak{U}$ gives $U\tau \neq 1$. Let $N = U\tau$; then N *flit* P is isomorphic to UT/T *flit* X. Use $U \cap W \leqq U \cap T$, Dedekind's Law, and the operator form of the Isomorphism Theorem, to get the following equalities and X-isomorphisms:

$$\begin{aligned} UT/T &\cong U/(U \cap T) = U/(U \cap T)(U \cap W) = U/(U \cap (U \cap T)W) \\ &\cong U(U \cap T)W/(U \cap T)W = UW/(U \cap T)W. \end{aligned}$$

Consequently

$$N \textit{ flit } P \cong UT/T \textit{ flit } X \cong UW/(U \cap T)W \textit{ flit } X/W.$$

It follows that N *flit* P, and hence also P, is contained in the variety generated by X/W, that is, in $\mathfrak{W}$. This completes the proof.

https://doi.org/10.1017/S1446788700018103 Published online by Cambridge University Press

3. Example

This is to show that $\mathfrak{A}_8\mathfrak{A}_3$ is not join-inaccessible. In fact, $\mathfrak{A}_8\mathfrak{A}_3$ is contained in the join of $(\mathfrak{B}_4 \wedge \mathfrak{N}_3)\mathfrak{A}_3$ and another soluble Cross variety $\mathfrak{W}$ defined below, without being contained in either.

Let H be a free group of rank two of the variety $\mathfrak{A}_2\mathfrak{A}_4 \wedge \mathfrak{N}_3$, freely generated by a and b; and let C be a group of order 3 generated by c. Consider the split extension CH of H by C according to the action $a^c = b$, $b^c = a^{-1}b^{-1}$. Let U and W be the verbal subgroups of CH corresponding to $(\mathfrak{B}_4 \wedge \mathfrak{N}_3)\mathfrak{A}_3$ and to the variety defined by the law

$$[y^8, x^3]^4[y^8, x^3, x^3, [y^8, x^3]],$$

respectively. Using that $(xy)^{-4}x^4y^4$ and $[y^4, x]$ are laws in H, it is not hard to calculate that U is generated by a^4 and b^4 while W is generated by $a^4[b, a, a]$ and $b^4[b, a, b]$, so $U \cap W = 1$. Let $\mathfrak{U}$ and $\mathfrak{W}$ be the varieties generated by CH/U and CH/W, respectively: as CH is a subdirect product of these factor groups, it lies in $\mathfrak{U} \vee \mathfrak{W}$. On the other hand, CH/H' generates $\mathfrak{A}_8\mathfrak{A}_3$ (this is well known, and follows, for instance, from 54.42 of [5]) but neither U nor W is contained in H': so $\mathfrak{A}_8\mathfrak{A}_3$ is contained in the join of $\mathfrak{U}$ and $\mathfrak{W}$ without being contained in either.

References

[1] J. M. Brady, 'On the classification of just-non-Cross varieties of groups', *Bull. Austral. Math. Soc.* 3 (1970), 293–311.

[2] J. M. Brady, 'On soluble just-non-Cross varieties of groups', *Bull. Austral. Math. Soc.* 3 (1970), 313–323.

[3] M. J. Dunwoody, 'On product varieties', *Math. Z.* 104 (1968), 91–97.

[4] L. G. Kovács and M. F. Newman, 'Hanna Neumann's problems', to appear in *Proc. Second Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra*, 1973; Lecture Notes in Mathematics 372, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[5] Hanna Neumann, *Varieties of groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

Australian National University
Canberra

https://doi.org/10.1017/S1446788700018103 Published online by Cambridge University Press