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The Essential Norm of a Bloch-to-Q_p Composition Operator

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Abstract. The Q_p spaces coincide with the Bloch space for $p > 1$ and are subspaces of BMOA for $0 < p \le 1$. We obtain lower and upper estimates for the essential norm of a composition operator from the Bloch space into Q_p, in particular from the Bloch space into BMOA.

1 Introduction

We denote by $H(D)$ the space of holomorphic functions on the unit disc D. The Q_p spaces, which were introduced in [AXZ], consists of functions in $H(D)$ such that

$$\|f\|_{Q_p}^2 := \sup_{a\in D} \int_D |f'(z)|^2 g^p(z,a)\, dA(z) < \infty \quad \text{for } 0 < p < \infty,$$

where dA denotes the Lebesgue area measure on the plane normalized so that $A(D) = 1$ and $g(z,a) := \log\big(|(1-\bar{a}z)/(z-a)|\big)$. The subspace $Q_{p,0}$ of Q_p consists of those functions f such that the above integral tends to zero when $|a| \to 1$.

We have $Q_p = \mathcal{B}$ for $1 < p < \infty$ and $Q_1 = \text{BMOA}$, where $\mathcal{B}$ is the the classical Bloch space and BMOA is the space of analytic functions on ∂D with bounded mean oscillation on the boundary. Hence $\mathcal{B}$ is the space of functions $f \in H(D)$ satisfying

$$\|f\|_{\mathcal{B}} := \sup_{z\in D} (1-|z|^2)\, |f'(z)| < \infty.$$

Also $Q_s \subset Q_p$ if $0 < s < p \le 1$ (see [AXZ]). Further, we have $Q_{1,0} = \text{VMOA}$, where VMOA is the subspace of BMOA consisting of functions of vanishing mean oscillation and for $p > 1$ and $\mathcal{B}_0$ the classical little Bloch space, $Q_{p,0} = \mathcal{B}_0$. Finally, Q_p is a Banach space with norm $\|f\| = |f(0)| + \|f\|_{Q_p}$, and $Q_{p,0}$ is a closed subspace of Q_p. In the definition of Q_p, $g(z,a)$ can be replaced by $h(z,a) := 1 - |(z-a)/(1-\bar{a}z)|^2$, since this results in the same space and an equivalent norm (see [SZ, Lemma 2.2], [ASX, Theorem 2.2]).

Let $\varphi\colon D \to D$ be an analytic self map of the complex unit disc D. Then the equation $C_\varphi f = f \circ \varphi$ defines a composition operator on the space of all holomorphic

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functions on D. Many results have been obtained concerning boundedness and compactness for composition operators on Hardy spaces, weighted Bergman spaces and weighted Bergman spaces of infinite order (see [BDL], [BDLT], [CM], [Sh1], [Sh2]). The investigation of composition operators from the Bloch space into Q_p has only recently taken place. More precisely, in [SZ] W. Smith and R. Zhao have characterized boundedness of $C_\varphi\colon \mathcal{B} \to Q_p$, $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ and $C_\varphi\colon \mathcal{B} \to Q_{p,0}$. They also show that boundedness of $C_\varphi\colon \mathcal{B} \to Q_{p,0}$ is equivalent to the compactness of the operator. When $Q_p = \mathrm{BMOA}$ and $Q_{p,0} = \mathrm{VMOA}$ a similar study has been made by S. Makhmutov and M. Tjani [MT]. Moreover, [MM] contains a characterization of symbols φ inducing compact composition operators on $\mathcal{B}$ and $\mathcal{B}_0$. This result was recently generalized by A. Montes-Rodriquez [MR] who computed the essential norm of C_φ on Bloch spaces. The main aim of this paper is to give lower and upper estimates for the essential norm of a composition operator from $\mathcal{B}$ into Q_p. Using this result we obtain a function theoretic characterization of the compactness of $C_\varphi\colon \mathcal{B} \to Q_p$ for $p \leq 1$. This answers a question of W. Smith and R. Zhao [SZ]. They only provide a sufficient condition in [SZ, Proposition 6.5]. The characterization of compactness was also independently obtained by J. Xiao in his recently published work [X, Theorem 1.2]. Our work also extends the result of A. Montes-Rodriquez, since for $p > 1$ the bound for the essential norm of C_φ in Theorem 6 is equivalent to the essential norm of C_φ in [MR, Theorem 2.1].

A map $T \in \mathcal{L}(X, Y)$ from the Banach space X into a Banach space Y is called *compact* (weakly compact), if it maps the closed unit ball of X onto a relatively compact (a relatively weakly compact) set in Y. The essential norm of a $T \in \mathcal{L}(X, Y)$ is defined by

$$\|T\|_e = \inf\{\|T - S\| : S \text{ is compact}\}.$$

Since $\|T\|_e = 0$ if and only if T is compact, estimates on $\|T\|_e$ give conditions for T to be compact.

For two quantities A and B we write $A \sim B$ if there exist strictly positive constants C and c such that $cB \leq A \leq CB$.

2 Results

First of all, we show that a general argument can be applied to composition operators from $\mathcal{B}$ into $Q_{p,0}$ showing that boundedness coincides with compactness. W. Smith and R. Zhao [SZ] obtained this result by a direct proof.

Proposition 1 *Let $0 < p < \infty$. Then*

(a) *$C_\varphi\colon \mathcal{B} \to Q_{p,0}$ is bounded if and only if $C_\varphi\colon \mathcal{B} \to Q_{p,0}$ is compact.*
(b) *$C_\varphi\colon \mathcal{B} \to Q_p$ is weakly compact if and only if $C_\varphi\colon \mathcal{B} \to Q_p$ is compact.*

Proof of (a) and (b) Since $Q_{p,0}$ is separable, it does not contain a copy of l^∞. Further, $\mathcal{B}$ is isomorphic to l^∞. Then $C_\varphi\colon l^\infty \to Q_{p,0}$ is weakly compact by a Theorem of Rosenthal (see [R]). Thus both in (a) and (b) we can consider $C_\varphi\colon \mathcal{B} \to Q_p$ as a weakly compact operator.

Let us now show that the closed unit ball of Q_p is compact for the compact open topology. We will work with the equivalent norm on Q_p, defined by replacing $g^p(z, a)$

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by $h^p(z,a) := \big(1 - |(z-a)/(1-\bar{a}z)|^2\big)^p$. Indeed, since the inclusion $Q_p \subset \mathcal{B}$ is continuous, any f in the closed unit ball of Q_p has the following growth:

$$|f(z)| \leq C \log(2/1-|z|), \quad z \in D.$$

Therefore we conclude that the closed unit ball of Q_p is a normal family by Montel's theorem. Moreover, let (f_n) be a sequence in Q_p with $\|f_n\| \leq 1$ such that $f_n \to f$ with respect to the compact-open topology. Let $a \in D$ be fixed. Then

$$\int_0^{2\pi} |f_n'(re^{i\theta})|^2 h^p(re^{i\theta}, a)\, d\theta \to \int_0^{2\pi} |f'(re^{i\theta})|^2 h^p(re^{i\theta}, a)\, d\theta$$

for all $0 < r < 1$. Now, by Fatou's lemma,

$$\begin{aligned}\int_0^1 r\,dr \int_0^{2\pi} &|f'(re^{i\theta})|^2 h^p(re^{i\theta}, a)\frac{d\theta}{\pi} \\ &\leq \liminf_n \int_0^1 r\,dr \int_0^{2\pi} |f_n'(re^{i\theta})|^2 h^p(re^{i\theta}, a)\frac{d\theta}{\pi} \\ &\leq \liminf_n \|f_n\|_{Q_p}^2 = \liminf_n \big(\|f_n\| - |f_n(0)|\big)^2 \leq \big(1 - |f(0)|\big)^2.\end{aligned}$$

Consequently, $\|f\| \leq 1$ and the closed unit ball of Q_p is closed with respect to the compact-open topology.

Therefore, by Dixmier-Ng theorem [N], there exists a Banach space P_p which is a predual of Q_p. The space P_p is defined as the subspace of Q_p^* of those functionals which are compact-open continuous when restricted to the unit ball of Q_p or equivalently to the bounded subsets. We show that $C_\varphi\colon \mathcal{B} \to Q_p$ is w^*-w^* continuous. Indeed, let $u \in P_p$. Since the predual of $\mathcal{B}$ is separable, we have by Corollary V.12.8 in [C] that $u \circ C_\varphi$ is w^* continuous if and only if $u \circ C_\varphi$ is w^* sequentially continuous. Therefore let $f_n \to f$ in the w^*-topology of $\mathcal{B}$. Then the sequence $(f_n - f)_n$ in $\mathcal{B}$ is norm bounded and w^* convergence in $\mathcal{B}$ implies pointwise convergence, so $f_n \to f$ in the compact-open topology. Hence $C_\varphi(f_n) \to C_\varphi(f)$ in the compact-open topology and consequently $\lim_{n\to\infty} \big|u\big(C_\phi(f_n - f)\big)\big| = 0$.

Since the composition operator is w^*-w^*-continuous and $Q_p = (P_p)^*$, there exists a continuous operator $T\colon P_p \to l_1$ such that $T^* = C_\varphi$. Hence T is compact and consequently also C_φ is compact. ■

Corollary 2 *Q_p is a dual space.*

This result is contained in the proof above.

For $p \in (0,\infty)$, boundedness of $C_\varphi\colon \mathcal{B} \to Q_p$ is characterized in [SZ] by the condition

$$\sup_{a\in D} \int_D \frac{|\varphi'(z)|^2}{\big(1-|\varphi(z)|^2\big)^2} g^p(z,a)\, dA(z) < \infty.$$

Further, W. Smith and R. Zhao showed that $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ is bounded if and only if $C_\varphi\colon \mathcal{B} \to Q_p$ is bounded and $\varphi \in Q_{p,0}$.

Example 3 There exists an analytic univalent map $\varphi\colon D \to D$ such that $C_\varphi\colon \mathcal{B} \to Q_p$ and $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ are bounded but non-compact for all $p \in (0,\infty)$.

Proof By Example 4.3 in [SZ] we can find an analytic univalent self-map φ of D such that $C_\varphi\colon \mathcal{B} \to Q_p$ and $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ are bounded but $C_\varphi(\mathcal{B}) \not\subset Q_{p,0}$ for all $p \in (0,\infty)$.

Let us assume that $C_\varphi\colon \mathcal{B} \to Q_p$ is compact. Then $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ is also compact. Since $\mathcal{B}_0^{**} = \mathcal{B}$ and $C_\varphi(\mathcal{B}_0^{**}) \subset Q_{p,0}$ by weak compactness, we have a contradiction. ∎

It is well known that under the usual integral pairing the dual of $\mathcal{B}_0$ is isomorphic to the Bergman space A^1 of analytic functions f on the unit disc such that

$$\int_D |f(z)|\, dA(z) < \infty.$$

We will need such a result with another natural pairing.

Lemma 4 *The map $f \mapsto \langle f, \cdot\rangle_{\mathcal{B}}$ defines an isomorphism from $\mathcal{B}$ onto the dual of $A^1 \oplus \mathbb{C}$. Here*

$$\langle f, h\rangle_{\mathcal{B}} := \int_D f'(z) g(\bar{z})(1-|z|^2)\, dA(z) + c f(0)$$

for $f \in \mathcal{B}$ and $h = (g,c) \in A^1 \oplus \mathbb{C}$.

Moreover, the map $h \mapsto \langle \cdot, h\rangle_{\mathcal{B}}$ defines an isomorphism from $A^1 \oplus \mathbb{C}$ onto the dual of $\mathcal{B}_0$.

Proof Let us define the space

$$A_w^\infty := \left\{ f \in H(D) : \|f\|_w := \sup_{z\in D} |f(z)|(1-|z|^2) < \infty \right\},$$

and its closed subspace A_w^0 consisting of functions f with $\lim_{|z|\to 1} |f(z)|(1-|z|^2) = 0$ (uniform limit). These are the same spaces as $A_\infty(\varphi)$ and $A_0(\varphi)$ of [SW] with $\varphi(z) := (1-|z|^2)$. Moreover, choosing $\psi(z) := 1$, the pair $\{\varphi, \psi\}$ is a *normal pair of weight functions* in the sense of [SW, p. 291]. Hence, Theorem 2 of that paper applies and we obtain the following results:

Lemma *The map $f \mapsto \langle f, \cdot\rangle_w$ defines an isomorphism from A_w^∞ onto the dual of A^1, where*

$$\langle f, g\rangle_w := \int_D f(z) g(\bar{z})(1-|z|^2)\, dA(z)$$

for $f \in A_w^\infty$ and $g \in A^1$. Moreover, the map $g \mapsto \langle \cdot, g\rangle_w$ defines an isomorphism from A^1 onto the dual of A_w^0.

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It is now elementary that the dualities

$$(A^0_w \oplus \mathbb{C})^* = (A^0_w)^* \oplus \mathbb{C}^* = A^1 \oplus \mathbb{C} \quad \text{and}$$
$$(A^1 \oplus \mathbb{C})^* = (A^1)^* \oplus \mathbb{C}^* = A^\infty_w \oplus \mathbb{C}$$

hold with respect to the pairing

$$\langle y, h\rangle_\oplus := \langle f, g\rangle_w + bc,$$

where $y = (f, b) \in A^0_w \oplus \mathbb{C}$ or $A^\infty_w \oplus \mathbb{C}$ and $h = (g, c) \in A^1 \oplus \mathbb{C}$.

On the other hand, taking into account the definitions of the norms of the relevant spaces, the map $I\colon f \mapsto \big(f', f(0)\big)$ is a linear isometric bijection

$$\mathcal{B} \to A^\infty_w \oplus \mathbb{C} \quad \text{and} \quad \mathcal{B}_0 \to A^0_w \oplus \mathbb{C},$$

the direct sums endowed with the sum-norm. So, Lemma 4 follows from the above remarks and

$$\langle f, h\rangle_{\mathcal{B}} = \langle If, h\rangle_\oplus,$$

valid for $f \in \mathcal{B}$ and $h \in A^1 \oplus \mathbb{C}$. ■

Next we introduce the test functions that will be used in the proof of our main result. Let $\alpha_m \in (1/2, 1)$ be such that $\alpha_m \to 1$ when $m \to \infty$ and let

$$f_{n,m,\theta}(z) := \frac{1}{\alpha_m}\sum_{k=0}^{\infty} \frac{2^k}{2^k + 2^n} z^{2^k+2^n}(\alpha_m e^{i\theta})^{2^k}, \quad n, m \in \mathbb{N},\ \theta \in [0, 2\pi[.$$

Then $f_{n,m,\theta} \in \mathcal{B}_0$ and $\|f_{n,m,\theta}\| \le C$, where C is a constant independent of n, m and θ (see [P], [Z, p. 101]).

Lemma 5 *For every $u \in \mathcal{B}^*_0$ we have*

$$\lim_{n\to\infty} \sup_{m,\theta} |u(f_{n,m,\theta})| = 0.$$

Proof For given $u \in \mathcal{B}^*_0$, let $h = (g, c) \in A^1 \oplus \mathbb{C}$ be such that

$$\begin{aligned}\sup_{m,\theta} |u(f_{n,m,\theta})| &= \sup_{m,\theta} \big|\langle f_{n,m,\theta}, h\rangle_{\mathcal{B}}\big| \le \sup_{m,\theta} \int_D |f'_{n,m,\theta}(z) g(\bar z)|(1 - |z|^2)\, dA(z)\\ &\le \sup_{m,\theta} \int_D 2|z|^{2^n-1}\Big|\sum_{k=0}^{\infty} 2^k(\alpha_m e^{i\theta})^{2^k} z^{2^k}\Big|\ |g(\bar z)|(1 - |z|^2)\, dA(z).\end{aligned}$$

By [JR, p. 436]

$$\Big|\sum_{k=0}^{\infty} 2^k(\alpha_m e^{i\theta})^{2^k} z^{2^k}\Big| \le \frac{\text{const.}}{1 - |z|},$$

so

$$\sup_{m,\theta} |u(f_{n,m,\theta})| \leq \text{const.} \int_D |z|^{2^n-1} |g(z)| \, dA(z).$$

Since $g \in A^1$, the Lebesgue Dominated Convergence Theorem gives that the last integral converges to zero when $n \to \infty$. ■

Theorem 6 *Suppose that C_φ defines a bounded operator from $\mathcal{B}$ into Q_p or from $\mathcal{B}_0$ into $Q_{p,0}$, where $0 < p < \infty$. Then we have*

$$(*) \qquad \|C_\varphi\|_e^2 \sim \lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\big(1-|\varphi(z)|^2\big)^2} g^p(z,a) \, dA(z).$$

In particular, C_φ is compact if and only if

$$\lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\big(1-|\varphi(z)|^2\big)^2} g^p(z,a) \, dA(z) = 0.$$

The formula $(*)$ with $p = 1$ holds especially for a bounded $C_\varphi \colon \mathcal{B} \to$ BMOA.

Proof We first show the lower estimate of the essential norm. We set $g_{n,m,\theta}(z) := f_{n,m,\theta}(z)/C$, $z \in D$, where C is as in the definition of $f_{n,m,\theta}$. Then $g_{n,m,\theta}$ is contained in the closed unit ball of $\mathcal{B}_0$. By Lemma 5 we obtain for each u in $\mathcal{B}_0^*$ that

$$(1) \qquad \lim_{n\to\infty} \sup_{m,\theta} |u(g_{n,m,\theta})| = 0.$$

For any compact operator $T \colon \mathcal{B} \to Q_p$ or $T \colon \mathcal{B}_0 \to Q_{p,0}$, we have that

$$\lim_{n\to\infty} \sup_{m,\theta} \|T g_{n,m,\theta}\| = 0.$$

Indeed, suppose that this is not true. Then there exists a subsequence $(n_j)_{j=1}^\infty$ such that for each j we can find m_j and θ_j and

$$(2) \qquad \|T g_{n_j,m_j,\theta_j}\| \geq c > 0 \quad \text{for all } j.$$

Because of (1) we have that $g_{n_j,m_j,\theta_j} \to 0$ weakly in $\mathcal{B}_0$ when $j \to \infty$. But since T is compact we obtain a contradiction with (2).

Hence, if T is an arbitrary compact operator,

$$\begin{aligned} \|C_\varphi - T\| &\geq \limsup_{n\to\infty} \sup_{m,\theta} \|(C_\varphi - T) g_{n,m,\theta}\| \\ &\geq \limsup_{n\to\infty} \sup_{m,\theta} \big(\|C_\varphi g_{n,m,\theta}\| - \|T g_{n,m,\theta}\|\big) = \limsup_{n\to\infty} \sup_{m,\theta} \|C_\varphi g_{n,m,\theta}\|. \end{aligned}$$

Thus we obtain

$$\|C_\varphi\|_e^2 \geq \frac{1}{C^2} \limsup_{n\to\infty} \sup_{m,\theta} \sup_{a\in D} \int_D \big| f'_{n,m,\theta}\big(\varphi(z)\big) \big|^2 |\varphi'(z)|^2 g^p(z,a) \, dA(z).$$

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Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$C^2\|C_\varphi\|_e^2 + \varepsilon \geq \sup_{a\in D} \int_D |\varphi(z)|^{2^{n+1}} \Big| \sum_{k=0}^{\infty} 2^k \big(\alpha_m \varphi(z)\big)^{2^k-1} (e^{i\theta})^{2^k} \Big|^2 |\varphi'(z)|^2 g^p(z,a)\, dA(z)$$

for all θ and all m. Let $a \in D$ be fixed. Integrating with respect to θ and using Fubini's theorem, we obtain

$$\begin{aligned} C^2\|C_\varphi\|_e^2 + \varepsilon & \\ &\geq \frac{1}{2\pi} \int_D |\varphi(z)|^{2^{n+1}} \Big(\int_0^{2\pi} \Big| \sum_{k=0}^{\infty} 2^k \big(\alpha_m \varphi(z)\big)^{2^k-1} (e^{i\theta})^{2^k} \Big|^2 d\theta \Big) |\varphi'(z)|^2 g^p(z,a)\, dA(z) \\ &= \int_D |\varphi(z)|^{2^{n+1}} \Big(\sum_{k=0}^{\infty} 2^{2k} |\alpha_m \varphi(z)|^{2(2^k-1)} \Big) |\varphi'(z)|^2 g^p(z,a)\, dA(z). \end{aligned}$$

The equality follows by Parseval's formula. By [JR, p. 437],

$$\sum_{k=0}^{\infty} 2^{2k} |\alpha_m \varphi(z)|^{2(2^k-1)} \geq \frac{1}{2} \frac{1}{\big(1 - |\alpha_m \varphi(z)|^2\big)^2}$$

for all $z \in D$ with $|\varphi(z)| > 0$. Thus by Fatou's lemma,

$$\begin{aligned} 2(C^2\|C_\varphi\|_e^2 + \varepsilon) &\geq \liminf_{m\to\infty} \int_D |\varphi(z)|^{2^{n+1}} \frac{|\varphi'(z)|^2}{\big(1 - |\alpha_m \varphi(z)|^2\big)^2} g^p(z,a)\, dA(z) \\ &\geq \int_D |\varphi(z)|^{2^{n+1}} \frac{|\varphi'(z)|^2}{\big(1 - |\varphi(z)|^2\big)^2} g^p(z,a)\, dA(z). \end{aligned}$$

Since $a \in D$ was arbitrary, we obtain that

$$2(C^2\|C_\varphi\|_e^2 + \varepsilon) \geq \frac{1}{e} \lim_{n\to\infty} \sup_{a\in D} \int_{\{z:|\varphi(z)|>1-2^{-(n+1)}\}} \frac{|\varphi'(z)|^2}{\big(1 - |\varphi(z)|^2\big)^2} g^p(z,a)\, dA(z).$$

Thus we have

$$2e(C^2\|C_\varphi\|_e^2 + \varepsilon) \geq \lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\big(1 - |\varphi(z)|^2\big)^2} g^p(z,a)\, dA(z).$$

Since ε was arbitrary, the lower estimate follows.

We now calculate the upper estimate. To do this, we consider a sequence of compact linear operators $C_k \colon \mathcal{B} \to \mathcal{B}$ or $C_k \colon \mathcal{B}_0 \to \mathcal{B}_0$, $k \in \mathbb{N}$, defined by $C_k f(z) = f(\frac{k}{k+1} z)$, $z \in D$. Let $\psi_k(z) := \frac{k}{k+1} z$, so that $C_k f = f \circ \psi_k$. For $k \in \mathbb{N}$ fixed we have

$$\begin{aligned} \|C_\varphi\|_e^2 &\leq \|C_\varphi - C_\varphi C_k\|^2 = \|C_\varphi(\mathrm{Id} - C_k)\|^2 \\ &= \sup_{\|f\|\leq 1} \sup_{a\in D} \int_D \big| (f - f\circ\psi_k)'\big(\varphi(z)\big) \big|^2 |\varphi'(z)|^2 g^p(z,a)\, dA(z). \end{aligned} \tag{3}$$

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Let $0 < r < 1$ be fixed. Then (3) is less than

$$\sup_{\|f\|\le 1}\sup_{a\in D}\int_{\{z:|\varphi(z)|>r\}}\Big|\big(f-f\circ\psi_k)'(\varphi(z))\Big|^2|\varphi'(z)|^2g^p(z,a)\,dA(z)$$
$$+\sup_{\|f\|\le 1}\sup_{a\in D}\int_{\{z:|\varphi(z)|\le r\}}\Big|(f-f\circ\psi_k)'(\varphi(z))\Big|^2|\varphi'(z)|^2g^p(z,a)\,dA(z)=:I_k+J_k.$$

To estimate the first term I_k observe that, for $\|f\|_{\mathcal{B}}\le 1$ and $z\in D$,

$$|f'(z)|\le\frac{1}{1-|z|^2}.$$

Since $\|f\circ\psi_k\|_{\mathcal{B}}\le\|f\|_{\mathcal{B}}$, we obtain

$$I_k\le\sup_{a\in D}\int_{\{z:|\varphi(z)|>r\}}\Big(\frac{2}{1-|\varphi(z)|^2}\Big)^2|\varphi'(z)|^2g^p(z,a)\,dA(z).$$

For the second term J_k, since $C_\varphi z=\varphi\in Q_p$ we get that

$$M:=\sup_{a\in D}\int_D|\varphi'(z)|^2g^p(z,a)\,dA(z)<\infty.$$

Thus,

$$J_k\le M\sup_{\|f\|\le 1}\sup_{\{z:|\varphi(z)|\le r\}}\Big|(f-f\circ\psi_k)'(\varphi(z))\Big|^2.$$

The sequence of operators $(\mathrm{Id}-C_k)_k$ satisfies $\lim_{k\to\infty}(\mathrm{Id}-C_k)g=0$ for each g in $H(D)$, and the space $H(D)$ endowed with the compact open topology co is a Fréchet space. Further, $D\colon\big(H(D),\mathrm{co}\big)\to\big(H(D),\mathrm{co}\big)$ defined by $Df=f'$ is a continuous linear operator. Therefore, by the Banach-Steinhaus theorem, the sequence $D\circ(\mathrm{Id}-C_k)_k$ converges to zero uniformly on the compact subsets of $\big(H(D),\mathrm{co}\big)$. Since the closed unit ball of $\mathcal{B}$ is a compact subset of $\big(H(D),\mathrm{co}\big)$ we conclude that

$$\lim_{k\to\infty}\sup_{\|f\|\le 1}\sup_{\{z:|\varphi(z)|\le r\}}\Big|(f-f\circ\psi_k)'(\varphi(z))\Big|=0.$$

Consequently,

$$\|C_\varphi\|_e^2\le\limsup_{k\to\infty}\|C_\varphi-C_\varphi C_k\|^2\le\limsup_{k\to\infty}I_k+\limsup_{k\to\infty}J_k$$
$$\le 4\sup_{a\in D}\int_{\{z:|\varphi(z)|>r\}}\frac{|\varphi'(z)|^2}{\big(1-|\varphi(z)|^2\big)^2}g^p(z,a)\,dA(z),$$

and the proof is complete. ■

Lemma 7

(a) *Assume that C_φ defines a bounded operator from $\mathcal{B}$ into Q_p, where $0 < p < \infty$. Then*

$$\limsup_{|a|\to 1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \geq \lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z).$$

(b) *Suppose that C_φ defines a bounded operator from $\mathcal{B}_0$ into $Q_{p,0}$, where $0 < p < \infty$. Then*

$$\lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) = \limsup_{|a|\to 1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z).$$

Proof (a) Let $0 < \delta < 1$ be fixed. By Theorem 1.8 in [SZ] boundedness of C_φ implies that

$$\sup_{a\in D} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) < \infty.$$

By Lemma 2.3 in [SZ] this integral is a continuous function at any $a \in D$. Thus it follows by compactness of $\{a : |a| \leq 1-\delta\}$ that

$$\lim_{r\to 1} \sup_{|a|\leq 1-\delta} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) = 0.$$

For any $r \in (0,1)$,

$$\begin{aligned}\sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) &\leq \sup_{1-\delta<|a|<1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \\ &\quad + \sup_{|a|\leq 1-\delta} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z).\end{aligned}$$

By letting $r \to 1$ in the above inequality, we get

$$\lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \leq \sup_{1-\delta<|a|<1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z),$$

which is valid for all $\delta \in (0,1)$. Thus the result follows as $\delta \to 0$.

(b) Since $\varphi \in Q_{p,0}$,

$$\lim_{|a|\to 1} \int_D |\varphi'(z)|^2 h^p(z,a)\, dA(z) = 0.$$

Let $0 < r < 1$ be fixed. Then

$$\lim_{|a|\to 1} \int_{\{z:|\varphi(z)|\le r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \le (1-r^2)^{-2} \lim_{|a|\to 1} \int_D |\varphi'(z)|^2 h^p(z,a)\, dA(z) = 0.$$

Thus

$$\begin{aligned}\lim_{r\to 1} \sup_{a\in D} \int_{\{z:|\varphi(z)|>r\}} &\frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \\ &\ge \lim_{r\to 1} \limsup_{|a|\to 1} \int_{\{z:|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) \\ &= \limsup_{|a|\to 1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z).\end{aligned}$$

Since part (a) is also valid for bounded composition operators from $\mathcal{B}_0$ into $Q_{p,0}$, the statement follows. ■

Corollary 8 *Assume that C_φ defines a bounded operator from $\mathcal{B}$ into Q_p or from $\mathcal{B}_0$ into $Q_{p,0}$, where $0 < p < \infty$. Then C_φ is compact if*

$$\lim_{|a|\to 1} \int_D \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} h^p(z,a)\, dA(z) = 0. \tag{4}$$

As a partial converse, if $C_\varphi\colon \mathcal{B}_0 \to Q_{p,0}$ is compact, then (4) holds.

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