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NOTES

A note on variable markup, knowledge spillover, and multiple steady states in the variety expansion model*

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Abstract

This paper incorporates the variable elasticity of substitution preferences in the variety expansion model developed by Grossman and Helpman [(1991) *Innovation and Growth in the Global Economy*, MIT Press]. There exists a balanced growth path when the elasticity of substitution is constant with knowledge spillover. When the elasticity of substitution is variable and the knowledge spillover is sufficiently small, a unique and stable steady state exists. When the knowledge spillover is sufficiently large, the steady state is unique and unstable. When the size of the knowledge externality is moderate, multiple equilibria exist.

Keywords: Variable markup; knowledge spillover; multiple steady states; variety expansion model

1. Introduction

According to Kaldor (1961), the per-capita output growth rate is constant at the equilibrium. To clarify this fact, the variety expansion model is developed by Grossman and Helpman (1991), which succeeds in endogenizing the growth rate. To obtain a balanced growth path, the variety expansion model assumes that there exists a knowledge spillover in research and development (R&D) activities and that there is a constant elasticity of substitution preference. Even though the constancy of elasticity of substitution between goods is widely assumed, the markup rates of monopolistic competitive firms are constant and are independent of their output level due to this assumption. However, recent empirical studies find that firms' markup rate depends on their level of output, with larger firms setting a higher markup rate.[1] Following these empirical studies, I extend the Grossman and Helpman (1991) model to consider the variable elasticity of substitution (VES) preferences, and hence, the existence of the balanced growth path is investigated.

Some studies examine the influence of the VES between goods on economic activities. As stated by Dixit and Stiglitz (1977) and Zhelobodko et al. (2012), if the elasticity of substitution between goods varies, the markup rate of firms depends on the consumption per capita in monopolistic competition models. When the elasticity of substitution increases (decreases) in the consumption per capita, the markup rates of monopolistic competitive firms also increase (decrease). Because an increase in the number of firms reduces the consumption per capita, the entry of new firms lowers (raises) the elasticity of substitution between goods and their markup rates.

Considering the knowledge spillover and the VES, three effects generate an increase in the number of firms as the economy grows. The first effect is the *spillover effect* as the number of firms generates an externality that reduces the cost of R&D investment, which encourages additional

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firms to enter the market. The second effect is the *consumption effect* as an increase in the number of firms reduces the profits through decreased the consumption per capita and increased market competition. The last effect is the *markup effect* as an increase in the number of firms reduces the markup rates and profit level. According to empirical studies of Dhyne et al. (2011), De Loecker et al. (2012), and De Loecker and Warzynski (2012), an increase in the output level of firms increases the markup rates, and an increase in the number of firms decreases the markup rates and profit level. Thus, the *markup effect* has a negative effect on the profit level.

In this study, I incorporate the VES in the variety expansion model developed by Grossman and Helpman (1991). When firms utilize the R&D investment to create a new product, they acquire patents that allow them to secure monopolistic profits. Due to the VES, as more firms enter the market, both the profit and markup of firms change. Further, in this study, I assume that the markup increases in the output level following the results of Dhyne et al. (2011), De Loecker et al. (2012), and De Loecker and Warzynski (2012). Then, an increase in the number of firms decreases the output level and the markup rates. Additionally, this study assumes a knowledge spillover exists in R&D activities. This spillover implies that an increase in the number of R&D investments reduces the costs of R&D investment.

Three main results are obtained from this study. The first outcome is to show the conditions for the balanced growth path when the elasticity of substitution is variable.[2] If a balanced growth path does exist, the number of firms grows at a constant rate. Also, the return on the R&D investment and labor supply in the R&D sector must be constant. When the elasticity of substitution is constant because the *spillover effect* is equal to the *consumption effect*, in tandem with the absence of the *markup effect*, the return on the R&D investment and the labor supply engaged in the R&D sector is constant, and there exists a balanced growth path. Nevertheless, an increase in the number of firms decreases the return on the R&D investment due to the VES, and the *spillover effect* is not equal to the sum of the *competition effect* and the *markup effect*. Accordingly, when the sum of the three effects is constant, the economy grows at a constant rate.[3]

The second finding of this study is that multiple equilibria may exist. When the knowledge spillover is sufficiently small, a unique and stable steady state is achieved. As the sum of the *consumption effect* and the *markup effect* is larger than the spillover effect, the R&D investment level becomes zero, and the economy goes into a unique steady state. Contrarily, when the knowledge spillover is sufficiently large, there exists a unique and unstable steady state since the *spillover effect* overcomes the sum of the *consumption effect* and the *markup effect*. Once the firms utilize the R&D investment in the steady state, R&D costs are drastically reduced and the R&D investment is intensively utilized. Subsequently, this steady state becomes unstable. When the knowledge spillover is at the intermediate value, the influence of an increase in the number of firms on the return on the R&D investment is ambiguous. Then, there exist multiple equilibria. Some steady states are stable while others are unstable.

Lastly, I investigate the first-best equilibrium. As Grossman and Helpman (1991) and Dhingra and Morrow (2019) highlight, the first-best equilibrium coincides with the competitive equilibrium when the elasticity of substitution is constant, and there is no knowledge spillover. However, when the elasticity of substitution is variable, the allocation of first-best equilibrium does not coincide with that of the competitive equilibrium.

Many previous papers study the impacts of VES on economic consequences. Behrens and Murata (2007) and Zhelobodko et al (2012) reveal the impact of an increase in the population size on the markup rate through pro-competitive effects, where the equilibrium prices are decreasing in the number of competing firms. A series of studies by Bertoletti and Etro (2016, 2017, 2021, 2022) analyze the effects of productivity, population size, and income on markup in a general utility function. Bilbiie et al. (2012, 2019) investigate macroeconomic fluctuations when the number of firms affects markups. Kimball (1995) and Smets and Wouters (2007) focus on the elasticity of demand for intermediate goods by firms, rather than the elasticity of demand by consumers, which is variable.

Some studies extend the variety expansion model with variable elasticity. The analysis of Boucekkine et al. (2017) and Latzer et al. (2019) is closely related to this study, as they verify the existence of a balanced growth path even if the elasticity of substitution is not constant. Boucekkine et al. (2017) make a special assumption regarding the indirect utility function, whereas Latzer et al. (2019) assume the knowledge spillover on the R&D investment and the marginal costs of production to obtain a balanced growth path. Besides, in Latzer et al. (2019), an increase in the number of firms reduces both the R&D and the marginal costs. Then, the consumption per capita and the return on the R&D investment are constant throughout time.

2. The model

We develop a dynamic general equilibrium model with VES between goods. Our model has a similar structure to that of Grossman and Helpman (1991, Ch. 3) and Acemoglu (2008, Ch.13). In this model, the population size is L, which is constant throughout time. Individuals have identical preferences:

$$W_0 = \int_0^\infty e^{-\rho t} U(\tilde{u}_t) dt, \quad 0 < \rho < 1, \tag{1}$$

where ρ is the constant subjective discount rate and $U(\tilde{u}_t)$ is the instantaneous utility per person at time t. I assume that $U(0) = 0$, $U'(\tilde{u}_t) > 0$, and $U''(\tilde{u}_t) < 0$. $\tilde{u}_t$ is given by

$$\tilde{u}_t = \sum_{i=1}^{n_t} u(q_t(i)), \tag{2}$$

where $q_t(i)$ represents consumption of differentiated goods i at time t and n_t denotes the number of goods. $u(q_t(i))$ is strictly increasing, concave, and four times continuously differentiable. Furthermore, I assume that the Inada condition $lim_{q_t(i)\to 0} u'(q_t(i)) = +\infty$ is satisfied and $u(0) = 0$.[4] The utility maximization problem of an individual can be solved in two steps. The first step is to address the static optimization problem, wherein the budget constraint at time t is

$$\sum_{i=1}^{n_t} p_t(i) q_t(i) = E_t, \tag{3}$$

where $p_t(i)$ is the price of product i and E_t is the total expenditure at time t. From the first-order condition, we derive the following inverse demand function of product i at time t:

$$p_t(i) = \frac{u'(q_t(i))E_t}{\sum_{i=1}^{n_t} u'(q_t(i))q_t(i)}. \tag{4}$$

Totally differentiating (4), the relationship between the demand of product i and the expenditure level is (See Appendix for proof)

$$\frac{\partial q_t(i)}{\partial E_t} = \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{(-\beta_i)\left(1 - \sum_{j=1}^{n_t} \frac{\alpha_j}{\beta_j}\right) E_t}, \tag{5}$$

where

$$\alpha_j = u''(q_t(j))q_t(j) + u'(q_t(j)),$$

$$\beta_j = \frac{u''(q_t(j))}{u'(q_t(j))} \sum_{k=1}^{n_t} u'(q_t(k)) q_t(k).$$

The second step is to solve the intertemporal optimization problem. The intertemporal budget constraint is given by

$$\int_0^\infty e^{-R(t)t} E_t dt = H(0) + \int_0^\infty e^{-R(t)t} w_t dt,$$

where

$$R(t) = \frac{1}{t} \int_0^t r_\omega d\omega.$$

r_ω is the interest rate at time ω, w is the wage rate, and $H(0)$ is the asset holding at time 0. Solving the intertemporal optimization problem, we can obtain the following equation: (See Appendix for proof)

$$e^{-\rho t} \frac{A_t}{E_t} = e^{-R(t)t}, \tag{6}$$

where

$$A_t = U'(\tilde{u}_t) \sum_{j=1}^{n_t} u'(q_t(j)) q_t(j) \sum_{k=1}^{n_t} \frac{u'(q_t(k))}{(-\beta_j)\left(1 - \sum_{j=1}^{n_t} \frac{\alpha_j}{\beta_j}\right)}.$$

A_t represents the marginal utility of the total expenditure at time t. Taking the logarithm and differentiating t and setting $E_t = 1$, I can obtain the following equation:

$$r_t = \rho - \frac{\dot{A}_t}{A_t}, \tag{7}$$

where $\dot{A}_t = \frac{dA_t}{dt}$. When the marginal utility of the total expenditure increases in t, individuals reduce the consumption and increase saving at time t. Then, the interest rate is less than the subjective discount rate.

The goods market is monopolistically competitive. To produce one unit of product, firms hire ψ unit of labor. Therefore, the profit of firm i is

$$\pi_t(i) = p_t(i) q_t(i) L - w_t \psi q_t(i) L. \tag{8}$$

Using (4), profit maximization yields the price of firm i as

$$p_t(i) = \frac{\psi w_t}{1 - \mu(q_t(i))}, \tag{9}$$

where

$$\mu(q_t(i)) = -\frac{u''(q_t(i)) q_t(i)}{u'(q_t(i))} > 0. \tag{10}$$

The markup of firm i is equal to the elasticity of marginal utility $\mu(q_t(i))$ and $0 < \mu(q_t(i)) < 1$ following Dhingra and Morrow (2019). I also assume that $\mu(q_t(i))$ is increasing in $q_t(i)$ (Dhyne et al. (2011), De Loecker et al. (2012), and De Loecker and Warzynski (2012)). The output of firms are identical; hence, I can write $q_t(i) = q_t$, and the variety label i can be dropped. Substituting (9) into (8), the firms' profit levels can be obtained as

$$\pi_t = \frac{w_t \psi \mu(q_t) q_t L}{1 - \mu(q_t)}. \tag{11}$$

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The R&D activities of the present model follow that of the model of Grossman and Helpman (1991, Ch. 3). The final good producers enter the R&D race and finance the cost of R&D by issuing equity in the stock market. The stock value of the final good producers at time t is equal to the present discounted value of its profit stream after t as follows:

$$v_t = \int_t^{\infty} e^{-\int_t^{\omega} r_s ds} \pi_{\omega} d\omega.$$

Differentiation of v_t with respect to t yields the following no-arbitrage condition:

$$\dot{v}_t = r_t v_t - \pi_t. \tag{12}$$

Goods producers hire relevant labor to generate blueprints. I presume that there exists knowledge spillover in R&D activities. I assume that L_t^A units of labor for R&D activity for the time interval dt produce a new variety of final good according to:

$$dn_t = \frac{L_t^A n_t^{\gamma}}{a} dt, \tag{13}$$

where γ represents the degree of knowledge spillover in R&D investment.[5] The cost of the R&D activities is $w_t L_t^A dt$. The production of innovative blueprints creates the value of $v_t dn_t$ since each blueprint has a market value of v_t. There is free entry into the R&D race; thus, the following free-entry condition must hold:

$$v_t \leqq \frac{a w_t}{n_t^{\gamma}}. \tag{14}$$

The labor market equilibrium condition is

$$L = L_t^A + \psi n_t q_t L. \tag{15}$$

Labor demand consists of R&D investment and the production of goods.

3. Equilibrium

In the equilibrium, since $q_t(i) = q_t$, the marginal utility of the total expenditure at time t, A_t, is

$$A_t = U'(\tilde{u}_t) n_t u'(q_t) q_t.$$

Then, (7) becomes

$$r_t = \rho - \left[1 + \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)}\right] \frac{\dot{n}_t}{n_t} + \left[1 - \mu(q_t) + \eta(q_t)\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)}\right] \frac{\dot{q}_t}{q_t}, \tag{16}$$

where $\eta(q_t) = \frac{u'(q_t)q_t}{u(q_t)}$ is the elasticity of utility and we assume that $0 < \eta(q_t) < 1$.

Substituting (4) for (9) and using $q_t(i) = q_t$ and $E_t = 1$, the following equation can be obtained:

$$\frac{\psi w_t}{1 - \mu(q_t)} = \frac{1}{n_t q_t}.$$

Using (14), the above equation becomes

$$\frac{\psi v_t n_t^{\gamma}}{a(1 - \mu(q_t))} = \frac{1}{n_t q_t}. \tag{17}$$

Differentiating (17) with respect to t, we can obtain the following dynamics:

$$\frac{\dot{v}_t}{v_t} = -(1+\gamma)\frac{\dot{n}_t}{n_t} - \left[1 + \frac{\mu'(q_t)q_t}{1 - \mu(q_t)}\right] \frac{\dot{q}_t}{q_t}. \tag{18}$$

From (11), (12), and (14), the dynamics of the stock value of the final good producers is

$$\frac{\dot{v}_t}{v_t} = r_t - \frac{n_t^{\gamma}\psi\mu(q_t)q_tL}{a(1-\mu(q_t))}. \tag{19}$$

Substituting (16) into (19), the dynamics of the value becomes

$$\frac{\dot{v}_t}{v_t} = \rho - \frac{n_t^{\gamma}\psi\mu(q_t)q_tL}{a(1-\mu(q_t))} - \left[1 + \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)}\right]\frac{\dot{n}_t}{n_t}$$
$$- \left[1-\mu(q_t) + \eta(q_t)\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)}\right]\frac{\dot{q}_t}{q_t}. \tag{20}$$

From (18) and (20), the following dynamics of q_t are obtained:

$$B_t\frac{\dot{q}_t}{q_t} = \rho - \frac{n_t^{\gamma}\psi\mu(q_t)q_tL}{a(1-\mu(q_t))} + C_t\frac{\dot{n}_t}{n_t}, \tag{21}$$

where

$$B_t = \eta(q_t)\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} - \frac{\mu'(q_t)q_t}{1-\mu(q_t)} - \mu(q_t) < 0,$$

$$C_t = \gamma - \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} > 0.$$

The second term of (21) represents the return on R&D investment. When q_t and n_t are given at time t, an increase in the growth rate of the number of firms decreases the consumption per capita.

From (13) and (15), the dynamics of the number of firms becomes

$$\dot{n}_t = \frac{n_t^{\gamma}L}{a}(1-\psi n_t q_t). \tag{22}$$

(21) and (22) constitute the dynamic system in this economy.

I investigate how many steady states exist. In the steady state, the consumption per capita and the number of firms are given by

$$n^* = \frac{1}{\psi q^*}, \tag{23}$$

$$\rho = \frac{(n^*)^{\gamma}\psi\mu(q^*)q^*L}{a(1-\mu(q^*))}, \tag{24}$$

where the superscript "$*$" represents the steady states. Substituting (23) into the right-hand side of (24), the following equation can be obtained:

$$RHS(q) \equiv \frac{\mu(q)\psi^{1-\gamma}L}{aq^{\gamma-1}(1-\mu(q))}. \tag{25}$$

$RHS(q)$ represents the return on R&D investment. In the steady state, the return on R&D investment is equal to the subjective discount rate, ρ. When $\gamma < 1$ ($\gamma > 1$) and q goes to zero, $RHS(q)$ goes to zero (infinity). When $\gamma < 1$ ($\gamma > 1$), and q goes to infinity, $RHS(q)$ goes to infinity (zero). Differentiating $RHS(q)$ with respect to q, I can derive the following equation:

$$RHS'(q) = \frac{\mu(q)\psi^{1-\gamma}L}{aq^{\gamma}(1-\mu(q))}\left[\Psi(q) + 1 - \gamma\right], \tag{26}$$

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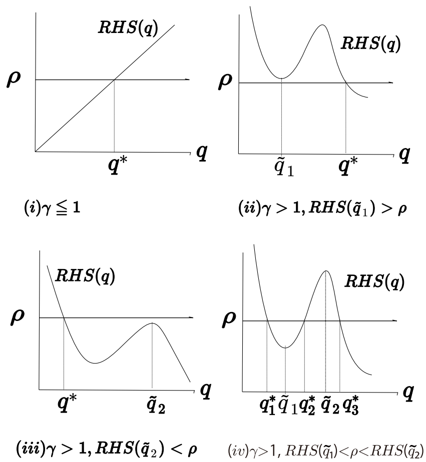

Figure 1. Steady states when $\mu'(q) > 0$.

where

$$\Psi(q) \equiv \frac{\mu'(q)q}{\mu(q)(1-\mu(q))} > 0.$$

When $\gamma \leqq 1$, $RHS'(q)$ is positive, and the return on R&D investment monotonically increases in the output level, as depicted in Figure 1. Therefore, when $\gamma \leqq 1$, there exists a unique steady state.

Conversely, when $\gamma > 1$, the sign of $RHS'(q)$ is ambiguous and there may be multiple steady states. I differentiate $\Psi(q)$ with respect to q as follows:

$$\Psi'(q) = \frac{\mu''(q)q\mu(q)(1-\mu(q)) + \mu'(q)\left[\mu(q)(1-\mu(q)) - \mu'(q)q + 2\mu(q)q\mu'(q)\right]}{\mu(q)^2(1-\mu(q))^2}.$$

Notably, the sign of $\Psi'(q)$ is ambiguous. Hereafter, I assume that there exists a unique $q = \hat{q}$ that $\Psi'(\hat{q}) = 0$ holds. Additionally, I assume that $\Psi(q)$ is inverted U-shaped and $\Psi'(q) > 0$ ($\Psi'(q) < 0$) when $q < \hat{q}$ ($q > \hat{q}$) and $\Psi(\hat{q}) > \gamma - 1$. From these assumptions, there exist $\tilde{q}_1$ and $\tilde{q}_2$ ($\tilde{q}_1 < \tilde{q}_2$) where $RHS'(\tilde{q}_1) = 0$ and $RHS'(\tilde{q}_2) = 0$ hold. Then, in $0 < q < \tilde{q}_1$ and $q > \tilde{q}_2$, $RHS(q^*)$ is decreasing in q, and $RHS(q)$ is increasing in $\tilde{q}_1 < q < \tilde{q}_2$. In Figure 1, when $RHS(\tilde{q}_1) < \rho < RHS(\tilde{q}_2)$, there exist three steady states, which are labeled as q_1^*, q_2^*, and q_3^* ($q_1^* < q_2^* < q_3^*$). When $RHS(\tilde{q}_1) > \rho$ or $RHS(\tilde{q}_2) < \rho$, there exists a unique steady state.

Next, I investigate the stability of the steady states. The condition that a steady state is saddle path stable is (See Appendix for the proof)

$$\gamma < 1 + \Psi(q^*).$$

The left-hand side of γ is the knowledge spillover in the R&D activity, and the right-hand side signifies the elasticity of the returns on R&D investment. When this condition holds, the steady state is saddle path stable. When $\gamma \leqq 1$, the steady state is always saddle path stable.[6] When $\gamma > 1$ and $RHS(\tilde{q}_1) > \rho$ or $RHS(\tilde{q}_2) < \rho$ at the steady state, the slope of $RHS(q^*)$ is negative and $\Psi(q^*) + 1 - \gamma < 0$ holds. Subsequently, the steady state is unstable. In Figure 1, when $\gamma > 1$ and $RHS(\tilde{q}_1) < \rho < RHS(\tilde{q}_2)$, there are three steady states. In two of the three steady states, q_1^* and q_3^*, the slope of RHS is negative. Therefore, these steady states become unstable. Contrarily, the slope of $RHS(q_2^*)$

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is positive. Then, at q_2^*, the steady state is saddle path stable. Summarizing the above result, we can obtain the following proposition:[7]

Proposition 1. *Suppose that there exists a unique* $q = \hat{q}$ *where* $\Psi'(\hat{q}) = 0$ *holds,* $\Psi'(q) > 0$ $(\Psi'(q) < 0)$ *when* $q < \hat{q}$ $(q > \hat{q})$*, and* $\Psi(\hat{q}) > \rho$*. When* $\gamma \leqq 1$*, there exists a unique and stable steady state. When* $\gamma > 1$ *and* $RHS(\tilde{q}_1) > \rho$ *or* $RHS(\tilde{q}_2) < \rho$*, there exists a unique and unstable steady state. When* $\gamma > 1$ *and* $RHS(\tilde{q}_1) < \rho < RHS(\tilde{q}_2)$*, there exist three steady states. Two of the three steady states are unstable and the other one is stable.*

In this model, there exist multiple steady states and balanced growth does not exist when the elasticity of substitution is not constant. This outcome differed from those of Grossman and Helpman (1991) and Latzer et al. (2019). From the right-hand side of (24), the return on the R&D investment is

$$RR \equiv \frac{\pi_t}{wL_t^A/\dot{n}_t} = \left(\frac{n_t^\gamma}{a}\right)(q_t\psi L)\left(\frac{\mu(q_t)}{1-\mu(q_t)}\right).$$

In this model, when firms enter the market, there are three effects on the return on R&D investment: one is ambiguous and the other two are negative. The ambiguous effect is the *spillover effect*, which is denoted by the first parentheses of $\frac{n_t^\gamma}{a}$ and is positive (negative) when $\gamma > 0$ $(\gamma < 0)$. Moreover, the *spillover effect* outlines that an increase in the number of firms reduces the costs of R&D investment, thus increasing the return on R&D investment when $\gamma > 0$. The first negative effect is the *consumption effect*, $q_t\psi L$. An increase in the number of firms reduces the consumption per good which reduces the profit of firms. The second negative effect is the *markup effect*, $\frac{\mu(q_t)}{1-\mu(q_t)}$. Since the elasticity of substitution is variable in this model, an increase in the number of firms reduces the consumption per good and the markup rates of firms as $\mu'(q_t)$.[8] This correlation diminishes the profit of firms and the return on R&D investment.

When the knowledge spillover of R&D investment is small, the *spillover effect* is smaller than the sum of the *competition effect* and the *markup effect*, and the return on the R&D investment results in a decrease in the number of firms. Therefore, the return on R&D investment is equal to the subjective discount rate and the economy goes to a steady state. When the knowledge spillover of the R&D investment is sufficiently large, the *spillover effect* is larger than the sum of the *consumption effect* and the *markup effect*, and the return on the R&D investment results in an increase in the number of firms. Similarly, the return on R&D investment is larger than the subjective discount rate and the steady state is unstable. When the knowledge spillover of the R&D investment is in the middle, there are some equilibria in n and q, where the rate of return on R&D investment is equal to the subjective discount rate. At some equilibria, because an increase in the number of firms decreases the return on the R&D investment, the return on the R&D investment is equal to the subjective discount rate and the steady states are stable. At other equilibria, because an increase in the number of firms raises the return on R&D investment, the steady states are unstable.

In contrast, in Grossman and Helpman (1991), the elasticity of the marginal utility of $\mu(q_t)$ is $\frac{1}{\sigma}$ and $\gamma = 1$. Then, RR becomes

$$RR = \left(\frac{n_t}{a}\right)(q_t\psi L)\left(\frac{1}{\sigma-1}\right).$$

In Grossman and Helpman (1991), there is no *markup effect* and the *spillover effect* is equal to the *consumption effect*. Thereafter, the total output of $q_t n_t$ is formulated so that the return on the R&D investment is equal to the subjective discount rate, $RR = \rho$ at time 0, and there exists a balanced growth path.

Latzer et al. (2019) assume an externality to marginal costs that the labor requirement is decreasing in the number of firms, that is, $\psi = \frac{\hat{\psi}}{n_t}$.[9] Due to this assumption, labor engaged in

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production is $\psi q_t n_t = \hat{\psi} q_t$ and is independent of the number of firms. Besides, an increase in the number of firms does not affect the output level of q_t. Then, in Latzer et al. (2019), RR becomes

$$RR = \left(\frac{n_t}{a}\right)(q_t L)\left(\frac{\mu(q_t)}{1-\mu(q_t)}\right)\left(\frac{\hat{\psi}}{n_t}\right) = \frac{\hat{\psi}}{a}(q_t L)\left(\frac{\mu(q_t)}{1-\mu(q_t)}\right).$$

The fourth parenthesis represents the additional externality to marginal costs. As the *spillover effect* is nullified by the additional eternality to marginal costs, the return on the R&D investment does not depend on the number of firms. The consumption per product q_t is determined so that the return on the R&D investment is equal to the subjective discount rate at time 0. Likewise, the number of workers engaged in production remains constant. Therefore, a balanced growth path exists in Latzer et al. (2019).

4. First-best equilibrium

In this section, I inspect the first-best equilibrium whereby the central planner distributes the labor for production and R&D investment and distinguishes the allocation of consumption and saving to maximize the welfare level (1). The resource constraint is

$$a\frac{\dot{n}_t}{n_t} + \psi n_t q_t L = L.$$

Accordingly, the dynamics of the number of firms and consumption per capita are

$$\left(B_t + \frac{\mu'(q_t)q_t}{1-\mu(q_t)}\right)\frac{\dot{q}_t}{q_t} = \rho - \frac{\gamma n_t^{\gamma-1}}{a} - \frac{n_t^{\gamma}\psi q_t L}{a}\left(\frac{1}{\eta(q_t)} - 1 - \gamma\right) + C_t\frac{\dot{n}_t}{n_t}, \tag{27}$$

$$\dot{n}_t = \frac{n_t^{\gamma} L}{a}(1 - \psi n_t q_t). \tag{28}$$

The dynamics of the number of firms in the first-best equilibrium are the same as that in the competitive equilibrium. When no knowledge spillover exists, $\gamma = 0$, the dynamics of the consumption per capita are

$$\left(B_t + \frac{\mu'(q_t)q_t}{1-\mu(q_t)}\right)\frac{\dot{q}_t}{q_t} = \rho - \frac{\psi q_t L}{a}\frac{1-\eta(q_t)}{\eta(q_t)} + C_t\frac{\dot{n}_t}{n_t}. \tag{29}$$

Comparing (29) with (21), the consumption per capita in the competitive equilibrium is equal to that in the first-best equilibrium when $\mu'(q_t) = 0$, and the following equation holds:

$$\frac{\mu(q_t)}{1-\mu(q_t)} = \frac{1-\eta(q_t)}{\eta(q_t)}.$$

As discussed in Grossman and Helpman (1991) and Dhingra and Morrow (2019), when the elasticity of substitution is constant, this equation holds and the competitive equilibrium coincides with the first-best equilibrium.

When there exists a knowledge spillover and $\gamma = 1$, the consumption per capita of the steady state in the first-best equilibrium is

$$\rho = \frac{1}{a} + \frac{n^F \psi q^F L}{a}\left(\frac{1}{\eta(q^F)} - 2\right), \tag{30}$$

where the superscript of F represents the first-best equilibrium. Comparing (24) with (30), the consumption per capita in the first-best equilibrium does not coincide that in the competitive equilibrium. Summarizing the above results, the following proposition can be obtained:

Proposition 2. *When there is no knowledge spillover and the elasticity of substitution is constant, the competitive equilibrium coincides with the first-best equilibrium. When $\gamma = 1$ or the elasticity of substitution is not constant, the competitive equilibrium does not coincide with the first-best equilibrium.*

5. Conclusion

In this study, I construct a variety expansion model with the VES. When the firms invest in the R&D, they obtain patents and produce products to obtain monopolistic profits. Then, the R&D investment leads to an increase in the number of firms in the economy. Following the recent empirical studies, this paper assumes that an increase in the output of firms raises their markup rates. As the number of firms increases, three effects on firm activity become apparent. The first effect is that an increase in the number of firms reduces the R&D costs, which in turn expands the return on the R&D investment when there exists the positive knowledge spillover. The second one is that the market competition becomes severe, and this reduces the profit of firms. The third effect is that as the number of firms increases, the consumption per capita decreases. Consequently, the firms lower their markup rates.

This study reveals that when the knowledge spillover is in the middle, the effect of an increase in the number of firms on the return on R&D investment is ambiguous. Then, there exist multiple equilibria where the return on the R&D investment is equal to the subjective discount rate. Some steady states are stable whereas the other steady states are unstable. Moreover, when the knowledge spillover is sufficiently small, there is a unique and stable steady state. Since the negative effect of an increase in the number of firms on the profit level is larger than the decrease in the innovation costs, the economy ceases the R&D investment and goes to a unique steady state. When the knowledge spillover is sufficiently large, there exists a unique and unstable steady state. Once the firms conduct R&D investment in a steady state, the R&D costs are drastically reduced. Accordingly, this steady state is unstable. Finally, when the elasticity of substitution is variable, the first-best equilibrium does not coincide with the competitive equilibrium.

Notes

1 See Dhyne et al. (2011), De Loecker et al. (2012), and De Loecker and Warzynski (2012).

2 Following Grossman and Helpman (1991), when the elasticity of substitution is constant and the knowledge spillover in R&D investment is unity, there exists a balanced growth path.

3 In this paper, since new entrants have the same productivity as existing firms, average productivity does not increase and the economy converges to a steady state. This result is consistent with Haltiwanger (2012) showing that the rate of productivity growth is slow if low-productivity firms do not exit and high-productivity firms do not enter.

4 Boucekkine et al. (2017) assume that $v(z) = -\frac{v(1/z)}{v'(1/z)} z$ where $v(z)$ represents the indirect utility function. This study does not impose the same assumption on the indirect utility function as Boucekkine et al. (2017).

5 From a meta-analysis of Neves and Sequeira (2018), the degree of knowledge spillover in R&D investment is from −0.1 to 2.

6 Bloom et al. (2020) show that the research productivity declines sharply and the value of γ can be interpreted as negative. Jones (2022) also assumes that the exponential growth of idea is getting harder to achieve. Following these studies, the steady state is saddle path stable.

7 In the Online Appendix, when $\mu'(q) < 0$, I investigate the existence of steady states and the stability of the steady states.

8 Differentiating the *markup effect* with respect to q_t is $\frac{\mu'(q_t)}{(1-\mu(q_t))} > 0$.

9 Since in Latzer et al. (2019), the labor requirement depends on time t and $\gamma = 1$, the dynamics of q_t of (21) becomes

$$B_t \frac{\dot{q}_t}{q_t} = \rho - \frac{n_t^{\gamma} \psi \mu(q_t) q_t L}{a(1 - \mu(q_t))} - \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} \frac{\dot{n}_t}{n_t}.$$

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A. APPENDIX

A.1. DERIVATION OF (5)

In this Appendix, we derive $\frac{\partial q_t(i)}{\partial E_t}$. When totally differentiating the (4) of product i, we can obtain the following equation:

$$\sum_{j=1}^{n_t} \alpha_j \frac{dq_t(j)}{dE_t} - \beta_i \frac{dq_t(i)}{dE_t} = \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{E_t}, i = 1, \ldots, n,$$

where

$$\alpha_j = u''(q_t(j))q_t(j) + u'(q_t(j)),$$

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$$\beta_j = \frac{u''(q_t(j))}{u'(q_t(j))}\sum_{j=1}^{n_t} u'(q_t(k))q_t(k).$$

These equations can be expressed in matrix as follows:

$$\Phi D = \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{E_t} I,$$

where

$$\Phi = \begin{pmatrix} \alpha_1 - \beta_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1 & \alpha_2 - \beta_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n - \beta_n \end{pmatrix},$$

$$D^T = \begin{pmatrix} \frac{dq_t(1)}{dE_t} & \frac{dq_t(2)}{dE_t} & \dots & \frac{dq_t(n)}{dE_t} \end{pmatrix},$$

$$I^T = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix},$$

where T represents the transpose. Using Cramer's rule, $\frac{dq_t(i)}{dE_t}$ is given by

$$\frac{dq_t(i)}{dE_t} = \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{E_t}\frac{\det(\Phi_i)}{\det(\Phi)},$$

where Φ_i is the matrix formed by replacing the i-th column of Φ through I. $\det(\Phi)$ and $\det(\Phi_i)$ are

$$\det(\Phi) = \Pi_{k=1}^{n_t}(-\beta_k)\left(1 - \sum_{k=1}^{n_t}\frac{\alpha_k}{\beta_k}\right),$$

$$\det(\Phi_i) = \Pi_{k\neq i}^{n_t}(-\beta_k).$$

Then, $\frac{dq_t(i)}{dE_t}$ becomes

$$\frac{dq_t(i)}{dE_t} = \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{E_t}\frac{\Pi_{k\neq i}^{n_t}(-\beta_k)}{\Pi_{k=1}^{n_t}(-\beta_k)\left(1 - \sum_{k=1}^{n_t}\frac{\alpha_k}{\beta_k}\right)}$$
$$= \frac{\sum_{j=1}^{n_t} u'(q_t(j))q_t(j)}{(-\beta_i)\left(1 - \sum_{k=1}^{n_t}\frac{\alpha_k}{\beta_k}\right)E_t}.$$

A.2. DERIVATION OF (6)

I solve the intertemporal utility maximization problem. The Lagrange function of $\pounds$ can be defined as follows:

$$\pounds = \int_0^\infty e^{-\rho t}U(\tilde{u}_t)dt + \lambda\left[a(0) + \int_0^\infty e^{-R(t)t}w_t dt - \int_0^\infty e^{-R(t)t}E_t dt\right],$$

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where λ is the Lagrange multiplier. When differentiating £ with respect to E_t, the following first-order condition can be obtained:

$$\frac{\partial £}{\partial E_t} = e^{-\rho t} U'(\tilde{u}_t) \sum_{i=1}^{n_t} u'(q_t(i)) \frac{dq_t(i)}{dE_t} - \lambda e^{-R(t)t} = 0.$$

Substituting (5) into the above equation, the first-order condition is

$$\frac{\partial £}{\partial E_t} = \frac{e^{-\rho t} U'(\tilde{u}_t)}{E_t} \sum_{j=1}^{n_t} u'(q_t(j)) q_t(j) \sum_{i=1}^{n_t} \frac{u'(q_t(i))}{(-\beta_i)(1 - \sum_{k=1}^{n_t} \frac{\alpha_k}{\beta_k})} - \lambda e^{-R(t)t} = 0.$$

Then, Equation (6) can be obtained.

In the symmetric equilibrium, $q_t(i) = q_t$, $\alpha = u''(q_t)q_t + u'(q_t)$, and $\beta = n_t u''(q_t(j))q_t$ hold. Then, in the symmetric equilibrium, the first-order condition becomes

$$\frac{\partial £}{\partial E_t} = \frac{e^{-\rho t} U'(\tilde{u}_t)}{E_t} n_t u'(q_t) q_t - \lambda e^{-R(t)t} = 0.$$

Taking logarithm and differentiating t, I can obtain the following equation:

$$-\rho - \frac{\dot{E}_t}{E_t} + \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} \left[\frac{\dot{n}_t}{n_t} + \frac{u'(q_t)q_t}{u(q_t)} \frac{\dot{q}_t}{q_t} \right] + \frac{\dot{n}_t}{n_t} + \frac{u''(q_t)q_t}{u'(q_t)} \frac{\dot{q}_t}{q_t} + \frac{\dot{q}_t}{q_t} = -r_t.$$

Taking $E_t = 1$, the interest rate at time t in the symmetric equilibrium is

$$r_t = \rho - \left(1 + \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} \right) \frac{\dot{n}_t}{n_t} - \left[1 + \frac{u'(q_t)q_t}{u(q_t)} \frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} + \frac{u''(q_t)q_t}{u'(q_t)} \right] \frac{\dot{q}_t}{q_t}.$$

A.3. STABILITY OF THE STEADY STATES

In this Appendix, I investigate the stability of the steady states. The linearized systems of (22) and (21) around the steady state are

$$\begin{pmatrix} \dot{n}_t \\ \dot{q}_t \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ \frac{q^*}{B}\phi_3 & \frac{q^*}{B}\phi_4 \end{pmatrix} \begin{pmatrix} n_t - n^* \\ q_t - q^* \end{pmatrix},$$

where

$$\phi_1 = -\frac{L(q^*)^{1-\gamma}}{a} < 0,$$

$$\phi_2 = -\frac{L(q^*)^{-(1+\gamma)}}{a} < 0,$$

$$\phi_3 = -\frac{(n^*)^{\gamma-1} q^* L}{a} \left[\frac{\gamma \mu(q^*)}{1 - \mu(q^*)} + C \right] < 0,$$

$$\phi_4 = -\frac{(n^*)^{\gamma} L}{a} \left[\frac{\mu'(q^*)q^* + \mu(q^*)(1 - \mu(q^*))}{(1 - \mu(q^*))^2} + C \right] < 0.$$

The determinant of the characteristic matrix becomes

$$\frac{q^*}{B}\phi_1\phi_4 - \frac{q^*}{B}\phi_2\phi_3 = \frac{(q^*)^{2(1-\gamma)}\mu(q^*)}{a^2 B} \frac{\Psi(q^*) + 1 - \gamma}{1 - \mu(q^*)}. \tag{31}$$

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The negative condition of the determinant of the characteristic matrix is

$$\gamma < 1 + \Psi(q^*).$$

Finally, I investigate the sign of the trace of the characteristic matrix when $\gamma > 1 + \Psi(q^*)$. The trace of the characteristic matrix is

$$\phi_1 + \frac{q^*}{B}\phi_4 = -\frac{\rho}{aB}\left[B + C + \frac{\mu'(q^*)q^* + \mu(q^*)(1-\mu(q^*))}{(1-\mu(q^*))^2}\right]$$
$$= -\frac{\rho}{\mu(q^*)B}\left[\gamma - (1-\eta(q^*))\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} + \frac{\mu'(q^*)q^* + \mu(q^*)(1-\mu(q^*))}{(1-\mu(q^*))^2}\right] > 0,$$

since $0 < \eta(q^*) < 1$ and $B < 0$. Thereafter, the trace of the characteristic matrix has a positive value. When $\gamma > 1 + \Psi(q^*)$ holds, both eigenvalues are positive, and the steady state is unstable.

When $\gamma \leqq 1$, the determinant of the characteristic matrix is negative and the steady state is saddle path stable. Notably, for $\gamma > 1$ and $RHS(\tilde{q}_1) > \rho$ or $RHS(\tilde{q}_2) < \rho$, there exists a unique steady state. At the steady-state equilibrium, the slope of $RHS(q^*)$ is negative, that is $\gamma > 1 + \Psi(q^*)$ holds. Then, the steady state is unstable. When $\gamma > 1$ and $RHS(\tilde{q}_1) < \rho < RHS(\tilde{q}_2)$, there exist three steady states q_1^*, q_2^*, q_3^*, and $q_1^* < q_2^* < q_3^*$. The slope of $RHS(q_1^*)$ and $RHS(q_3^*)$ is negative and that of $RHS(q_2^*)$ is positive. Therefore, the steady states of q_1^* and q_3^* are unstable. Contrarily, the steady state of q_2^* is saddle path stable.

A.4. WHEN $\mu'(q_t) < 0$

In this Appendix, the number and stability of steady states are discussed when the elasticity of substitution is decreasing in the consumption level, that is $\mu'(q_t) < 0$. Since $\mu(q_t) = -\frac{u''(q_t)q_t}{u'(q_t)}$, $\mu'(q_t)$ becomes

$$\mu'(q_t) = \frac{\mu(q_t)}{q_t}\left(1 + \frac{u'''(q_t)q_t}{u''(q_t)} + \mu(q_t)\right). \tag{32}$$

We assume $\frac{u'''(q_t)q_t}{u''(q_t)} < -2 < -1 - \mu(q_t)$ in this Appendix.

The consumption per capita in the steady state is determined by (25). From (26), when $\gamma \geq 1$, $RHS'(q) < 0$ for all q since $\mu'(q_t) < 0$ and $\Psi(q) < 0$. Therefore, when $\gamma \geq 1$, there exists a unique steady state since $\lim_{q\to 0} RHS(q) = \infty$ and $\lim_{q\to\infty} RHS(q) = 0$. On the contrary, when $\gamma < 1$, $\lim_{q\to 0} RHS(q) = 0$, $\lim_{q\to\infty} RHS(q) = \infty$, and the sign of $RHS'(q)$ is ambiguous. Then, there may be multiple steady states. Assuming that $\Psi'(q) < 0$ ($\Psi'(q) > 0$) when $q < \hat{q}$ ($q > \hat{q}$), there exist $\tilde{q}_1$ and $\tilde{q}_2$ ($\tilde{q}_1 < \tilde{q}_2$) where $RHS'(\tilde{q}_1) = 0$ and $RHS'(\tilde{q}_2) = 0$. Then, in $0 < q < \tilde{q}_1$ and $q > \tilde{q}_2$, $RHS(q)$ is increasing in q, and in $\tilde{q}_1 < q < \tilde{q}_2$, $RHS(q)$ is decreasing in q. When $RHS(\tilde{q}_1) < \rho$ and $RHS(\tilde{q}_2) > \rho$, there exists a unique steady state. When $RHS(\tilde{q}_2) < \rho < RHS(\tilde{q}_1)$, there exist three steady states.

Next, I investigate the stability condition when $\mu'(q) < 0$. From Appendix A.3, the condition that a steady state is saddle path stable is $\frac{1+\Psi(q^*)-\gamma}{B} < 0$. Substituting $\mu'(q_t)$ into B, the sign of B is given by

$$B = \eta(q_t)\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} - \frac{\mu(q_t)}{1-\mu(q_t)}(2 + \frac{u'''(q_t)q_t}{u''(q_t)}) < 0, \tag{33}$$

since $\frac{u'''(q_t)q_t}{u''(q_t)} < -2$. Then, since $B < 0$, the negative condition of the determinant of the characteristic matrix is $\gamma < 1 + \Psi(q^*)$ when $\mu'(q_t) < 0$. The trace of the characteristic matrix is

$$\phi_1 + \frac{q^*}{B}\phi_4 = -\frac{\rho}{\mu(q^*)B}\left(\gamma - (1-\eta(q^*))\frac{U''(\tilde{u}_t)\tilde{u}_t}{U'(\tilde{u}_t)} + \frac{\mu(q^*)}{(1-\mu(q^*))^2}(2 + \frac{u'''(q_t)q_t}{u''(q_t)})\right) > 0,$$

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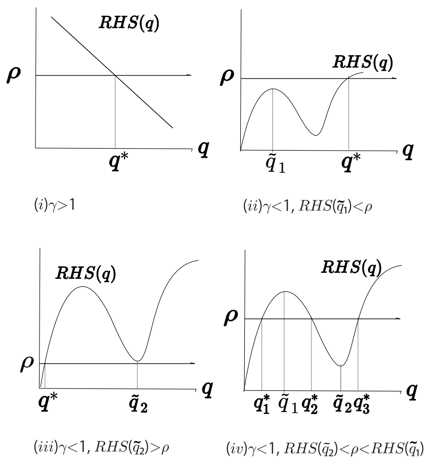

Figure 2. Steady states when $\mu'(q) < 0$.

since $U''(\tilde{u}_t) < 0$, $\frac{u'''(q_t)q_t}{u''(q_t)} < -2$ and $B < 0$. Therefore, the trace of the characteristic matrix has a positive value when $\mu'(q_t) < 0$.

When $1 + \Psi(q^*) < 1 < \gamma$, there exists a unique unstable steady state since $RHS'(q^*) < 0$ and $\Psi(q^*) + 1 < \gamma$ in Figure 2. When $\gamma < 1$ and $RHS(\tilde{q}_1) < \rho$ or $RHS(\tilde{q}_2) > \rho$, there exists a unique steady state. At the steady-state equilibrium, the slope of $RHS(q^*)$ is positive, that is $\gamma < 1 + \Psi(q^*)$, the steady state is stable. When $\gamma < 1$, $RHS(\tilde{q}_1) > \rho$, and $RHS(\tilde{q}_2) < \rho$ hold, there exist three steady states q_1^*, q_2^*, and q_3^* where $RHS'(q_1^*) > 0$, $RHS'(q_2^*) < 0$, and $RHS'(q_3^*) > 0$ hold. Therefore, the steady states of q_1^* and q_3^* are saddle path stable. Contrarily, the steady state of q_2^* is unstable.

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