INCOMPATIBILITY OF GENERIC HUGENESS PRINCIPLES

MONROE ESKEW

Abstract. We show that the weakest versions of Foreman's minimal generic hugeness axioms cannot hold simultaneously on adjacent cardinals. Moreover, conventional forcing techniques cannot produce a model of one of these axioms.

§1. Introduction. In [5–8], Foreman proposed generic large cardinals as new axioms for mathematics. These principles are similar to strong kinds of traditional large cardinal axioms but speak directly about small uncountable objects like ω_1, ω_2, etc. Because of this, they are able to answer many classical questions that are not settled by ZFC plus traditional large cardinals. For example, if ω_1 is minimally generically huge, then the Continuum Hypothesis holds and there is a Suslin line [8].

For a poset $\mathbb{P}$, let us say that a cardinal κ is $\mathbb{P}$-*generically huge* if $\mathbb{P}$ forces that there is an elementary embedding $j : V \to M \subseteq V[G]$ with critical point κ, where M is a transitive class closed under $j(\kappa)$-sequences from $V[G]$. If $\mathbb{P}$ forces that $j(\kappa) = \lambda$, we call λ the *target*. We say that κ is $\mathbb{P}$-*generically n-huge* when the requirement on M is strengthened to closure under $j^n(\kappa)$-sequences (where j^n is the composition of j with itself n times), and we say κ is $\mathbb{P}$-*generically almost-huge* if the requirement is weakened to closure under $<j(\kappa)$-sequences. We say that a cardinal κ is $\mathbb{P}$-*generically measurable* if $\mathbb{P}$ forces that there is an elementary embedding $j : V \to M \subseteq V[G]$ with critical point κ, where M is transitive.

If κ is the successor of an infinite cardinal μ, we say that κ is *minimally generically n-huge* if it is $\mathrm{Col}(\mu, \kappa)$-generically n-huge, where $\mathrm{Col}(\mu, \kappa)$ is the poset of functions from initial segments of μ into κ ordered by end-extension. The main result of this note is that for a successor cardinal κ, it is inconsistent for both κ and κ^+ to be minimally generically huge.

THEOREM 1. *Suppose $0 < m \leq n$ and κ is a regular cardinal that is $\mathbb{P}$-generically n-huge with target λ, where $\mathbb{P}$ is nontrivial and strongly λ-c.c. Then κ^{+m} is not $\mathbb{Q}$-generically measurable for any κ-closed $\mathbb{Q}$.*

Here, "nontrivial" means that forcing with $\mathbb{P}$ necessarily adds a new set. Usuba [12] introduced the *strong λ-chain condition* (strong λ-c.c.), which means that $\mathbb{P}$ has no antichain of size λ and forcing with $\mathbb{P}$ does not add branches to λ-Suslin trees. As Usuba observed, $\mathbb{P}$ having the strong λ-c.c. is implied by $\mathbb{P}$ having the μ-c.c. for $\mu < \lambda$ and by $\mathbb{P} \times \mathbb{P}$ having the λ-c.c. In particular, if $\theta = \kappa^{<\mu}$, then $\mathrm{Col}(\mu, \kappa)$ collapses θ to μ and is strongly θ^+-c.c. Let us also remark that in Theorem 1, κ-closure can be weakened to κ-strategic-closure without change to the arguments.

Regarding the history: Woodin proved, in unpublished work mentioned in [8, p. 1126], that it is inconsistent for ω_1 to be minimally generically 3-huge while ω_3 is minimally generically 1-huge. Subsequently, the author [3] improved this to show the inconsistency of a successor cardinal κ being minimally generically n-huge while κ^{+m} is minimally generically almost-huge, where $0 < m < n$. The weakening of the hypothesis to κ being only generically 1-huge uses an idea from the author's work with Cox [1].

In contrast to Theorem 1, Foreman [4] exhibited a model where for all $n > 0$, ω_n is $\mathbb{P}$-generically almost-huge with target ω_{n+1} for some ω_{n-1}-closed, strongly ω_{n+1}-c.c. poset $\mathbb{P}$. A simplified construction was given by Shioya [11].

We prove Theorem 1 in Section 2 via a generalization that is less elegant to state. In Section 3, we discuss what is known about the consistency of generic hugeness by itself and present a corollary of Theorem 1 showing that the usual forcing strategies cannot produce models where ω_1 is generically huge with target ω_2 by a strongly ω_2-c.c. poset. Our notations and terminology are standard. We assume the reader is familiar with the basics of forcing and elementary embeddings.

§2. Generic huge embeddings and approximation. The relevance of the strong κ-c.c. is its connection to the approximation property of Hamkins [9]. Suppose $\mathcal{F} \subseteq \mathcal{P}(\lambda)$. We say that a set $X \subseteq \lambda$ is *approximated by* $\mathcal{F}$ when $X \cap z \in \mathcal{F}$ for all $z \in \mathcal{F}$. If $V \subseteq W$ are models of set theory, then we say that the pair (V, W) satisfies the *κ-approximation property* for a V-cardinal κ when for all $\lambda \in V$ and all $X \subseteq \lambda$ in W, if X is approximated by $\mathcal{P}_\kappa(\lambda)^V$, then $X \in V$. We say that a forcing $\mathbb{P}$ *has the κ-approximation property* when the κ-approximation property is forced to hold of the pair $(V, V[G])$. The following result appears as Lemma 1.5 and Note 1.11 in [12]:

THEOREM 2 (Usuba). *If $\mathbb{P}$ is a nontrivial κ-c.c. forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a κ-closed forcing, then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-approximation property if and only if $\mathbb{P}$ has the strong κ-c.c.*

Theorem 1 will follow from the more general lemma below.

LEMMA 3. *The following hypotheses are jointly inconsistent*:

(1) $\kappa_0 \leq \kappa_1$ *and* $\lambda_0 \leq \lambda_1$ *are regular cardinals.*
(2) $\mathbb{P}$ *is a nontrivial strongly* λ_0-*c.c. poset that forces an elementary embedding* $j : V \to M \subseteq V[G]$ *with* $j(\kappa_0) = \lambda_0$, $j(\kappa_1) = \lambda_1$, $\mathcal{P}(\lambda_1)^V \subseteq M$, *and* $M^{<\lambda_0} \cap V[G] \subseteq M$.
(3) κ_1^+ *is* $\mathbb{Q}$-*generically measurable for a* κ_0-*closed* $\mathbb{Q}$.

PROOF. We will need a first-order version of (3) that can be carried through the embedding of (2). Replace it by the (possibly weaker) hypothesis that $\mathbb{Q}$ is a κ_0-closed poset and for some $\theta \gg \lambda_1$, $\mathbb{Q}$ forces an elementary embedding $j : H_\theta^V \to N$ with critical point κ_1^+, where $N \in V^{\mathbb{Q}}$ is a transitive set.

CLAIM 4. $\kappa_1^{<\kappa_0} = \kappa_1$. ⊣

PROOF. Let $G \subseteq \mathbb{Q}$ be generic over V, and let $j : H_\theta^V \to N$ be an elementary embedding with critical point κ_1^+, where $N \in V[G]$ is a transitive set. By $<\kappa_0$-distributivity, $\mathcal{P}_{\kappa_0}(\kappa_1)^N \subseteq \mathcal{P}_{\kappa_0}(\kappa_1)^V$, so the cardinality of $\mathcal{P}_{\kappa_0}(\kappa_1)^V$ must be below the critical point of j. ⊣

CLAIM 5. $\lambda_1^{<\lambda_0} = \lambda_1$.

PROOF. Let $G \subseteq \mathbb{P}$ be generic over V, and let $j : V \to M$ be as hypothesized in (2). By the closure of M, $\mathcal{P}_{\lambda_0}(\lambda_1)^M = \mathcal{P}_{\lambda_0}(\lambda_1)^{V[G]}$. By elementarity and Claim 4, $M \models \lambda_1^{<\lambda_0} = \lambda_1$. Thus M has a surjection $f : \lambda_1 \to \mathcal{P}_{\lambda_0}(\lambda_1)^{V[G]} \supseteq \mathcal{P}_{\lambda_0}(\lambda_1)^V$. If $\lambda_1^{<\lambda_0} > \lambda_1$ in V, then f would witnesses a collapse of λ_1^+, contrary to the λ_0-c.c. ⊣

Now let $\mathcal{F} = \mathcal{P}_{\lambda_0}(\lambda_1)^V$. Let $j : V \to M \subseteq V[G]$ be as in hypothesis (2). Claim 5 implies that $\mathcal{F}$ is coded by a single subset of λ_1 in V, so $\mathcal{F} \in M$. In M, let $\mathcal{A}$ be the collection of subsets of λ_1 that are approximated by $\mathcal{F}$. Since $\mathcal{P}(\lambda_1)^V \subseteq M$, it is clear that $\mathcal{P}(\lambda_1)^V \subseteq \mathcal{A}$.

For each $\alpha < \lambda_1^+$, there exists an $X \in \mathcal{A} \cap V$ that codes a surjection from λ_1 to α in some canonical way. Working in M, choose for each $\alpha < \lambda_1^+$ an $X_\alpha \in \mathcal{A}$ that codes a surjection from λ_1 to α.

By elementarity, λ_1^+ is $j(\mathbb{Q})$-generically measurable in M, witnessed by generic embeddings with domain $H_{j(\theta)}^M$. By the closure of M, $j(\mathbb{Q})$ is λ_0-closed in $V[G]$. Let $H \subseteq j(\mathbb{Q})$ be generic over $V[G]$. Let $i : H_{j(\theta)}^M \to N \in M[H] \subseteq V[G][H]$ be given by the $j(\mathbb{Q})$-generic measurability of λ_1^+ in M, with $\mathrm{crit}(i) = \delta = \lambda_1^+$.

Let $\langle X_\alpha' : \alpha < i(\delta) \rangle = i(\langle X_\alpha : \alpha < \delta \rangle)$. By elementarity, X_δ' is approximated by $i(\mathcal{F}) = \mathcal{F}$. Since $\mathbb{P} * j(\dot{\mathbb{Q}})$ is a nontrivial strongly λ_0-c.c. forcing followed by a λ_0-closed forcing, it has the λ_0-approximation property by

Usuba's theorem. Therefore, $X'_\delta \in V$. But this is a contradiction, since X'_δ codes a surjection from λ_1 to $(\lambda_1^+)^V$.

Let us now complete the proof of Theorem 1. Suppose $n \geq 1$, $\kappa < \lambda$, $\mathbb{P}$ is strongly λ-c.c., and $\mathbb{P}$ forces an embedding $j : V \to M \subseteq V[G]$ such that $j(\kappa) = \lambda$ and M is closed under $j^n(\kappa)$-sequences from $V[G]$. By the λ-c.c. of $\mathbb{P}$ and the λ-closure of M, $(\lambda^+)^M = (\lambda^+)^V$. Suppose inductively that $i < n$ and $(\lambda^{+i})^M = (\lambda^{+i})^V \leq j^{i+1}(\kappa)$. Again, by the chain condition and the $j^{i+1}(\kappa)$-closure of M, $(\lambda^{+i+1})^M = (\lambda^{+i+1})^V$. Since $\kappa^{+i} < \lambda^{+i} = j(\kappa^{+i})$, $j(\lambda^{+i})$ must be an M-cardinal greater than λ^{+i}, so $\lambda^{+i+1} \leq j(\lambda^{+i})$. By elementarity applied to the induction hypothesis, $j(\lambda^{+i}) \leq j^{i+2}(\kappa)$. Thus the induction hypothesis carries through up to n. Now suppose $0 < m \leq n$ and set $\kappa_0 = \kappa$, $\lambda_0 = \lambda$, $\kappa_1 = \kappa^{+m-1}$, and $\lambda_1 = \lambda_0^{+m-1}$. Then we have $j(\kappa_0) = \lambda_0$ and $j(\kappa_1) = \lambda_1 \leq j^n(\kappa)$. If κ^{+m} is also generically measurable by a κ-closed forcing, then this assignment of variables satisfies the hypotheses of the lemma, which we have shown to be inconsistent.

REMARK 6. Suppose ω_1 is $\mathbb{P}$-generically almost-huge and ω_2 is $\mathbb{Q}$-generically measurable, where $\mathbb{P}$ is strongly ω_2-c.c. and $\mathbb{Q}$ is countably closed. This holds, for example, in Foreman's model [4]. Let $j : V \to M$ be an embedding witnessing the $\mathbb{P}$-generic almost-hugeness of ω_1. Put $\kappa_0 = \kappa_1 = \omega_1$ and $\lambda_0 = \lambda_1 = \omega_2$. The only hypothesis of Lemma 3 that fails is $\mathcal{P}(\omega_2)^V \subseteq M$.

§3. On the consistency of generic hugeness. It is not known whether any successor cardinal can be minimally generically huge. Moreover, it is not known whether ω_1 can be $\mathbb{P}$-generically huge with target ω_2 for an ω_2-c.c. forcing $\mathbb{P}$. But we do not think that Theorem 1 is evidence that this hypothesis by itself is inconsistent, since there are other versions of generic hugeness for ω_1 that satisfy the hypothesis of Theorem 1 and are known to be consistent relative to huge cardinals. Magidor [10] showed that if there is a huge cardinal, then in a generic extension, ω_1 is $\mathbb{P}$-generically huge with target ω_3, where $\mathbb{P}$ is strongly ω_3-c.c. Shioya [11] observed that if κ is huge with target λ, then Magidor's result can be obtained from a two-step iteration of Easton collapses, $\mathbb{E}(\omega, \kappa) * \dot{\mathbb{E}}(\kappa^+, \lambda)$. An easier argument shows that after the first step of the iteration, or even in the extension by the Levy collapse $\mathrm{Col}(\omega, {<}\kappa)$, ω_1 is $\mathbb{P}$-generically huge with target λ by a strongly λ-c.c. forcing $\mathbb{P}$.

Theorem 1 shows that in these models, ω_2 is not $\mathbb{Q}$-generically measurable for a countably closed $\mathbb{Q}$. It also shows that if it is consistent for ω_1 to be generically huge with target ω_2 by a strongly ω_2-c.c. forcing, then this cannot be demonstrated by a standard method resembling Magidor's:

COROLLARY 7. *Suppose κ is a huge cardinal with target λ. Suppose $\mathbb{P}$ is such that:*

(1) $\mathbb{P}$ *is* λ-*c.c. and contained in* V_λ.
(2) $\mathbb{P}$ *preserves* κ *and collapses* λ *to become* κ^+.
(3) *For all sufficiently large* $\alpha < \lambda$ (*for example*, *all Mahlo* α *beyond a certain point*), $\mathbb{P} \cong (\mathbb{P} \cap V_\alpha) * \dot{\mathbb{Q}}_\alpha$, *where* $\dot{\mathbb{Q}}_\alpha$ *is forced to be* κ-*closed*.

Then in any generic extension by $\mathbb{P}$, κ *is not generically huge with target* λ *by a strongly* λ-*c.c. forcing*.

Furthermore, *suppose* λ *is supercompact in* V, *and* (3) *is strengthened to*:

(4) *For all sufficiently large* $\alpha < \beta < \lambda$, $\mathbb{P} \cong (\mathbb{P} \cap V_\alpha) * \dot{\mathrm{Col}}(\kappa, \beta) * \dot{\mathbb{Q}}_{\alpha,\beta}$, *where* $\dot{\mathbb{Q}}_{\alpha,\beta}$ *is forced to be* κ-*closed*.

Then κ *is not generically huge with target* λ *by a strongly* λ-*c.c. forcing in any* λ-*directed-closed forcing extension of* $V^{\mathbb{P}}$.

PROOF. Let $j : V \to M$ witness that κ is huge with target λ. By elementarity and the fact that $\mathcal{P}(\lambda) \subseteq M$, λ is measurable in V. Let $\mathcal{U}$ be a normal ultrafilter on λ, and let $i : V \to N$ be the ultrapower embedding.

Since the decomposition of (3) holds for all "sufficiently large" α, $N \models i(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is forced to be κ-closed. By the closure of N, V also believes that $\dot{\mathbb{Q}}$ is forced by $\mathbb{P}$ to be κ-closed. Thus if we take $G \subseteq \mathbb{P}$ generic over V, then the embedding i can be lifted by forcing with $\mathbb{Q}$. This means that in $V[G]$, λ is $\mathbb{Q}$-generically measurable, $\mathbb{Q}$ is κ-closed, and $\lambda = \kappa^+$. Theorem 1 implies that in $V[G]$, κ cannot be generically huge with target λ by a strongly λ-c.c. forcing.

For the final claim, suppose λ is supercompact in V, and let $\dot{\mathbb{R}}$ be a $\mathbb{P}$-name for a λ-directed-closed forcing. Let γ be such that $\Vdash_{\mathbb{P}} |\dot{\mathbb{R}}| \leq \gamma$. By [2, Theorem 14.1], $\mathrm{Col}(\kappa, \gamma) \cong \mathrm{Col}(\kappa, \gamma) \times \mathbb{R}$ in $V^{\mathbb{P}}$. Let $i : V \to N$ be an elementary embedding such that $\mathrm{crit}(i) = \lambda$, $i(\lambda) > \gamma$, and $N^\gamma \subseteq N$. By applying (4) in N, there is in N a complete embedding of $\mathbb{P} * \dot{\mathbb{R}}$ into $i(\mathbb{P})$, such that the quotient forcing is equivalent to something of the form $\mathrm{Col}(\kappa, \gamma) * \dot{\mathbb{Q}}_{\lambda,\gamma}$, where $\dot{\mathbb{Q}}_{\lambda,\gamma}$ is forced to be κ-closed in $N^{\mathbb{P}*\dot{\mathbb{R}}*\dot{\mathrm{Col}}(\kappa,\gamma)}$. By the closure of N, the quotient is forced to be κ-closed in $V^{\mathbb{P}*\dot{\mathbb{R}}}$.

Let $G * H \subseteq \mathbb{P} * \dot{\mathbb{R}}$ be generic. Further κ-closed forcing yields a generic $G' \subseteq i(\mathbb{P})$ that projects to $G * H$. We can lift the embedding to $i : V[G] \to N[G']$. By elementarity, $i(\mathbb{R})$ is $i(\lambda)$-directed-closed in $N[G']$. Thus $i[H]$ has a lower bound $r \in i(\mathbb{R})$. By the closure of N, $i(\mathbb{R})$ is at least κ-closed in $V[G']$. Forcing below r yields a generic $H' \subseteq i(\mathbb{R})$ and a lifted embedding $i : V[G * H] \to N[G' * H']$. Hence in $V[G * H]$, λ is generically measurable via a κ-closed forcing. Theorem 1 implies that κ cannot be generically huge with target λ by a strongly λ-c.c. forcing. ⊣

REFERENCES

[1] S. Cox and M. Eskew, *Compactness versus hugeness at successor cardinals*. **Journal of Mathematical Logic**. vol. 23 (2023), no. 1, 2250016.

[2] J. Cummings, *Iterated forcing and elementary embeddings*, **Handbook of Set Theory** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 775–883.

[3] M. Eskew, *Generic large cardinals as axioms*. **Review of Symbolic Logic**, vol. 13 (2020), no. 2, pp. 375–387.

[4] M. Foreman, *More saturated ideals*, **Cabal Seminar 79–81** (A. S. Kechris, D. A. Martin, and Y. N. Maschovakis, editors), Lecture Notes in Mathematics, vol. 1019, Springer, Dordrecht, 1983, pp. 1–27.

[5] ———, *Potent axioms*. **Transactions of the American Mathematical Society**, vol. 294 (1986), no. 1, pp. 1–28.

[6] ———, *Generic large cardinals: New axioms for mathematics?* **Documenta Mathematica**, vol. II (1998), pp. 11–21.

[7] ———, *Has the continuum hypothesis been settled?* **Lecture Notes in Logic**, vol. 24 (2006), pp. 56–75.

[8] ———, *Ideals and generic elementary embeddings*, **Handbook of Set Theory** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 885–1147.

[9] J. D. Hamkins, *Extensions with the approximation and cover properties have no new large cardinals*. **Fundamenta Mathematicae**, vol. 180 (2003), no. 3, pp. 257–277.

[10] M. Magidor, *On the existence of nonregular ultrafilters and the cardinality of ultrapowers*. **Transactions of the American Mathematical Society**, vol. 249 (1979), no. 1, pp. 97–111.

[11] M. Shioya, *Easton collapses and a strongly saturated filter*. **Archive for Mathematical Logic**, vol. 59 (2020), nos. 7–8, pp. 1027–1036.

[12] T. Usuba, *The approximation property and the chain condition*. **Research Institute for Mathematical Sciences Kokyuroku**, vol. 1895 (2014), pp. 103–107.

KURT GÖDEL RESEARCH CENTER
INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN
KOLINGASSE 14-16
1090 WIEN, AUSTRIA
E-mail: monroe.eskew@univie.ac.at