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Endomorphisms of Kronecker Modules Regulated by Quadratic Algebra Extensions of a Function Field

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Abstract. The Kronecker modules $\mathbb{V}(m, h, \alpha)$, where m is a positive integer, h is a height function, and α is a K-linear functional on the space $K(X)$ of rational functions in one variable X over an algebraically closed field K, are models for the family of all torsion-free rank-2 modules that are extensions of finite-dimensional rank-1 modules. Every such module comes with a regulating polynomial f in $K(X)[Y]$. When the endomorphism algebra of $\mathbb{V}(m, h, \alpha)$ is commutative and non-trivial, the regulator f must be quadratic in Y. If f has one repeated root in $K(X)$, the endomorphism algebra is the trivial extension $K \ltimes S$ for some vector space S. If f has distinct roots in $K(X)$, then the endomorphisms form a structure that we call a bridge. These include the coordinate rings of some curves. Regardless of the number of roots in the regulator, those $\operatorname{End}\mathbb{V}(m, h, \alpha)$ that are domains have zero radical. In addition, each semi-local $\operatorname{End}\mathbb{V}(m, h, \alpha)$ must be either a trivial extension $K \ltimes S$ or the product $K \times K$.

Introduction

We work with an algebraically closed field K and an indeterminate X. The $K[X]$-submodules of $K(X)$ are tractable models for the torsion-free, rank-one $K[X]$-modules. The creation of workable models for $K[X]$-modules of rank greater than one has not enjoyed similar success. The measure of complexity of these higher rank modules is discussed in [27] using the parallel context of abelian groups. By a *Kronecker module* we mean a right module over the four-dimensional Kronecker algebra $\left[\begin{smallmatrix} K & K^2 \\ 0 & K \end{smallmatrix}\right]$. Every $K[X]$-module can be viewed as a Kronecker module with rank preserved. Thus the problem of classifying the Kronecker modules of rank more than one is at least as hard as that of $K[X]$-modules. However, the Kronecker modules form the testing ground for the study of representations of associative algebras that are not of finite type, see [1, 6, 7, 9, 21–26].

Just as with $K[X]$-modules, the construction of all possible rank two Kronecker modules is not an issue, see [11]. Unlike the case of $K[X]$-modules, there do exist non-zero, finite-dimensional, torsion-free Kronecker modules. Thus torsion-free, rank-two Kronecker modules that are extensions of non-zero, finite-dimensional modules are possible. These form a class of Kronecker modules with potential for results that are quite different from any that come up in the $K[X]$-module theory. For instance, some of these rank-two Kronecker modules embed in modules of rank-one, see [13, 14]. The endomorphism algebras of these rank-two Kronecker modules offer an abundance of surprises. Most notably, the endomorphism algebras for one class of such modules comprise the coordinate rings of all elliptic curves, see [16].

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These rank-two extensions of finite-dimensional modules can be constructed using a triple (m, h, α), where m is a positive integer, h is a height function, and α is a K-linear functional on the space of all rational functions in X. This construction, denoted by $\mathbb{V}(m, h, \alpha)$, will be reviewed in detail below.

To each pair (h, α) we attach a polynomial f in $K(X)[Y]$, either monic or zero, called the *regulator* of (h, α). The endomorphisms of $\mathbb{V}(m, h, \alpha)$ form a K-algebra. It is known that unless f is linear or quadratic in Y, the algebra $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is just K, what we call the *trivial* algebra, see [17]. If the regulator is linear, the module $\mathbb{V}(m, h, \alpha)$ has a finite-dimensional direct summand, and $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is non-commutative but well understood, see [17, p. 1568].

When the regulator is quadratic the story gets interesting. In this case the endomorphism algebra $\operatorname{End}\mathbb{V}(m, h, \alpha)$ must be commutative, and may well be non-trivial, see [15] for instance. The quadratic regulator $f(Y)$ may have no roots, two roots, or one repeated root in $K(X)$. Each possibility impinges on the structure of $\operatorname{End}\mathbb{V}(m, h, \alpha)$.

If the regulator has a single, repeated root, $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is shown to be the trivial extension $K \ltimes S$ for some K-linear space S. We recall from [5] that the trivial extension $K \ltimes S$ is the vector space $K \oplus S$ endowed with the multiplication

$$(\lambda, s)(\mu, t) = (\lambda\mu, \lambda t + \mu s) \quad \text{for all } (\lambda, s), (\mu, t) \text{ in } K \oplus S.$$

In the case of distinct roots we show that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is embedded in the product of two subalgebras of $K(X)$ as a structure that we call a *bridge*. We also classify the bridges that are so realized as endomorphism algebras. Thus, when the quadratic regulator has roots, the endomorphism algebras are fully understood.

When the regulator has no roots, it is known that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is a domain. Here we show that this domain has zero radical. This, along with our classification of realizable bridges, yields that the only possible semi-local endomorphism algebras of $\mathbb{V}(m, h, \alpha)$ are $K \ltimes S$ and $K \times K$.

We end this introduction by summarizing the role that the height function plays in these endomorphism algebras which we have been studying over several papers. In [18, 19] we saw that an indecomposable $\mathbb{V}(m, h, \alpha)$ with non-trivial endomorphism algebra is constructible if and only if h assumes a finite value at least once while at the same time $h \geq 2$ infinitely often. In [20] we showed that a purely simple $\mathbb{V}(m, h, \alpha)$ with non-trivial endomorphisms is constructible if and only if h assumes the value ∞ at least once but also assumes finite values at least twice. For a review of purely simple modules and some of their properties, see [12]. In [16] we built purely simple modules whose endomorphism algebras are the coordinate rings of elliptic curves, but an explicit construction of all possible endomorphism algebras of purely simple modules remains in the dark. However their endomorphism algebras in the case when the pole algebra is affine tie Kronecker mmodules to affine curves, see [10, 16].

1 Preliminaries

This stand-alone section introduces the synergistic mix of derivers on $K(X)$, the regulator as a polynomial over $K(X)$, two-by-two matrices over $K(X)$, and height functions. They provide the definition of the modules studied here in a way that makes

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them approachable by linear algebra. Throughout, K stands for an algebraically closed field, whose elements will be called *scalars*.

The Valuations ord_θ and ord_∞

Let $K(X)$ be the field of rational functions in an indeterminate X. For each scalar θ we adopt the shorthand $X_\theta = (X - \theta)^{-1}$. Every non-zero t in $K(X)$ has a unique factorization

$$t = \lambda \prod_{\theta \in K} X_\theta^{j_\theta},$$

where $\lambda \in K$, $j_\theta \in \mathbb{Z}$, and all but finitely many j_θ are 0. For each θ in K, the integer j_θ is denoted by $\mathrm{ord}_\theta(t)$, and the integer $-\sum_{\theta \in K} j_\theta$ is denoted by $\mathrm{ord}_\infty(t)$. If we agree that $\mathrm{ord}_\theta(0) = -\infty$ for all θ in $\mathrm{K} \cup \{\infty\}$, and s, t are rational functions, the familiar valuation properties hold:

$$\mathrm{ord}_\theta(st) = \mathrm{ord}_\theta(s) + \mathrm{ord}_\theta(t), \quad \mathrm{ord}_\theta(s+t) \leq \max\{\mathrm{ord}_\theta(s), \mathrm{ord}_\theta(t)\},$$
$$\mathrm{ord}_\theta(s+t) = \mathrm{ord}_\theta(t) \text{ when } \mathrm{ord}_\theta(s) < \mathrm{ord}_\theta(t).$$

If $\theta \in \mathrm{K} \cup \{\infty\}$ and $\mathrm{ord}_\theta(t) > 0$, the rational function t has a *pole* at θ and its order is $\mathrm{ord}_\theta(t)$. The functions X^n and X_θ^{n+1}, where $\theta \in K$ and $0 \leq n$, form the *standard basis* of $K(X)$ over K. The expansion of a rational function in terms of the standard basis is known as its partial fraction expansion. For θ in K, a positive power of X_θ appears in the partial fraction expansion of t if and only if t has a pole at θ. In that case the highest power of X_θ appearing is $X_\theta^{\mathrm{ord}_\theta(t)}$. A positive power of X appears in the partial fraction expansion of t if and only if t has a pole at ∞. Then the highest power of X that appears is $X^{\mathrm{ord}_\infty(t)}$.

Height Functions and Pole Spaces

Any function $h\colon \mathrm{K} \cup \{\infty\} \to \{\infty, 0, 1, 2, \dots\}$ is known as a *height function*. The attached K-linear space

$$R_h = \{s \in K(X) : \mathrm{ord}_\theta(s) \leq h(\theta) \text{ for all } \theta \text{ in } \mathrm{K} \cup \{\infty\}\}$$

is called a *pole space*. Pole spaces have an intrinsic definition as well. They are the non-zero subspaces R of $K(X)$ with the property that whenever $t \in R$, every function s, such that $\mathrm{ord}_\theta(s) \leq \max\{0, \mathrm{ord}_\theta(t)\}$ for all θ in $\mathrm{K} \cup \{\infty\}$, is also in R. Every pole space contains K, which is the smallest possible pole space. The biggest pole space is $K(X)$. Pole spaces form a lattice. The sum of two pole spaces R_h and R_ℓ is the pole space corresponding to the height function $\max\{h, \ell\}$. The intersection of R_h and R_ℓ corresponds to the height function $\min\{h, \ell\}$. Pole spaces have been crucial to the description of torsion-free, rank-one, Kronecker modules, see [3].

The Pole Algebra

Given a pole space R_h, the *pole algebra* for h is the algebra of functions in R_h having poles only at those θ in $\mathrm{K} \cup \{\infty\}$ where $h(\theta) = \infty$. We denote the pole algebra by A_h.

Of course, A_h is a pole space in its own right coming from the height function that agrees with h, when h takes the value ∞, but is 0 otherwise. If $\theta \in K$, we let $K[X_\theta]$ be the algebra of polynomials in X_θ. We can view A_h as the sum of all $K[X_\theta]$ taken over those θ where $h(\theta) = \infty$, plus possibly the space $K[X]$ of ordinary polynomials in X should it happen that $h(\infty) = \infty$. Another viewpoint is that A_h is the K-algebra inside R_h that contains all K-algebras inside R_h.

The pole algebra equals $K(X)$ when the height function always assumes the value ∞. The pole algebra A_h will equal K exactly when h never assumes the value ∞. The latter height functions have been treated in [17] under the name of *singular* height functions.

We should also note that a rational function t belongs to A_h if and only $tR_h \subseteq R_h$. Thus R_h is a module over the algebra A_h under the action of multiplication of rational functions.

Every ideal of a pole algebra is principal because a pole algebra properly containing K is the localization, using a suitable multiplicative set, of either $K[X]$ if $K[X]$ is in A_h, or of some $K[X_\theta]$ inside A_h. The non-zero ideals zA_h have finite codimension in A_h. Furthermore they have codimension one if and only if the generator z is a prime in A_h.

The Spur of a Pole Space

A pole space R_h is the sum of two pole subspaces. We have the pole algebra A_h, and what we might call the *spur*:

$$S_h = \{u \in R_h : \mathrm{ord}_\theta(u) \le 0 \text{ when } h(\theta) = \infty\}.$$

In other words, S_h is the pole space corresponding to the height function that agrees with h where h is finite valued, but is 0 where h takes the value ∞. The pole algebra A_h and the spur S_h are complementary in R_h, in the sense that

$$A_h \cap F_h = K \text{ while } A_h + S_h = R_h. \tag{1.1}$$

Equivalent Height Functions

Suppose that h is a height function and that t is a non-zero function in the pole space R_h. Since $\mathrm{ord}_\theta(t) \le h(\theta)$, for all θ in $\mathrm{K} \cup \{\infty\}$, the function ℓ given by

$$\ell(\theta) = h(\theta) - \mathrm{ord}_\theta(t), \text{ for all } \mathrm{K} \cup \{\infty\}, \tag{1.2}$$

is again a height function. Should the symbol $\infty - \mathrm{ord}_\theta(t)$ come up, it is taken to be ∞. A bit of reflection reveals that $tR_\ell = R_h$. Both h and ℓ assume the value ∞ on the same subset of $\mathrm{K} \cup \{\infty\}$. Thus $A_h = A_\ell$. In particular h is singular if and only if ℓ is singular. When two height functions h, ℓ are equivalent in this way, their pole spaces R_h, R_ℓ are both of finite co-dimension in $R_h + R_\ell$. This equivalence of height functions gives the isomorphism types of all rank-one Kronecker modules, see [3]. We will henceforth denote this equivalent ℓ by $h - \operatorname{ord}(t)$, which suggests how it was obtained.

The Pole Space of a Rational Function

A pole space is finite-dimensional over K if and only if its height function assumes positive values on only a finite set and never assumes the value ∞. The objects of our attention will arise from infinite-dimensional pole spaces, yet we shall encounter finite-dimensional ones as follows. Given a rational function s, let P_s be the pole space coming from the height function h given by

$$h(\theta) = \max\{0, \operatorname{ord}_\theta(s)\} \text{ for every } \theta \text{ in } \mathrm{K} \cup \{\infty\}.$$

This finite-dimensional P_s is called the *pole space of the function s*. It is the smallest pole space containing s. A function belongs to P_s if and only if its partial fraction expansion extends no further than the partial fraction expansion of s. If s is a scalar, then its pole space is K.

Functionals and Derivers

We shall need to work with K-linear functionals $\alpha\colon K(X) \to K$. If α is such a functional and $r \in K(X)$, let $\langle \alpha, r\rangle$ denote the value in K that α takes at r. Given a functional α and a rational function r, it is shown in [8, Proposition 3.4] that there is a unique rational function $\partial_\alpha(r)$ so that $\partial_\alpha(r)(\theta)$ is defined at all θ in K where $r(\theta)$ is defined, and for all such θ

$$\partial_\alpha(r)(\theta) = \langle \alpha, (r - r(\theta))X_\theta\rangle. \tag{1.3}$$

From this it is easy to see that the mapping $(\alpha, r) \mapsto \partial_\alpha(r)$ is K-linear in both α and r. The K-linear map $\partial_\alpha\colon K(X) \to K(X)$ will be called a *deriver*. For a functional α and a rational function r, the functional given by $t \mapsto \langle \alpha, rt\rangle$ will be denoted by $\alpha * r$. The name *deriver* is motivated by the following derivation-like property which can be deduced from (1.3):

$$\partial_\alpha(st) = s\partial_\alpha(t) + \partial_{\alpha * t}(s) \text{ for any functional } \alpha \text{ and rational functions } s, t. \tag{1.4}$$

The explicit calculation of ∂_α on the standard basis of $K(X)$ is as follows:

$$\begin{gathered} \partial_\alpha(1) = 0, \quad \partial_\alpha(X^n) = \langle \alpha, X^{n-1}\rangle + \langle \alpha, X^{n-2}\rangle X + \cdots + \langle \alpha, 1\rangle X^{n-1}, \\ \partial_\alpha(X_\theta^n) = -\langle \alpha, X_\theta^n\rangle X_\theta - \langle \alpha, X_\theta^{n-1}\rangle X_\theta^2 - \cdots - \langle \alpha, X_\theta\rangle X_\theta^n, \end{gathered} \tag{1.5}$$

for all θ in K and all $n \geq 1$. The formulas (1.5) will be used often. In conjunction with the partial fraction expansion of r, they reveal that

$$\operatorname{ord}_\theta(\partial_\alpha(r)) \leq \max\{0, \operatorname{ord}_\theta(r)\} \text{ for all } \theta \text{ in } K, \tag{1.6}$$

$$\operatorname{ord}_\infty(\partial_\alpha(r)) < \max\{0, \operatorname{ord}_\infty(r)\}. \tag{1.7}$$

Consequently every pole of $\partial_\alpha(r)$ is a pole of r. Furthermore, if R_h is a pole space, then $\partial_\alpha(R_h) \subseteq R_h$, *i.e.*, derivers leave pole spaces invariant.

Some Deriver Identities

In [19, Proposition 2.3] it is proved that the composition of derivers is a commutative operation, and furthermore that the composite of two derivers is again a deriver. Thus the space of derivers is a commutative algebra without an identity element. Here we exploit this commutativity, in conjunction with (1.4), to develop some deriver identities for use in Section 2.

Lemma 1.1 *For any functional α and any rational functions y, z,*

$$\partial_\alpha(\partial_\alpha(yz)) = \partial_\alpha(\partial_\alpha(y)z + y\partial_\alpha(z)) - \partial_\alpha(y)\partial_\alpha(z).$$

Proof On the left of the identity we have

$$\begin{aligned} \partial_\alpha(\partial_\alpha(yz)) =& \partial_\alpha(y\partial_\alpha(z) + \partial_{\alpha * z}(y)) && \text{using (1.4)} \\ =& \partial_\alpha(y\partial_\alpha(z)) + \partial_\alpha(\partial_{\alpha * z}(y)). \end{aligned}$$

On the right we have

$$\begin{aligned} &\partial_\alpha(\partial_\alpha(y)z + y\partial_\alpha(z)) - \partial_\alpha(y)\partial_\alpha(z) \\ &\quad = \partial_\alpha(y)\partial_\alpha(z) + \partial_{\alpha * z}(\partial_\alpha(y)) + \partial_\alpha(y\partial_\alpha(z)) - \partial_\alpha(y)\partial_\alpha(z) \quad \text{using (1.4)} \\ &\quad = \partial_{\alpha * z}(\partial_\alpha(y)) + \partial_\alpha(y\partial_\alpha(z)). \end{aligned}$$

Because derivers commute under composition, the left- and right-sides of the desired identity have come down to the same thing. ■

Lemma 1.2 *For any functional α and any rational functions t, s, r,*

$$\begin{aligned} \partial_\alpha(\partial_\alpha(t)sr + t\partial_\alpha(sr)) - \partial_\alpha(t)\partial_\alpha(sr) =& \partial_\alpha(\partial_\alpha(tsr)) \\ =& \partial_\alpha(\partial_\alpha(ts)r + ts\partial_\alpha(r)) - \partial_\alpha(ts)\partial_\alpha(r). \end{aligned}$$

Proof The first equality follows from Lemma 1.1 by putting $y = t, z = sr$. To get the second equality, put $y = ts, z = r$. ■

The Regulator of a Pair (h, α)

Take a height function h with infinite-dimensional pole space R_h, and a functional $\alpha \colon K(X) \to K$. The deriver ∂_α is a K-linear operator on the space $K(X)$. Every rational function t acts on $K(X)$ as the *multiplier* $s \mapsto ts$. We identify t with its multiplier. The deriver ∂_α leaves R_h invariant, but a multilpier t need not. Nevertheless, the space tR_h lies inside the finite-dimensional extension $R_h + P_t$ of R_h. Let $\mathcal{A}$ denote the subalgebra of $\operatorname{End}_K K(X)$ that is generated by ∂_α and by all multipliers. Put

$$\mathcal{I} = \{\sigma \in \mathcal{A} : \sigma(R_h) \text{ is finite-dimensional over } K\}.$$

The operators in $\mathcal{I}$ are said to have *finite rank on* R_h. Since the dimension of R_h is infinite, $\mathcal{I}$ is a proper left ideal of $\mathcal{A}$. One can check that $\mathcal{I}$ is also a right ideal using the fact that, for every σ in $\mathcal{A}$, the image $\sigma(R_h)$ is inside a finite-dimensional extension of R_h, see also [15, Lemma 2.1].

While $\mathcal{A}$ is typically a non-commutative algebra containing $K(X)$, the quotient algebra $\mathcal{A}/\mathcal{I}$ is a commutative $K(X)$-algebra. For the proof see [15, Lemma 2.2]. Briefly, it suffices to check that a multiplier t commutes with ∂_α modulo the ideal $\mathcal{I}$. From the deriver property (1.4) we have $\partial_\alpha(tr)t - t\partial_\alpha(r) = \partial_{\alpha * r}(t)$ for all r in R_h. Derivers leave pole spaces invariant. Thus for all r in R_h the functions $\partial_{\alpha * r}(t)$ lie in the finite-dimensional pole space P_t, and thereby $\partial_\alpha \circ t - t \circ \partial_\alpha \in \mathcal{I}$.

Clearly $\mathcal{A}/\mathcal{I}$ is generated as a $K(X)$-algebra by the image $\partial_\alpha + \mathcal{I}$ of ∂_α under the canonical projection $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. If $K(X)[Y]$ is the algebra of polynomials in Y over $K(X)$, we are entitled to the substitution map

$$\epsilon\colon K(X)[Y] \to \mathcal{A}/\mathcal{I} \text{ where } Y \mapsto \partial_\alpha + \mathcal{I}. \tag{1.8}$$

The unique monic generator $f(Y)$ of $\ker \epsilon$ is the polynomial in $K(X)[Y]$ that we call the *regulator* of the pair (h, α). Given a polynomial $g(Y)$ in $K(X)[Y]$, let $g(\partial_\alpha)$ stand for any preimage in $\mathcal{A}$ of $g(\partial_\alpha + \mathcal{I})$ under the projection $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. Although this notation is ambiguous, we have just seen that no matter what preimage $g(\partial_\alpha)$ is taken, we always have

$$f(Y) \text{ divides } g(Y) \Leftrightarrow g(\partial_\alpha + \mathcal{I}) = 0 \Leftrightarrow g(\partial_\alpha) \text{ has finite rank on } R_h.$$

Convenient preimages $g(\partial_\alpha)$ will be taken as needed. To summarize, the regulator of (h, α) is the polynomial $f(Y)$ in $K(X)[Y]$ uniquely specified by the following properties:

- $f(Y)$ is monic or zero,
- $f(\partial_\alpha)$ has finite rank on R_h,
- $f(Y)$ divides $g(Y)$ in $K(X)[Y]$ if and only if $g(\partial_\alpha)$ has finite rank on R_h.

If ℓ is a height function equivalent to h as in (1.2), a polynomial $f(Y)$ regulates (h, α) if and only if $f(Y)$ regulates (ℓ, α). This can be seen by observing that both R_h and R_ℓ have finite codimension in $R_h + R_\ell$. Thus the regulator of (h, α) is unchanged over the equivalence class of h.

Equivalence of (h, α) Pairs

We shall say that the pair (h, α), where h is a height function and α is a functional, is equivalent to another such pair (ℓ, β) provided that $(\ell, \beta) = (h - \operatorname{ord}(t), \alpha * t)$ for some non-zero rational function t. It is not hard to see that this is the same as having $(h, \alpha) = (\ell - \operatorname{ord}(t^{-1}), \beta * t^{-1})$. We shall need the connection between the regulators of equivalent pairs.

Proposition 1.3 *Let the pair (h, α) be equivalent to the pair $(h - \operatorname{ord}(t), \alpha * t)$ using some non-zero rational function t. If $f(Y)$ is the regulator of (h, α) in $K(X)[Y]$ and n is the degree of $f(Y)$, then the regulator of $(h - \operatorname{ord}(t), \alpha * t)$ is*

$$t^n f\Big(\frac{Y + \partial_\alpha(t)}{t}\Big).$$

Proof The height function $h - \operatorname{ord}(t)$ is equivalent to the height function h. Hence the regulator of the pair $(h - \operatorname{ord}(t), \alpha * t)$ is the same as the regulator of $(h, \alpha * t)$. Recall the algebra $\mathcal{A}$ generated by all multipliers and the deriver ∂_α, that is used to define the regulator of (h, α). By using (1.4) it can be seen that

$$\partial_{\alpha * t} = \partial_\alpha \circ t - \partial_\alpha(t) \text{ as operators on } K(X). \tag{1.9}$$

Thus $\partial_{\alpha*t}$, along with all multipliers, generates the same algebra $\mathcal{A}$. Referring to the substitution map of (1.8), we put $a = \partial_\alpha + \mathcal{I}$ and $b = \partial_{\alpha*t} + \mathcal{I}$. Using (1.9) it follows that

$$b = at - \partial_\alpha(t) \text{ or equivalently } a = \frac{b + \partial_\alpha(t)}{t}.$$

Since $f(Y)$ is the monic polynomial in $K(X)[Y]$ of least degree to vanish at a, the monic polynomial of least degree to vanish at b is $t^n f\bigl(\frac{Y+\partial_\alpha(t)}{t}\bigr)$. Hence this latter polynomial is the regulator of $(h, \alpha * t)$. That also makes it the regulator of $(h - \operatorname{ord}(t), \alpha * t)$. ■

In due course we will need to observe that the regulator of (h, α) has no roots, distinct roots, or repeated roots in $K(X)$, if and only if the regulator of the equivalent pair $(h - \operatorname{ord}(t), \alpha * t)$ has the same corresponding properties.

The Sum of a Deriver and a Multiplier

The operator $\partial_\alpha + r$, for some functional α and some multiplier r, will surface at various points. If $\theta \in K$, and t is in the pole algebra $K[X_\theta]$, then

$$\partial_\alpha(t) + rt \in K[X_\theta] + P_r \tag{1.10}$$

because derivers leave pole spaces invariant. Similarly if $t \in K[X]$, then

$$\partial_\alpha(t) + rt \in K[X] + P_r.$$

For $\partial_\alpha + r$ to have finite rank on $K[X_\theta]$ it suffices that its kernel be infinite-dimensional, as we now show.

Lemma 1.4 *Let α be a functional and r a multiplier. If $\ker(\partial_\alpha + r) \cap K[X_\theta]$ is infinite-dimensional, then $(\partial_\alpha + r)K[X_\theta] \subseteq P_r$. Likewise if $\ker(\partial_\alpha + r) \cap K[X]$ is infinite-dimensional, then $(\partial_\alpha + r)K[X] \subseteq P_r$.*

Proof Suppose to the contrary that $\partial_\alpha(t) + rt \notin P_r$ for some t in $K[X_\theta]$. In that case $\operatorname{ord}_\theta(\partial_\alpha(t) + rt) > 0$ by virtue of (1.10). For any non-zero s in $K[X_\theta]$ the deriver property (1.4) reveals that $(\partial_\alpha + r)(st) = s(\partial_\alpha(t) + rt) + \partial_{\alpha*t}(s)$. Since

$$\operatorname{ord}_\theta(\partial_\alpha(t) + rt) > 0,$$

and because of (1.6), we have

$$\operatorname{ord}_\theta(s(\partial_\alpha(t) + rt)) = \operatorname{ord}_\theta(s) + \operatorname{ord}_\theta(\partial_\alpha(t) + rt) > \operatorname{ord}_\theta(s) \geq \operatorname{ord}_\theta(\partial_{\alpha*t}(s)).$$

As $s(\partial_\alpha(t)+rt)$ and $\partial_{\alpha * t}(s)$ have poles of different order at θ, they cannot add up to 0. Thus $(\partial_\alpha + r)(st) \neq 0$. This shows that $\partial_\alpha + r$ is injective on the subspace $tK[X_\theta]$. However $tK[X_\theta]$ has finite codimension in $K[X_\theta]$, leaving only room for $\ker(\partial_\alpha + r)$ to be finite-dimensional inside $K[X_\theta]$.

The proof of the claim involving $K[X]$ is the same with X_θ replaced by X. ■

Next we proceed to define the Kronecker modules of interest to us.

1.1 The Modules $\mathbb{V}(m, h, \alpha)$

A right module over the Kronecker algebra $\left[\begin{smallmatrix} K & K^2 \\ 0 & K \end{smallmatrix}\right]$ is called a *Kronecker module*, but in practice it can be viewed as a pair of linear transformations between a pair of K-linear spaces:

$$U \overset{a}{\underset{b}{\rightrightarrows}} V.$$

From such a perspective an *endomorphism* of $U \overset{a}{\underset{b}{\rightrightarrows}} V$ is a pair of K-linear maps $U \xrightarrow{\psi} U$, $V \xrightarrow{\varphi} V$ for which the following diagrams commute:

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "\psi"] & V \arrow[d, "\varphi"] \\ U \arrow[r, "a"] & V \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "b"] \arrow[d, "\psi"] & V \arrow[d, "\varphi"] \\ U \arrow[r, "b"] & V \end{tikzcd}$$

A general problem goes as follows: given a module $U \overset{a}{\underset{b}{\rightrightarrows}} V$, find its endomorphisms and give the structure of its endomorphism algebra. For instance, if $U = V$ and a is the identity mapping on V, the problem becomes to find the commutant algebra of b.

Rank-One Modules

Given a pole space R_h, put $R_h^- = \{r \in R_h : \operatorname{ord}_\infty(r) < h(\infty)\}$. We see that $XR_h^- \subseteq R_h$, and R_h^- is the biggest subspace of R_h to tolerate such inclusion. The modules

$$\mathbb{F}_h = (R_h^- \overset{a}{\underset{b}{\rightrightarrows}} R_h, \text{ where } a\colon r \mapsto r \text{ and } b\colon r \mapsto Xr)$$

are interesting because they provide working models for the class of all torsion-free, indecomposable, rank-one modules, see [3]. In [3] it is shown that the endomorphism algebra of $\mathbb{F}_h$ is the pole algebra A_h. It is also shown in [3] that the complete isomorphism invariants for such rank-one modules are the equivalence classes of height functions.

The Space $V(m, h, \alpha)$

For a positive integer m, let P_m be the space of polynomials of degree strictly less than m. This is nothing but the pole space $P_{X^{m-1}}$.

We shall be working with K-linear subspaces of the space $K(X)^2$ of pairs of rational functions. These pairs will be written in column notation.

Given a triplet (m, h, α) where m is a positive integer, h is a height function and α is a functional, put

$$V(m, h, \alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in K(X)^2 : r \in R_h \text{ and } \partial_\alpha(r) + s \in P_m \right\}, \tag{1.11}$$

$$V^-(m, h, \alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in V : r \in R_h^- \text{ and } \partial_\alpha(r) + s \in P_{m-1} \right\}.$$

We may also think of $V(m, h, \alpha)$ as the space of all vectors in $K(X)^2$ that take the form

$$\begin{pmatrix} r \\ \ell - \partial_\alpha(r) \end{pmatrix} \text{ where } r \in R_h \text{ and } \ell \in P_m.$$

The computations of this paper will rely heavily on the definition of $V(m, h, \alpha)$.

Observe that $XV^-(m, h, \alpha) \subseteq V(m, h, \alpha)$. Indeed, if $\binom{r}{s} \in V^-(m, h, \alpha)$, then $r \in R_h^-$ and $\partial_\alpha(r) + s \in P_{m-1}$. Therefore $Xr \in R_h$, and using (1.4) and (1.5) we get that

$$\partial_\alpha(Xr) + Xs = X\partial_\alpha(r) + \partial_{\alpha * r}(X) + Xs = X(\partial_\alpha(r) + s) + \langle \alpha, r \rangle \in P_m.$$

Definition of the Module $\mathbb{V}(m, h, \alpha)$

The rank-two modules we now present comprise exactly all extensions of finite-dimensional $\mathbb{F}_k$ by infinite-dimensional $\mathbb{F}_h$. We introduce our modules in a way that makes them approachable by means of linear algebra. For a demonstration that they pick up all such extensions, see [15, Section 2].

The Kronecker module $\mathbb{V}(m, h, \alpha)$ is

$$V^-(m, h, \alpha) \overset{a}{\underset{b}{\rightrightarrows}} V(m, h, \alpha) \tag{1.12}$$

where

$$a\colon \begin{pmatrix} r \\ s \end{pmatrix} \mapsto \begin{pmatrix} r \\ s \end{pmatrix} \text{ and } b : \begin{pmatrix} r \\ s \end{pmatrix} \mapsto X\begin{pmatrix} r \\ s \end{pmatrix} \text{ for each } \begin{pmatrix} r \\ s \end{pmatrix} \text{ in } V^-(m, h, \alpha).$$

The space R_h is infinite-dimensional over K exactly when h is positive on an infinite subset of $\mathrm{K} \cup \{\infty\}$ or h is infinite-valued at some θ of $\mathrm{K} \cup \{\infty\}$. When R_h is finite-dimensional, the modules $\mathbb{V}(m, h, \alpha)$ are completely understood thanks to Kronecker's work, see [1, p. 302]. *So, we make the blanket assumption that in defining* $\mathbb{V}(m, h, \alpha)$*, the pole space* R_h *is infinite-dimensional.*

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1.2 The Algebra $\operatorname{End}\mathbb{V}(m,h,\alpha)$

The endomorphism algebras of $\mathbb{V}(m,h,\alpha)$ are the primary concern of this paper. Our study of $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is based on [17, Theorem 2.2] which facilitates the use of linear algebra. It says that

> *the endomorphisms of $\mathbb{V}(m,h,\alpha)$ are the $K(X)$-linear operators on $K(X)^2$ which leave the K-linear subspace $V(m,h,\alpha)$ invariant.*

Our discussion of endomorphisms of $\mathbb{V}(m,h,\alpha)$ will be based on the above point of view. Consequently, in order to follow our proofs it will be necessary to maintain familiarity with the definition of the space $V(m,h,\alpha)$ in (1.11). The definition of the module $\mathbb{V}(m,h,\alpha)$ above was for the purpose of reference only, and will not play any further direct role.

We represent the endomorphisms of $\mathbb{V}(m,h,\alpha)$ as 2×2 matrices of rational functions

$$\varphi = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \tag{1.13}$$

acting on $K(X)^2$ in the usual way. If I is the identity matrix and $\lambda \in K$, the scalar matrix λI is an endomorphism, obviously. If these are the only endomorphisms, we say that $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is *trivial*. It is a matter of some intricacy to discern which non-scalar matrices give endomorphisms, and subsequently to delineate the structure of the endomorphism algebra.

Non-Trivial $\operatorname{End}\mathbb{V}(m,h,\alpha)$ with Quadratic Regulator and Generic Matrix

When $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is non-commutative, the module has a finite-dimensional direct summand and $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is fully understood, see [15]. Thus our interest diverts to modules $\mathbb{V}(m,h,\alpha)$ whose endomorphism algebra is commutative. Whenever such algebras are non-trivial, it is seen from [20, Propositions 2.2 and 2.3] that the regulator of the pair (h,α) must be a quadratic in $K(X)[Y]$. Explicitly,

$$Y^2 + pY + q, \quad \text{where } p \text{ and } q \text{ are in } K(X). \tag{1.14}$$

Attached to such (h,α) there is also what we call the *generic matrix*

$$D = \begin{bmatrix} p & -1 \\ q & 0 \end{bmatrix}, \tag{1.15}$$

which may or may not be an endomorphism of $\mathbb{V}(m,h,\alpha)$. In Section 2 we illustrate how the generic matrix controls $\operatorname{End}\mathbb{V}(m,h,\alpha)$ when this algebra is non-trivial and commutative.

2 The Generic Matrix

Henceforth we operate under the understanding that there is a module $\mathbb{V}(m,h,\alpha)$ as in (1.12). An endomorphism is a 2×2 matrix of rational functions, as in (1.13), that leaves the space $V(m,h,\alpha)$ of (1.11) invariant. We assume that (h,α) has a quadratic regulator as in (1.14) along with its generic matrix D as in (1.15).

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The Determinant and Trace of an Endomorphism

We begin this section with a look at the determinant and the trace of endomorphisms.

Proposition 2.1 *If φ is endomorphism of $\mathbb{V}(m,h,\alpha)$, then*

$$\det\varphi \in A_h \quad \textit{and} \quad \operatorname{trace}\varphi \in A_h.$$

Proof Let φ be an endomorphism expressed as in (1.13). Since $\binom{1}{0}$ and $\binom{0}{1}$ lie in the space $V(m,h,\alpha)$, their images $\binom{s}{u}$ and $\binom{t}{v}$ under φ also lie in $V(m,h,\alpha)$. From (1.11) we deduce that

$$s \in R_h, \quad t \in R_h, \quad u \in R_h + P_m, \quad v \in R_h + P_m.$$

Let k be the height function given by $k(\theta) = 2h(\theta)$ when $\theta \in K$, and $h(\infty) = m + 2h(\infty)$. When the symbolisms 2∞ or $m+\infty$ come up in the definition of k, we take that to mean just ∞ again. A little reflection makes it clear that

$$\det\varphi = sv - ut \in R_k.$$

The height functions k and h have the same pole algebra, because both h and k take the value ∞ at exactly the same θ in $\mathrm{K} \cup \{\infty\}$. Since φ^n is an endomorphism for all positive integers n, and since $\det(\varphi^n) = (\det\varphi)^n$, the rational functions $(\det\varphi)^n$ all lie in R_k. If $\det\varphi$ has a pole at some θ in $\mathrm{K} \cup \{\infty\}$, then the relations

$$0 < \operatorname{ord}_\theta(\det\varphi) \quad \text{and} \quad \operatorname{ord}_\theta((\det\varphi)^n) = n\operatorname{ord}_\theta(\det\varphi) \le k(\theta),$$

for all positive integers n, reveal that $k(\theta) = \infty$. Since $\det\varphi$ only has poles where k, and thereby h, take on the value ∞, the determinant of φ lies in A_h.

To see that $\operatorname{trace}\varphi \in A_h$, we must prove that $h(\theta) = \infty$ for every pole θ of $\operatorname{trace}\varphi$. Since $\det\varphi \in A_h$ as just shown, it suffices to consider only those poles θ of $\operatorname{trace}\varphi$ that are not poles of $\det\varphi$. Now if x, y are the eigenvalues of φ in some algebraic closure of $K(X)$, the so-called Newton identities

$$x^n + y^n = (x+y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2}) \quad \text{for } n = 2, 3, \dots,$$

tell us that

$$\text{(2.1)} \qquad \operatorname{trace}(\varphi^n) = \operatorname{trace}\varphi \operatorname{trace}\varphi^{n-1} - \det\varphi \operatorname{trace}\varphi^{n-2} \quad \text{for } n = 2, 3, \dots.$$

We can now argue by induction that

$$\text{(2.2)} \qquad \operatorname{ord}_\theta(\operatorname{trace}\varphi^n) = n\operatorname{ord}_\theta(\operatorname{trace}\varphi) \quad \text{for } n = 1, 2, 3, \dots.$$

Indeed, supposing (2.2) holds for positive integers prior to n and using the fact θ is not a pole of $\det\varphi$, we obtain

$$\begin{aligned}\operatorname{ord}_\theta(\operatorname{trace}\varphi \operatorname{trace}\varphi^{n-1}) &= n\operatorname{ord}_\theta(\operatorname{trace}\varphi)\\ &> (n-2)\operatorname{ord}_\theta(\operatorname{trace}\varphi) = \operatorname{ord}_\theta(\det\varphi \operatorname{trace}\varphi^{n-2}),\end{aligned}$$

and then from (2.1) we deduce (2.2) for n. As with the determinant, the trace of any endomorphism lies in the pole space R_k defined above. Then (2.2) reveals that R_k contains functions whose order at θ is arbitrarily high. Hence $k(\theta) = \infty$ and then $h(\theta) = \infty$, by the definition of k. ∎

Why the Generic Matrix Matters

Next we explain how the generic matrix anchors the endomorphism algebra.

Proposition 2.2 *If* (h, α) *has quadratic regulator as in* (1.14) *and generic matrix* D *as in* (1.15)*, then every endomorphism of* $\mathbb{V}(m, h, \alpha)$ *takes the form*

$$tD + \partial_\alpha(t)I + \mu I, \text{ for some } t \text{ in } R_h \text{ and some } \mu \text{ in } K. \tag{2.3}$$

Proof Let φ be an endomorphism as in (1.13). Since column vectors $\binom{0}{g}$ belong to $V(m, h, \alpha)$ for all polynomials g in P_m, the vectors

$$\begin{pmatrix} tg \\ vg \end{pmatrix} = \varphi \begin{pmatrix} 0 \\ g \end{pmatrix} \in V(m, h, \alpha) \text{ for all polynomials } g \text{ in } P_m.$$

From the definition (1.11) of $V(m, h, \alpha)$ this implies that $\partial_\alpha(tg) + vg \in P_m$ for all polynomials g in P_m. By the deriver property (1.4) we get $\partial_\alpha(tg) = g\partial_\alpha(t) + \partial_{\alpha * t}(g)$. Hence $g(\partial_\alpha(t) + v) + \partial_{\alpha * t}(g) \in P_m$ for all g in P_m. Putting $g = 1$, gives $\partial_{\alpha * t}(1) = 0$ by (1.5), and thus $\partial_\alpha(t) + v \in P_m$. Since

$$\deg \partial_{\alpha * t}(g) < \deg g \leq \deg g + \deg(\partial_\alpha(t) + v) = \deg g(\partial_\alpha(t) + v),$$

it follows that

$$\deg\big(g(\partial_\alpha(t) + v) + \partial_{\alpha * t}(g)\big) = \deg g + \deg(\partial_\alpha(t) + v) < m,$$

for all polynomials g of degree less than m. The only way for this to happen is with $\partial_\alpha(t) + v \in K$.

According to [20, Proposition 2.4], an endomorphism φ of $\mathbb{V}(m, h, \alpha)$ takes the form

$$\varphi = tD + \partial_\alpha(t)I + gI = \begin{bmatrix} tp + \partial_\alpha(t) + g & -t \\ tq & \partial_\alpha(t) + g \end{bmatrix},$$

for some t in R_h and some g in P_m. From the prior argument, with $\binom{t}{v}$ replaced by $\binom{-t}{\partial_\alpha(t)+g}$, we get that $g = \partial_\alpha(-t) + \partial_\alpha(t) + g \in K$. Putting $\mu = g$, all endomorphisms take the desired form. ∎

We caution that not every matrix taking the form (2.3) need be an endomorphism. Even D, which arises when $t = 1$ and $\mu = 0$, might not be an endomorphism.

Three Options When the Regulator Is Quadratic

Proposition 2.2 shows that in case of a quadratic regulator (1.14), with generic matrix D as in (1.15), the algebra $\operatorname{End} \mathbb{V}(m, h, \alpha)$ embeds in $K(X)[D]$ as a K-subalgebra. The characteristic polynomial of D is $Y^2 - pY + q$, a close relative of the regulator. It is plain to see that the regulator $Y^2 + pY + q$ has no roots, a repeated root, or distinct roots in $K(X)$ if and only if the characteristic poynomial of D has the same respective properties. The algebra $K(X)[D]$ is isomorphic to $K(X)[Y]/(Y^2 - pY + q)$, and up to isomorphism this is

- the trivial extension $K(X) \ltimes K(X)$ when $Y^2 - pY + q$ has just one root,
- the product algebra $K(X) \times K(X)$ when $Y^2 - pY + q$ has two roots,
- a quadratic field extension of $K(X)$ when $Y^2 - pY + q$ has no roots.

Thus the regulator anticipates what $\operatorname{End} \mathbb{V}(m, h, \alpha)$ will look like. In Sections 3 and 4 we describe $\operatorname{End} \mathbb{V}(m, h, \alpha)$ fully when the regulator has one repeated root and two distinct roots, respectively. When the regulator has no roots, the algebra $\operatorname{End} \mathbb{V}(m, h, \alpha)$ is a domain with subtle possibilities, see [17, 20]. We content ourselves in Section 5 by showing in this last case that the algebra $\operatorname{End} \mathbb{V}(m, h, \alpha)$ has zero radical.

The A_h-Module of Endomorphism Parameters: $\mathcal{P}(m, h, \alpha)$

It follows from Proposition 2.2 that $\mathbb{V}(m, h, \alpha)$ has non-trivial endomorphisms if and only if it has one of the form $tD + \partial_\alpha(t)I$ for some non-zero t in R_h. Any t in R_h that causes $tD + \partial_\alpha(t)I$ to be an endomorphism will be called an *endomorphism parameter* of $\mathbb{V}(m, h, \alpha)$. The set of endomorphism parameters is a subspace of R_h, and we shall denote it by $\mathcal{P}(m, h, \alpha)$. We recall that R_h is an A_h-module using multiplication of rational functions. Next we show that $\mathcal{P}(m, h, \alpha)$ is an A_h-submodule of R_h.

Proposition 2.3 *The space $\mathcal{P}(m, h, \alpha)$ of endomorphism parameters is a module over A_h.*

Proof The proof is routine but requires close and repeated attention to the definition of the space $V(m, h, \alpha)$ given in (1.11).

If $t \in \mathcal{P}(m, h, \alpha)$ and $s \in A_h$, we need to show that ts is in $\mathcal{P}(m, h, \alpha)$. That is, assuming the matrix

$$\varphi = tD + \partial_\alpha(t)I = \begin{bmatrix} tp + \partial_\alpha(t) & -t \\ tq & \partial_\alpha(t) \end{bmatrix}$$

is an endomorphism, we need to prove the matrix

$$\psi = tsD + \partial_\alpha(ts)I = \begin{bmatrix} tsp + \partial_\alpha(ts) & -ts \\ tsq & \partial_\alpha(ts) \end{bmatrix}$$

is an endomorphism.

An endomorphism is a matrix that leaves $V(m, h, \alpha)$ invariant. Accordingly we have $\varphi V(m, h, \alpha) \subseteq V(m, h, \alpha)$ and we need $\psi V(m, h, \alpha) \subseteq V(m, h, \alpha)$. Given the definition of $V(m, h, \alpha)$ it is enough to check that

$$\psi \begin{pmatrix} 0 \\ g \end{pmatrix} \in V(m, h, \alpha) \quad \text{for all } g \text{ in } P_m, \tag{2.4}$$

and also

$$\psi \begin{pmatrix} r \\ -\partial_\alpha(r) \end{pmatrix} \in V(m, h, \alpha) \quad \text{for all } r \text{ in } R_h. \tag{2.5}$$

By direct calculation

$$\psi\begin{pmatrix}0\\ g\end{pmatrix}=\begin{pmatrix}-tsg\\ \partial_\alpha(ts)g\end{pmatrix}.$$

Since φ leaves $V(m,h,\alpha)$ invariant,

$$\varphi\begin{pmatrix}0\\ g\end{pmatrix}=\begin{pmatrix}-tg\\ \partial_\alpha(t)g\end{pmatrix}\in V(m,h,\alpha).$$

Thus $-tg\in R_h$, and using the fact $s\in A_h$, we get $-tsg\in R_h$. Furthermore by (1.4) and then (1.7):

$$\partial_\alpha(-tsg)+\partial_\alpha(ts)g=-\partial_\alpha(ts)g-\partial_{\alpha*ts}(g)+\partial_\alpha(ts)g=-\partial_{\alpha*ts}(g)\in P_m.$$

In light of definition (1.11) the information just derived yields (2.4).

Since

$$\psi\begin{pmatrix}r\\ -\partial_\alpha(r)\end{pmatrix}=\begin{pmatrix}tsrp+\partial_\alpha(ts)r+ts\partial_\alpha(r)\\ tsrq-\partial_\alpha(ts)\partial_\alpha(r)\end{pmatrix},$$

the requirement (2.5) comes down to proving

$$tsrp+\partial_\alpha(ts)r+ts\partial_\alpha(r)\in R_h, \tag{2.6}$$

as well as

$$\partial_\alpha(tsrp+\partial_\alpha(ts)r+ts\partial_\alpha(r))+tsrq-\partial_\alpha(ts)\partial_\alpha(r)\in P_m \tag{2.7}$$

for each r in R_h.

To work on (2.6), recall that φ leaves $V(m,h,\alpha)$ invariant, and so

$$\varphi\begin{pmatrix}r\\ -\partial_\alpha(r)\end{pmatrix}=\begin{pmatrix}trp+\partial_\alpha(t)r+t\partial_\alpha(r)\\ trq-\partial_\alpha(t)\partial_\alpha(r)\end{pmatrix}\in V(m,h,\alpha).$$

Definition (1.11) then yields $trp+\partial_\alpha(t)r+t\partial_\alpha(r)\in R_h$. Using (1.4) we have

$$\begin{aligned}tsrp+\partial_\alpha(ts)r+ts\partial_\alpha(r)&=tsrp+\partial_\alpha(t)sr+\partial_{\alpha*t}(s)r+ts\partial_\alpha(r)\\ &=s(trp+\partial_\alpha(t)r+t\partial_\alpha(r))+\partial_{\alpha*t}(s)r.\end{aligned}$$

Since $s\in A_h$ and derivers leave pole spaces invariant we get $\partial_{\alpha*t}(s)\in A_h$. Because r and $trp+\partial_\alpha(t)r+t\partial_\alpha(r)$ belong to the A_h-module R_h, the combination

$$s(tpr+\partial_\alpha(t)r+t\partial_\alpha(r))+\partial_{\alpha*t}(s)r \text{ is in } R_h.$$

From the calculation preceding just above we get (2.6).

To work on (2.7) notice that if $r\in R_h$, then $sr\in R_h$ because $s\in A_h$. Hence $\left(\begin{smallmatrix}sr\\ -\partial_\alpha(sr)\end{smallmatrix}\right)\in V(m,h,\alpha)$. Since φ leaves $V(m,h,\alpha)$ invariant, a direct calculation reveals that

$$\varphi\begin{pmatrix}sr\\ -\partial_\alpha(sr)\end{pmatrix}=\begin{pmatrix}tsrp+\partial_\alpha(t)sr+t\partial_\alpha(sr)\\ tsrq-\partial_\alpha(t)\partial_\alpha(sr)\end{pmatrix}\in V(m,h,\alpha).$$

From (1.11) this gives

$$\partial_\alpha(tsrp + \partial_\alpha(t)sr + t\partial_\alpha(sr)) + tsrq - \partial_\alpha(t)\partial_\alpha(sr) \in P_m.$$

An inspection of the identity in Lemma 1.2 gives

$$\begin{aligned}\partial_\alpha(tsrp + \partial_\alpha(ts)r + ts\partial_\alpha(r)) + tsrq - \partial_\alpha(ts)\partial_\alpha(r)&\\ = \partial_\alpha(tsrp + \partial_\alpha(t)sr + t\partial_\alpha(sr)) &+ tsrq - \partial_\alpha(t)\partial_\alpha(sr).\end{aligned}$$

Thus (2.7) follows as desired. ■

We shall also refer to $\mathcal{P}(m, h, \alpha)$ as the *parameter module*.

Arranging for the Generic Matrix to Be an Endomorphism

A desirable situation would be to have D itself be an endomorphism. This need not happen. However, we now check that if $\mathbb{V}(m, h, \alpha)$ has non-trivial endomorphisms, then $\mathbb{V}(m, h, \alpha)$ has an isomorphic copy whose generic matrix is an endomorphism of the copy.

An isomorphism between $\mathbb{V}(m, h, \alpha)$ and another $\mathbb{V}(m, \ell, \beta)$ comes down to a $K(X)$-linear operator on $K(X)^2$ that restricts to a K-linear isomorphism between the space $V(m, h, \alpha)$ and the space $V(m, \ell, \beta)$. Next we show how equivalent functional-height pairs lead to isomorphic Kronecker modules.

Proposition 2.4 *If $t \in R_h$ and $t \neq 0$, then the matrix*

$$\psi = \begin{bmatrix} t & 0 \\ -\partial_\alpha(t) & 1 \end{bmatrix} \tag{2.8}$$

implements an isomorphism from $\mathbb{V}(m, h - \operatorname{ord}(t), \alpha * t)$ to $\mathbb{V}(m, h, \alpha)$.

Proof Let $\binom{u}{v} \in V(m, h - \operatorname{ord}(t), \alpha * t)$. Since

$$\psi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} t & 0 \\ -\partial_\alpha(t) & 1 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} tu \\ v - \partial_\alpha(t)u \end{pmatrix},$$

we must prove that

$$\begin{pmatrix} tu \\ v - \partial_\alpha(t)u \end{pmatrix} \in V(m, h, \alpha).$$

By (1.11) we have $u \in R_{h-\operatorname{ord}(t)}$ and $\partial_{\alpha*t}(u) + v \in P_m$, and by (1.11) we want

$$tu \in R_h \quad \text{and} \quad \partial_\alpha(tu) + v - \partial_\alpha(t)u \in P_m.$$

The statement $u \in R_{h-\operatorname{ord}(t)}$ gives $\operatorname{ord}_\theta(u) \leq h(\theta) - \operatorname{ord}_\theta(t)$ for all θ in $\mathrm{K} \cup \{\infty\}$. Thus $\operatorname{ord}_\theta(tu) = \operatorname{ord}_\theta(t) + \operatorname{ord}_\theta(u) \leq h(\theta)$, which means $tu \in R_h$. Next using (1.4) we get

$$\partial_\alpha(tu) + v - \partial_\alpha(t)u = \partial_\alpha(t)u + \partial_{\alpha*t}(u) + v - \partial_\alpha(t)u = \partial_{\alpha*t}(u) + v \in P_m.$$

Therefore ψ maps $V(m, h - \operatorname{ord}(t), \alpha * t)$ into $V(m, h, \alpha)$.

The deriver property (1.4) along with (1.5) show that

$$0 = \partial_\alpha(1) = \partial_\alpha(tt^{-1}) = t^{-1}\partial_\alpha(t) + \partial_{\alpha * t}(t^{-1}),$$

and thus $t^{-1}\partial_\alpha(t) = -\partial_{\alpha*t}(t^{-1})$. Therefore the inverse of ψ is the matrix

$$\psi^{-1} = \begin{bmatrix} t^{-1} & 0 \\ t^{-1}\partial_\alpha(t) & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & 0 \\ -\partial_{\alpha*t}(t^{-1}) & 1 \end{bmatrix}.$$

Using this understanding of ψ^{-1} we can verify, as was done with ψ, that ψ^{-1} maps $V(m, h, \alpha)$ back into $V(m, h - \operatorname{ord}(t), \alpha * t)$. ■

We remind ourselves from Proposition 2.2 that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is trivial if and only if $\mathbb{V}(m, h, \alpha)$ has 0 as its only endomorphism parameter.

Proposition 2.5 *For a given $\mathbb{V}(m, h, \alpha)$ suppose that t is a non-zero function in the parameter module yielding the endomorphism $\varphi = tD + \partial_\alpha(t)I$. If ψ is the matrix given in* (2.8), *and*

$$E = \begin{bmatrix} \operatorname{trace}\varphi & -1 \\ \det\varphi & 0 \end{bmatrix},$$

then

- *E is the generic matrix for the pair $(h - \operatorname{ord}(t), \alpha * t)$ which is regulated by the polynomial $Y^2 + \operatorname{trace}\varphi Y + \det\varphi$,*
- $\operatorname{End}\mathbb{V}(m, h - \operatorname{ord}(t), \alpha * t) = \psi^{-1}(\operatorname{End}\mathbb{V}(m, h, \alpha))\psi$,
- *$E = \psi^{-1}\varphi\psi$ and E is an endomorphism of $\mathbb{V}(m, h - \operatorname{ord}(t), \alpha * t)$.*

Proof By straightforward calculation we see that $\operatorname{trace}\varphi = pt + 2\partial_\alpha(t)$ and $\det\varphi = qt^2 + pt\partial_\alpha(t) + \partial_\alpha(t)^2$. According to Proposition 1.3, the regulator of $(h - \operatorname{ord}(t), \alpha * t)$ is

$$\begin{aligned} t^2\Big[\Big(\frac{Y + \partial_\alpha(t)}{t}\Big)^2 + p\Big(\frac{Y + \partial_\alpha(t)}{t}\Big) + q\Big] \\ &= Y^2 + (pt + 2\partial_\alpha(t))Y + qt^2 + pt\partial_\alpha(t) + \partial_\alpha(t)^2 \\ &= Y^2 + \operatorname{trace}\varphi Y + \det\varphi \end{aligned}$$

as claimed above. The conjugacy of the endomorphism algebras follows because ψ maps the space $V(m, h - \operatorname{ord}(t), \alpha * t)$ isomorphically onto $V(m, h, \alpha)$ as in Proposition 2.4. The third fact is seen by performing the matrix multiplications indicated. ■

The import of Proposition 2.5 is that if a module $\mathbb{V}(m, h, \alpha)$ is commutative with non-trivial endomorphisms, then $\mathbb{V}(m, h, \alpha)$ is isomorphic to some module $\mathbb{V}(m, \ell, \beta)$ where the generic matrix for (ℓ, β) is a non-trivial endomorphism of $\mathbb{V}(m, \ell, \beta)$. Thus, for the purposes of studying the structure of $\operatorname{End}\mathbb{V}(m, h, \alpha)$, when this algebra is commutative and non-trivial

it does no harm to suppose that the generic matrix D of (h, α), as given in (1.15), *is an endomorphism of $\mathbb{V}(m, h, \alpha)$.*

An Advantage When D Is an Endomorphism

To say that D is an endomorphism means that 1 is an endomorphism parameter. In that case Proposition 2.3 forces $A_h \le \mathcal{P}(m,h,\alpha) \le R_h$. Immediately we get a significant family of known endomorphisms.

Here we record some useful constraints on D's that are endomorphisms.

Proposition 2.6 *If $Y^2 + pY + q$ regulates (h,α) and the generic matrix D is an endomorphism of $\mathbb{V}(m,h,\alpha)$, then $p \in A_h$, $q \in A_h$, and $\partial_\alpha(p) + q \in K$.*

Proof Since $p = \operatorname{trace} D$ and $q = \det D$, Proposition 2.1 gives that p, q are in A_h. The square of D is an endomorphism, and $D^2 = pD - qI$. By Proposition 2.2 it follows that $-q = \partial_\alpha(p) + \mu$ for some scalar μ, so that $\partial_\alpha(p) + q$ is a scalar. ■

3 When the Regulator Has One Repeated Root

In this section we obtain the structure of $\operatorname{End}\mathbb{V}(m,h,\alpha)$ when the regulator of (h,α) is quadratic with a repeated root in $K(X)$. Recall that $\mathcal{P}(m,h,\alpha)$ is the parameter module.

Proposition 3.1 *If the regulator of (h,α) is quadratic with repeated root r in $K(X)$ and the generic matrix D is an endomorphism of $\mathbb{V}(m,h,\alpha)$, then $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is isomorphic to the trivial extension algebra $K \ltimes \mathcal{P}(m,h,\alpha)$.*

Proof By (1.14) and (1.15) the regulator and generic matrix in this case are

$$(Y-r)^2 = Y^2 - 2rY + r^2 \quad \text{and} \quad D = \begin{bmatrix} -2r & -1 \\ r^2 & 0 \end{bmatrix}.$$

By Proposition 2.6, $\partial_\alpha(-2r) + r^2 \in K$. If r had a pole at some θ in $\mathrm{K} \cup \{\infty\}$, then

$$\operatorname{ord}_\theta(r^2) = 2\operatorname{ord}_\theta(r) > \operatorname{ord}_\theta(r) = \operatorname{ord}_\theta(-2r) \ge \operatorname{ord}_\theta(\partial_\alpha(-2r)),$$

which shows that $\partial_\alpha(-2r) + r^2$ would still have a pole at θ in contradiction to the fact this is a scalar. Having no poles, r must be a scalar.

Since r is a scalar, the matrix

$$D + rI = \begin{bmatrix} -r & -1 \\ r^2 & r \end{bmatrix}$$

is an endomorphism, and $(D + rI)^2 = 0$ by just checking. Hence the endomorphism algebra contains a nilpotent of index 2.

By Proposition 2.2, every endomorphism takes the form

$$tD + \partial_\alpha(t)I + \mu I = t(D + rI) + (\partial_\alpha(t) - rt)I + \mu I, \tag{3.1}$$

where $t \in \mathcal{P}(m,h,\alpha)$ and $\mu \in K$. We will show that

$$\partial_\alpha(t) - rt \in K. \tag{3.2}$$

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After that we will be done, because every endomorphism then takes the form (3.1) of a scalar plus a multiple of a nilpotent of index 2 by an endomorphism parameter.

We have the endomorphism

$$tD + \partial_\alpha(t)I = \begin{bmatrix} -2tr + \partial_\alpha(t) & -t \\ tr^2 & \partial_\alpha(t) \end{bmatrix}.$$

Thus $\det(tD + \partial_\alpha(t)I) = (\partial_\alpha(t) - rt)^2$, and according to Proposition 2.1 this determinant lies in the pole algebra A_h. Consequently $\partial_\alpha(t) - rt \in A_h$ as well.

Because of (1.1) it is enough to prove (3.2) when t is in the spur S_h and when t is in the pole algebra A_h. Suppose the endomorphism parameter t is in S_h. We have just seen that $\partial_\alpha(t) - rt \in A_h$. On the other hand, the pole space S_h remains invariant under derivers, and it is certainly invariant under multiplication by the scalar r. Thus $\partial_\alpha(t) - rt \in S_h$, so that $\partial_\alpha(t) - rt \in A_h \cap S_h = K$.

To see (3.2) when $t \in A_h$, recall that the regulator is $(Y - r)^2$. Hence the operator $(\partial_\alpha - r)^2$ maps R_h to a finite-dimensional space. The pole algebra A_h is the sum of all spaces $K[X_\theta]$ where $\theta \in K$ and $h(\theta) = \infty$, plus possibly the space $K[X]$, should it happen that $h(\infty) = \infty$. For each such θ where $h(\theta) = \infty$, the operator $(\partial_\alpha - r)^2$ maps $K[X_\theta]$ to a finite-dimensional space. Thus $\ker(\partial_\alpha - r)$ restricted to $K[X_\theta]$ is infinite-dimensional. By Lemma 1.4, $\partial_\alpha - r$ maps $K[X_\theta]$ into P_r, which is K since r is a scalar. If it should happen that $h(\infty) = \infty$, then $\partial_\alpha - r$ maps $K[X]$ into K by the same reasoning. Since $\partial_\alpha - r$ maps spanning subspaces of A_h into K, this operator maps all of A_h into K as desired in (3.2). ∎

Next we remove the assumption that D be an endomorphism.

Theorem 3.2 *If the quadratic regulator of* (h, α) *has a repeated root, then*

$$\operatorname{End} \mathbb{V}(m, h, \alpha) \cong K \ltimes S \text{ for some } K\text{-linear space } S.$$

Proof If the endomorphism algebra is trivial, the zero space will do the job of S. If the endomorphism algebra is not trivial, then by Proposition 2.5, $\operatorname{End} \mathbb{V}(m, h, \alpha)$ is isomorphic to some other $\operatorname{End} \mathbb{V}(m, \ell, \beta)$ which contains the generic matrix of (ℓ, β) as an endomorphism. The regulator of (ℓ, β) as given by Proposition 1.3 still has but one repeated root. By Proposition 3.1, this latter algebra, and thereby the former one too, is isomorphic to $K \ltimes S$ for some non-zero space S. ∎

Concrete examples of such endomorphism algebras $K \ltimes S$ have been constructed in [18, Proposition 3.3]. Thus all endomorphism algebras that are realized by means of a regulator with repeated root are known.

4 When the Regulator Has Distinct Roots

We now propose to examine closely endomorphisms when (h, α) is regulated by

$$(Y + u)(Y + v) = Y^2 + (u + v)Y + uv,$$

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where u, v are distinct functions in $K(X)$. The generic matrix is

$$D = \begin{bmatrix} u+v & -1 \\ uv & 0 \end{bmatrix},$$

whose characteristic polynomial is $Y^2 - (u+v)Y + uv = (Y-u)(Y-v)$. Thus u and v are the distinct eigenvalues of D as an operator on $K(X)^2$, and we get diagonalization of D over $K(X)$. The $K(X)$-algebra $K(X)[D]$ consists of all matrices $rD + sI$ where $r, s \in K(X)$, and there is a $K(X)$-algebra isomorphism $\Lambda\colon K(X)[D] \to K(X) \times K(X)$ given by $rD + sI \mapsto (s + ur, s + vr)$. The K-algebra $\operatorname{End} \mathbb{V}(m, h, \alpha)$ sits inside $K(X)[D]$ as the algebra of matrices $tD + \partial_\alpha(t)I + \mu I$, where t runs over the parameter module and μ is any scalar. Thus the mapping Λ restricted to these endomorphisms is given as

$$\Lambda : tD + \partial_\alpha(t)I + \mu I \mapsto (\partial_\alpha(t) + ut, \partial_\alpha(t) + vt) + (\mu, \mu), \tag{4.1}$$

where t runs through the parameter module and μ is any scalar.

Our job in this section is to determine the image of $\operatorname{End} \mathbb{V}(m, h, \alpha)$ under Λ. Given (4.1), a clearer understanding of the parameter module $\mathcal{P}(m, h, \alpha)$ and its images under the operators $\partial_\alpha + u$ and $\partial_\alpha + v$ is needed.

By Proposition 2.5 it is harmless to assume that D is an endomorphism, and we do so throughout this section unless otherwise specified. Taking D as an endomorphism, we have noted before Proposition 2.6 the advantage that $A_h \subseteq \mathcal{P}(m, h, \alpha) \subseteq R_h$.

The Parameter Module as Fractional Ideal

Proposition 4.1 *The functions u and v lie in A_h and have no pole in common.*

Proof If θ in $\mathrm{K} \cup \{\infty\}$ is a common pole of u and v, then θ is a pole of uv. By Proposition 2.6, $\partial_\alpha(u+v) + uv \in K$, and thus $\operatorname{ord}_\theta(uv) = \operatorname{ord}_\theta(\partial_\alpha(u+v))$. From this we obtain

$$\begin{aligned} \operatorname{ord}_\theta(u) + \operatorname{ord}_\theta(v) &= \operatorname{ord}_\theta(uv) \\ &= \operatorname{ord}_\theta(\partial_\alpha(u+v)) = \operatorname{ord}_\theta(\partial_\alpha(u) + \partial_\alpha(v)) \\ &\leq \max\{\operatorname{ord}_\theta(\partial_\alpha(u)), \operatorname{ord}_\theta(\partial_\alpha(v))\} \\ &\leq \max\{\operatorname{ord}_\theta(u), \operatorname{ord}_\theta(v)\} \end{aligned}$$

using (1.6) and (1.7). As the above contradicts the fact that the sum of two positive integers exceeds their maximum, u and v cannot share a pole. Furthermore, Proposition 2.6 says that $u + v \in A_h$. Since u and v share no poles, both u and v must be in A_h. ■

Pairs of functions u, v satisfying the properties of Proposition 4.1 arose in [19] under the name of *detached* functions. There, using quite different arguments, we showed that any quadratic regulator must have detached roots for the case where $R_h = K(X)$.

Proposition 4.2 *If t is in the parameter module $\mathcal{P}(m, h, \alpha)$ of $\mathbb{V}(m, h, \alpha)$, then*

$$\partial_\alpha(t) + ut \in A_h, \quad \partial_\alpha(t) + vt \in A_h \quad \text{and} \quad (u - v)t \in A_h.$$

Proof Since derivers leave pole spaces invariant, $\partial_\alpha(t) \in R_h$ for every t in R_h. Because u and v are in A_h as shown in Proposition 4.1, ut and vt lie in R_h. Hence $\partial_\alpha(t) + ut$ and $\partial_\alpha(t) + vt$ are in R_h for all t in R_h. As t runs over the parameter module and μ runs over K, the embedding Λ given in (4.1) causes $\partial_\alpha(t) + ut + \mu$ and $\partial_\alpha(t) + vt + \mu$ each to exhaust a respective K-algebra inside R_h. Since A_h is the algebra inside R_h that contains all K-algebras inside R_h, we conclude that $\partial_\alpha(t) + ut + \mu \in A_h$ and $\partial_\alpha(t) + vt + \mu \in A_h$. After that the scalar μ can be dropped.

By subtracting $\partial_\alpha(t) + vt$ from $\partial_\alpha(t) + ut$, we get $(u - v)t \in A_h$. ■

It follows from Proposition 4.2 that Λ in (4.1) embeds $\operatorname{End}\mathbb{V}(m, h, \alpha)$ into $A_h \times A_h$. A worthwhile insight into $\mathcal{P}(m, h, \alpha)$ emerges as follows.

Proposition 4.3 *The parameter module $\mathcal{P}(m, h, \alpha)$ is a fractional ideal of A_h taking the form $\mathcal{P}(m, h, \alpha) = \frac{w}{u-v}A_h$ for some non-zero w in A_h.*

Proof We see from Proposition 4.2 that the A_h-module $(u - v)\mathcal{P}(m, h, \alpha)$ is inside A_h, and is thus an ideal of A_h. Because all ideals of A_h are principal, we obtain $(u - v)\mathcal{P}(m, h, \alpha) = wA_h$ for some w in A_h. Now just divide by $u - v$. ■

Since D is an endomorphism, we can record as well that

$$A_h \subseteq \mathcal{P}(m, h, \alpha) \subseteq \frac{1}{u - v}A_h. \tag{4.2}$$

The Components of A_h Regulated by the Factors of the Regulator

The family of pole subalgebras C inside A_h for which $(\partial_\alpha + u)C \subseteq P_u$ certainly has K in it, and it forms a complete lattice. Thus we can define C_u to be the biggest pole subalgebra of A_h such that $(\partial_\alpha + u)C_u \subseteq P_u$. Likewise define C_v to be the biggest pole subalgebra of A_h such that $(\partial_\alpha + v)C_v \subseteq P_v$. We prove now that C_u and C_v decompose A_h in the sense that

$$C_u + C_v = A_h \quad \text{and} \quad C_u \cap C_v = K. \tag{4.3}$$

To see that $C_u \cap C_v = K$, suppose on the contrary that the pole algebra $C_u \cap C_v$ properly contains K. In that case $C_u \cap C_v$ is infinite-dimensional, and both $\partial_\alpha + u$ and $\partial_\alpha + v$ have finite rank on $C_u \cap C_v$. If ℓ is the height function that defines $C_u \cap C_v$, we see from the definition of regulators that both $Y + u$ and $Y + v$ regulate (ℓ, α). This is impossible because $u \neq v$.

The fact that $C_u + C_v = A_h$ follows directly from the next result which uses Lemma 1.4.

Proposition 4.4 *If $K[X_\theta] \subseteq A_h$, then either $K[X_\theta] \subseteq C_u$ or $K[X_\theta] \subseteq C_v$. Likewise, if $K[X] \subseteq A_h$, then either $K[X] \subseteq C_u$ or $K[X] \subseteq C_v$.*

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Proof Since $(Y + u)(Y + v)$ is the regulator, $(\partial_\alpha + u) \circ (\partial_\alpha + v)$ has finite rank on R_h. Thus for any θ in K where $K[X_\theta] \subseteq A_h$, the operator $(\partial_\alpha + u) \circ (\partial_\alpha + v)$ maps $K[X_\theta]$ to a finite-dimensional space. Consequently one of $\ker(\partial_\alpha + u)$ or $\ker(\partial_\alpha + v)$ restricted to $K[X_\theta]$ is infinite-dimensional. Indeed, if $\ker(\partial_\alpha + v)$ restricted to $K[X_\theta]$ is finite-dimensional, then the image $(\partial_\alpha + v)K[X_\theta]$ is infinite-dimensional. Since derivers leave pole spaces invariant, this image is inside the finite-dimensional extension $K[X_\theta] + P_v$ of $K[X_\theta]$. Thus $(\partial_\alpha + v)K[X_\theta]$ has an infinite-dimensional intersection with $K[X_\theta]$. Because $\partial_\alpha + u$ has to map this intersection to a finite-dimensional space, the restriction of $\partial_\alpha + u$ to $K[X_\theta]$ must have infinite-dimensional kernel. After that, Lemma 1.4 shows that either $\partial_\alpha + u$ maps $K[X_\theta]$ into P_u, or $\partial_\alpha + v$ maps $K[X_\theta]$ into P_v. In other words either $K[X_\theta] \subseteq C_u$ or $K[X_\theta] \subseteq C_v$.

To prove the statement in case $K[X] \subseteq R_h$, replace X_θ by X. ■

Since $\partial_\alpha + u, \partial_\alpha + v$ have finite rank on C_u, C_v, respectively we shall say that

C_u and C_v are the *components of A_h regulated by $Y + u$ and $Y + v$, respectively*.

Proposition 4.5 *The following crossover inclusions hold.*

$$P_v \subseteq C_u \quad \textit{and} \quad P_u \subseteq C_v. \tag{4.4}$$

Proof We work on $P_v \subseteq C_u$. By Proposition 4.1, $v \in A_h$ so that $K[X_\theta] \subseteq A_h$ for every pole θ of v. For every such θ and for every non-zero t in $K[X_\theta]$ we use the fact that derivers do not increase the order of poles to get

$$\operatorname{ord}_\theta(vt) = \operatorname{ord}_\theta(v) + \operatorname{ord}_\theta(t) > \operatorname{ord}_\theta(t) \geq \operatorname{ord}_\theta \partial_\alpha(t).$$

Since vt and $\partial_\alpha(t)$ have different order at θ, they cannot add up to 0. Hence $\partial_\alpha + v$ is injective on $K[X_\theta]$. Thus $K[X_\theta]$ cannot sit inside C_v. By Proposition 4.4, $K[X_\theta] \subseteq C_u$. The same argument applies using $K[X]$, if ∞ is a pole of v. Thus $P_v \subseteq C_u$. Similarly $P_u \subseteq C_v$. ■

Next we look at images of $\partial_\alpha + u$ and $\partial_\alpha + v$ acting on the portion of the parameter module $\mathcal{P}(m, h, \alpha)$ that sits in the spur S_h.

Proposition 4.6 *The following inclusions hold:*

$$(\partial_\alpha + u)(\mathcal{P}(m, h, \alpha) \cap S_h) \subseteq P_u, \quad (\partial_\alpha + v)(\mathcal{P}(m, h, \alpha) \cap S_h) \subseteq P_v. \tag{4.5}$$

Proof Let $t \in \mathcal{P}(m, h, \alpha) \cap S_h$. Since derivers leave pole spaces invariant,

$$\partial_\alpha(t) + ut \in S_h + P_{ut} = S_h + P_u + P_t = S_h + P_u.$$

The first equality above holds because u in A_h has no pole in common with t in S_h. The second equality follows because $t \in S_h$, so that $P_t \subseteq S_h$. Thus $\partial_\alpha(t)+ut \in S_h+P_u$. However $\partial_\alpha(t) + ut \in A_h$, as seen from Proposition 4.2. Since $P_u \subseteq A_h$, the modular law, along with the fact that $S_h \cap A_h = K$, gives

$$\partial_\alpha(t) + ut \in (S_h + P_u) \cap A_h = (S_h \cap A_h) + P_u = K + P_u = P_u.$$

Likewise $\partial_\alpha(t) + vt \in P_v$. ■

Proposition 4.7 *The operator $\partial_\alpha + u$ maps the parameter module $\mathcal{P}(m, h, \alpha)$ into C_v, and $\partial_\alpha + v$ maps $\mathcal{P}(m, h, \alpha)$ into C_u.*

Proof Recall that $A_h \subseteq \mathcal{P}(m, h, \alpha) \subseteq R_h$. Since $R_h = A_h + S_h$, we get using (4.3)

$$\mathcal{P}(m, h, \alpha) = A_h + \mathcal{P}(m, h, \alpha) \cap S_h = C_u + C_v + \mathcal{P}(m, h, \alpha) \cap S_h.$$

By (4.5) and (4.4) we have $(\partial_\alpha + u)(\mathcal{P}(m, h, \alpha) \cap S_h) \subseteq C_v$. By the definition of C_u and (4.4) we get $(\partial_\alpha + u)C_u \subseteq C_v$. If $t \in C_v$, then $\partial_\alpha(t) \in C_v$ because derivers leave pole spaces invariant. Also $ut \in C_v$ because $u \in C_v$ according to (4.4). Thus $(\partial_\alpha + u)C_v \subseteq C_v$, too. Therefore $\partial_\alpha + u$ maps all of $\mathcal{P}(m, h, \alpha)$ into C_v. Similarly $\partial_\alpha + v$ maps $\mathcal{P}(m, h, \alpha)$ into C_u. ■

The next result follows instantly from Proposition 4.7.

Proposition 4.8 *The embedding Λ of* (4.1) *sends* $\operatorname{End} \mathbb{V}(m, h, \alpha)$ *into $C_v \times C_u$.*

What remains is to capture the image of Λ inside $C_v \times C_u$.

Bridges Across a Pole Algebra

Take any pole algebra A and two pole subalgebras L and M that are complementary in A in the sense that $L + M = A$ and $L \cap M = K$. For any z in A define

$$L \times_z M = \{(r, s) \in L \times M : r - s \in zA\}. \tag{4.6}$$

It is evident that $L \times_z M$ is a K-subalgebra of the product $L \times M$. We call such a construction a *bridge across A*. This is a special case of what is sometimes known as a *fibre product* in the literature. For example, taking the component C_v as L, and C_u as M, and $z = w$ from Proposition 4.3, we get a good scenario for forming bridges. Our objective, when the regulator of (h, α) has distinct roots, is to show that the endomorphisms form the bridge $C_v \times_w C_u$ across A_h. Before getting to that, let us record some routine aspects of bridges.

Proposition 4.9 *Take a bridge $L \times_z M$ across a pole algebra A.*

(i) *$L \times_z M \cong K$ if and only if $z = 0$.*
(ii) *$L \times_z M$ has proper idempotents if and only if z is a unit in A.*
(iii) *$L \times_z M$ is an integral domain if and only if either $z = 0$ or at least one of L, M equals K with z a non-unit of A.*
(iv) *The bridge $L \times_z M$ is an integral domain if and only if z is a non-unit of A and $L \times_z M$ is isomorphic to the subalgebra $K + zA$ of A.*
(v) *When z is a non-zero, non-unit of A, the domain $K + zA$ is all of A if and only if z is a prime in A.*

Proof (i) If $z = 0$ and $(r, s) \in L \times_z M$ we get $r = s$, and since $L \cap M = K$ we see that $(r, s) \in K(1, 1)$. Conversely, if $z \neq 0$, use the fact $A = L + M$ to write $z = r - s$ for some r in L and some s in M. Then $(r, s) \in L \times_z M$, but it is not in $K(1, 1)$.

(ii) If z is a unit, then $zA = A$ and any pair (r, s) in $L \times M$ satisfies $r - s \in A$. Thus $L \times_z M = L \times M$ which is clearly an algebra with proper idempotents. Conversely, if

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$L \times_z M$ has proper idempotents, then they would have to be $(1,0)$ and $(0,1)$. In that case $1 = 1 - 0 \in zA$, and z must be a unit.

(iii) The case $z = 0$ is handled by (i).

Suppose $L = K$ and z is a non-unit of A. In that case $M = A$ and the bridge becomes $K \times_z A$. Now $(r,s) \in K \times_z A$ if and only if r is a scalar and $s \in r + zA$. Hence the map $(r,s) \mapsto s$ from $K \times_z A$ is onto $K + zA$. Since z is a non-unit and r is a scalar it is easy to see this map is injective. Hence $L \times_z M$ is isomorphic to the domain $K + zA$.

Now for the converse. If z is a unit, we do not get a domain because (ii) yields idempotents. If both L and M contain K properly, then both

$$L \cap zA \neq (0) \quad \text{and} \quad M \cap zA \neq (0).$$

This is because the ideal zA has finite-codimension in A while the pole algebras L and M are infinite-dimensional when they properly contain K. Hence there exist a non-zero r in L and non-zero s in M such that $(r,0) \in L \times_z M$ and $(0,s) \in L \times_z M$. Now the bridge $L \times_z M$ has ample amounts of zero divisors when both L and M properly contain K.

(iv) One direction is trivial and the other has been shown already in (iii).

(v) Since K is algebraically closed, our non-zero, non-unit z is a prime in A if and only if zA has co-dimension one in A. That is, if and only if $K + zA = A$. ■

Endomorphisms Make Bridges

Recall the components C_v, C_u of A_h which are regulated by the linear factors of $(Y + u)(Y + v)$ and the element w in A_h which specifies the parameter module $\mathcal{P}(m,h,\alpha)$ in Proposition 4.3.

Proposition 4.10 *The mapping* Λ *of* (4.1) *embeds* $\operatorname{End}\mathbb{V}(m,h,\alpha)$ *into* $C_v \times_w C_u$.

Proof Proposition 4.8 puts $\operatorname{End}\mathbb{V}(m,h,\alpha)$ into $C_v \times C_u$. Now if

$$(r,s) = (\partial_\alpha(t) + ut + \mu, \partial_\alpha(t) + vt + \mu)$$

for some t in $\mathcal{P}(m,h,\alpha)$ and some μ in K, then Propositions 4.3 yields

$$r - s \in (u - v)\mathcal{P}(m,h,\alpha) = (u - v)\frac{w}{u - v}A_h = wA_h.$$

Hence Λ embeds $\operatorname{End}\mathbb{V}(m,h,\alpha)$ into $C_v \times_w C_u$. ■

The matter of having Λ be onto $C_v \times_w C_u$ remains.

Proposition 4.11 *Suppose* $(r,s) \in C_v \times_w C_u$. *If* $t = \frac{r-s}{u-v}$, *then* $t \in \mathcal{P}(m,h,\alpha)$ *and* $\partial_\alpha(t) + ut - r = \partial_\alpha(t) + vt - s \in K$.

Proof By the definition of a bridge, $r - s \in wA_h$. From Proposition 4.3

$$t = \frac{r - s}{u - v} \in \frac{w}{u - v}A_h = \mathcal{P}(m,h,\alpha).$$

Equality of $\partial_\alpha(t) + ut - r$ and $\partial_\alpha(t) + vt - s$ follows directly from the choice of t. So it remains to see that this common function is a scalar. Since $t \in \mathcal{P}(m, h, \alpha)$, Proposition 4.7 yields $\partial_\alpha(t) + ut \in C_v$ and $\partial_\alpha(t) + vt \in C_u$. Thus $\partial_\alpha(t) + ut - r \in C_v$ while its alter ego $\partial_\alpha(t) + vt - s \in C_u$. We deduce from (4.3) that this common function lies in $C_v \cap C_u = K$. ■

We put all the pieces together for the following result.

Proposition 4.12 *If the regulator of* (h, α) *is* $(Y + u)(Y + v)$ *with* u *and* v *distinct, and if the generic matrix* D *is an endomorphism of* $\mathbb{V}(m, h, \alpha)$*, then the algebra* $\operatorname{End}\mathbb{V}(m, h, \alpha)$ *is isomorphic to the bridge* $C_v \times_w C_u$ *across* A_h*.*

Proof The only thing left to check is that Λ maps $\operatorname{End}\mathbb{V}(m, h, \alpha)$ onto $C_v \times_w C_u$. Take $(r, s) \in C_v \times_w C_u$. According to Proposition 4.11, the functions $\partial_\alpha(t) + ut - r$ and $\partial_\alpha(t)+vt-s$ are equal to the same scalar, say λ. We claim that the endomorphism $tD + \partial_\alpha(t) - \lambda I$ is mapped to (r, s) under Λ. Since Λ maps this endomorphism to $(\partial_\alpha(t) + ut - \lambda, \partial_\alpha(t) + vt - \lambda)$, it suffices to check that $\partial_\alpha(t) + ut - \lambda = r$ and $\partial_\alpha(t) + vt - \lambda = s$. However, that is precisely what λ achieves. ■

The cumulative result coming up next does not assume the generic matrix is an endomorphism.

Theorem 4.13 *If the quadratic regulator of* (h, α) *has distinct roots, then the algebra* $\operatorname{End}\mathbb{V}(m, h, \alpha)$ *is isomorphic to a bridge across* A_h*.*

Proof The only thing stopping us from using Proposition 4.12 is the possibility that the generic matrix D might not be an endomorphism of $\mathbb{V}(m, h, \alpha)$. But, if $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is non-trivial, then Proposition 2.5 lets us replace $\mathbb{V}(m, h, \alpha)$ with an isomorphic $\mathbb{V}(m, \ell, \beta)$ whose generic matrix is an endomorphism of $\mathbb{V}(m, \ell, \beta)$. Then we can apply Proposition 4.12. If $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is trivially just K, we recall from Proposition 4.9 that a bridge formed by using $w = 0$ is also just K. So trivial endomorphism algebras are bridges, too. ■

Realizable Bridges

Having just seen that in the presence of a quadratic regulator with distinct roots endomorphism algebras are isomorphic to bridges, we come to the matter of deciding which bridges come up. Any K-algebra isomorphic to some $\operatorname{End}\mathbb{V}(m, h, \alpha)$ in which the regulator of (h, α) is quadratic with distinct roots will be called a *realizable bridge*. We now propose to identify the bridges that are realizable.

We say that a pole algebra is *big* when all but finitely many X_θ lie in it, and *small* when infinitely many X_θ lie outside of it. We shall also say that θ in K *supports* A_h when $X_\theta \in A_h$.

Theorem 4.14 *If* $L \times_z M$ *is a bridge across a pole algebra* A *with either* L *or* M *a small pole algebra properly containing* K*, and* z *in* A *is non-zero, then the bridge* $L \times_z M$ *is realizable.*

Proof We may as well suppose that L is small and properly contains K. Without losing generality we can also suppose $K[X] \subseteq L$, for if this did not hold, we would have $X_\eta \in L$ for some η in K. Using the field automorphism $\sigma\colon K(X) \to K(X)$ given by $f(X) \mapsto f(\frac{1}{X} + \eta)$, we get that $\sigma\colon X_\eta \mapsto X$. Clearly σ moves pole algebras to pole algebras, and thus $\sigma(L)$ is a pole algebra containing $K[X]$. Then the algebra isomorphism on $K(X) \times K(X)$ given by

$$(f(X), g(X)) \mapsto \Big(f\Big(\frac{1}{X} + \eta\Big), g\Big(\frac{1}{X} + \eta\Big)\Big).$$

restricts to an isomorphism $L \times_z M \cong \sigma(L) \times_{\sigma(z)} \sigma(M)$.

Having made the simplification that $K[X] \subseteq L$, we show next that we can suppose $z \in K[X]$ without loss of generality. Indeed, write $z = f/g$ where f, g are coprime polynomials. Since $z \in A$ and A is a pole algebra, $1/g \in A$, and since $K[X] \subseteq A$, it follows that g is a unit of A. Thus the condition $(r, s) \in zA$ in (4.6), telling us when a pair (r, s) from $L \times M$ lies in $L \times_z M$, is the same as the condition $(r, s) \in fA$. Hence we can suppose z to be a polynomial.

We can also reduce to the case where no root of z supports A. For if θ in K were a root of z and θ supported A, we could write $z = (X - \theta)r$ for some polynomial f, then observe $X - \theta$ is a unit of A to get $zA = fA$, and then $L \times_z M = L \times_f M$. In this way we can remove all roots of z that support A.

Our job has come down to realizing $L \times_z M$ where

(4.7) $\quad 0 \neq z \in K[X] \subseteq L$ with no root of z supporting A, and L is small.

We split the realization into two cases: $K = M$ and $K \subsetneq M$.

Suppose $K = M$, in which case $L = A$ with A small. We need (m, h, α) so that (h, α) has a quadratic regulator with distinct roots and $\operatorname{End}\mathbb{V}(m, h, \alpha) \cong A \times_z K$. Take m to be any positive integer. Since A is small, the set Γ of all θ in K that support A has an infinite complement in K. Pick Δ to be any infinite subset of K that is disjoint from both Γ and from the finite set of roots of z. Define the height function h according to

$$h(\theta) = \begin{cases} \infty & \text{if } \theta \in \Gamma \text{ or } \theta = \infty, \\ 1 & \text{if } \theta \in \Delta, \\ 0 & \text{if } \theta \in K \setminus (\Delta \cup \Gamma). \end{cases}$$

For this choice of h, the pole algebra A_h is our original small A, while the pole space is $R_h = A + \sum_{\theta \in \Delta} KX_\theta$. Choose α to be any functional such that $\alpha = 0$ on A and $\langle \alpha, X_\theta \rangle = z(\theta)$ for all θ in Δ.

To get the regulator of (h, α) we need the deriver on R_h. Since $\alpha = 0$ on A, clearly $\partial_\alpha = 0$ on A. For each θ in Δ formula (1.5) yields $\partial_\alpha(X_\theta) = -\langle \alpha, X_\theta \rangle X_\theta = -z(\theta)X_\theta$. Hence for all θ in Δ

$$(\partial_\alpha + z)(X_\theta) = (z - z(\theta))X_\theta = \frac{z - z(\theta)}{X - \theta}.$$

The latter is a polynomial of degree less than $\deg z$. Therefore $\partial_\alpha + z$ maps $\sum_{\theta \in \Delta} KX_\theta$ into a finite-dimensional space. Consequently $(\partial_\alpha + z) \circ \partial_\alpha$ maps R_h into a finite-dimensional space. Since $z \neq 0$, the regulator is $(Y + z)(Y + 0) = Y^2 + zY + 0$.

Theorem 4.13 says that End $\mathbb{V}(m, h, \alpha)$ is a bridge across A_h. We now verify that this endomorphism algebra is our desired $A \times_z K$.

The generic matrix for (h, α) is $D = \left[\begin{smallmatrix} z & -1 \\ 0 & 0 \end{smallmatrix}\right]$. A spanning set for the space $V(m, h, \alpha)$ as defined in (1.11) consists of the vectors

$$\begin{pmatrix} 0 \\ s \end{pmatrix} \text{ where } s \in P_m, \quad \begin{pmatrix} r \\ 0 \end{pmatrix} \text{ where } r \in A, \quad \begin{pmatrix} X_\theta \\ z(\theta)X_\theta \end{pmatrix} \text{ where } \theta \in \Delta.$$

One can check routinely by using (1.11) and our computation of ∂_α above that D maps this spanning set back into $V(m, h, \alpha)$. Thus D is an endomorphism. Proposition 4.12 becomes fully applicable to the current construction.

Specializing the notations of Proposition 4.12 to our current construction, we have $u = z$ and $v = 0$. To compute the components C_z and C_0 as defined prior to (4.3), observe that since $z \neq 0$, the operator $\partial_\alpha + z$ is injective on A_h, and that $\partial_\alpha + 0$ maps A_h to zero. This tells us that $C_z = K$ while $C_0 = A_h$. Thus Proposition 4.12 yields End $\mathbb{V}(m, h, \alpha) \cong A_h \times_w K$ for some w in A_h. To find that w, we need the parameter module $\mathcal{P}(m, h, \alpha)$ so that Proposition 4.3 can be used. We know that $\mathcal{P}(m, h, \alpha)$ lies inside R_h, and by (4.2) we have $A_h \subseteq \mathcal{P}(m, h, \alpha) \subseteq \frac{1}{z}A_h$. It follows that the poles of any endomorphism parameter must either support A_h or lie among the roots of z. However, R_h does not have functions whose poles are roots of z, by its very construction. Consequently $\mathcal{P}(m, h, \alpha) = A_h$. Since $A_h = \frac{z}{z-0}A_h$, Proposition 4.12 tells us that z itself is a suitable w. Therefore End $\mathbb{V}(m, h, \alpha) \cong A_h \times_z K = A \times_z K$ as desired for the first case.

Suppose in the second case that $K \subsetneq M$. Here we will not need the fact L is small, but if L were big, then M would be small because the conditions prior to (4.6) reveal that two big algebras cannot form a bridge. We are back in the situation of (4.7), and we want to realize $L \times_z M$. Again let m be any positive integer. For the height function h put

$$h(\theta) = \begin{cases} \infty & \text{if } \theta = \infty \text{ or } X_\theta \in A, \\ 0 & \text{for all other } \theta \text{ in K} \cup \{\infty\}. \end{cases}$$

We see that $R_h = A_h = A = L + M$.

In order to define the functional α we need a little digression. For each r in $K[X]$ and θ in K, put

$$r^\star = \frac{r - r(\theta)}{X - \theta} = (r - r(\theta))X_\theta.$$

Clearly $r^\star$ is again a polynomial of degree less than deg r. We may iterate this process to get polynomials r,$r^\star$, $r^{\star\star}$, $r^{\star\star\star}$ and in general the k-th iterate denoted by $r^{\star k}$. The degrees of the iterates are strictly decreasing until $r^{\star k} = 0$ when k is beyond deg r. These polynomials are useful to us because of the identity

$$\text{(4.8)} \qquad rX_\theta^n = r(\theta)X_\theta^n + r^\star(\theta)X_\theta^{n-1} + r^{\star\star}(\theta)X_\theta^{n-2} + \cdots + r^{\star(n-1)}X_\theta + r^{\star n},$$

for each $n = 1, 2, \ldots$. A routine inductive argument based on $rX_\theta^n = r(\theta)X_\theta^n + r^\star X_\theta^{n-1}$ verifies formula (4.8).

Now we define α. Let Δ be the set of θ in K that support M, which is non-empty since M properly contains K. We can observe that

$$A = R_h = L \oplus \sum_{\theta \in \Delta} \sum_{n=1}^{\infty} K X_\theta^n.$$

On L simply put $\alpha = 0$, and for each θ in Δ and each $n = 1, 2, \dots$, put

$$\langle \alpha, X_\theta^n \rangle = z^{\star(n-1)}(\theta).$$

It does not matter how α is defined on the rest of the standard basis of $K(X)$.

We now compute the deriver ∂_α on each of L and M, and thence the regulator of (h, α). Since $\alpha = 0$ on L, so also is $\partial_\alpha = 0$ on L. For each $\theta \in \Delta$ and each $n = 1, 2, \dots$, we use the deriver formula (1.5) to get

$$\begin{aligned} \partial_\alpha(X_\theta^n) &= -\langle \alpha, X_\theta \rangle X_\theta^n - \langle \alpha, X_\theta^2 \rangle X_\theta^{n-1} - \langle \alpha, X_\theta^3 \rangle X_\theta^{n-2} - \cdots - \langle \alpha, X_\theta^n \rangle X_\theta. \\ &= -z(\theta) X_\theta^n - z^\star(\theta) X_\theta^{n-1} - z^{\star\star}(\theta) X_\theta^{n-2} - \cdots - z^{\star(n-1)}(\theta) X_\theta. \end{aligned}$$

By formula (4.8) with $r = z$, we see that for each θ in Δ and each $n = 1, 2, \dots$,

$$(\partial_\alpha + z)(X_\theta^n) = z^{\star n}.$$

Since $z^{\star n}$ is a polynomial of degree at most $\deg z$, and since

$$M = K + \sum_{\theta \in \Delta} \sum_{n=1}^{\infty} K X_\theta^n,$$

the operator $\partial_\alpha + z$ maps the infinite-dimensional pole algebra M into the finite-dimensional pole space P_z. The operator $\partial_\alpha + 0$ maps L into the finite-dimensional pole space $P_0 = K$, in fact to 0. Consequently $(\partial_\alpha + z) \circ (\partial_\alpha + 0)$ maps all of R_h into a finite-dimensional space, actually into $P_z + P_0 = P_z$. After recalling that $z \neq 0$, we can see that the regulator of (h, α) is $(Y + z)(Y + 0) = Y^2 + zY + 0$.

The generic matrix for (h, α) is $D = \left[\begin{smallmatrix} z & -1 \\ 0 & 0 \end{smallmatrix}\right]$ To check that D is an endomorphism, observe that a spanning set for $V(m, h, \alpha)$ as defined in (1.11) consists of the vectors $\binom{0}{s}$ where $s \in P_m$, and $\binom{r}{-\partial_\alpha(r)}$ where $r \in A_h$. Then

$$D\begin{pmatrix} 0 \\ s \end{pmatrix} = \begin{pmatrix} -s \\ 0 \end{pmatrix} \quad \text{and} \quad D\begin{pmatrix} r \\ -\partial_\alpha(r) \end{pmatrix} = \begin{pmatrix} (\partial_\alpha + z)(r) \\ 0 \end{pmatrix}.$$

We check that the first entries of these outputs of D lie in L. Since ∂_α vanishes on L, it will follow that these outputs of D belong to $V(m, h, \alpha)$. Well, $s \in P_m \subseteq K[X] \subseteq L$. Also $A_h = A = L + M$. If $r \in L$, then $(\partial_\alpha + z)(r) = zr \in L$ since $z \in K[X] \subseteq L$. If $r \in M$, we have seen before that $(\partial_\alpha + z)(r) \in P_z \subseteq L$. Since D leaves $V(m, h, \alpha)$ invariant, D is an endomorphism, and Proposition 4.12 becomes applicable.

Specializing the notations of Proposition 4.12 to this construction we have $u = z$ and $v = 0$. As we have seen $\partial_\alpha + z$ maps M into P_z and $\partial_\alpha + 0$ maps L into P_0, in fact, into the zero space. Thus C_z and C_0 as defined prior to (4.3) become $C_z = M$ and $C_0 = L$. Proposition 4.12 yields $\operatorname{End} \mathbb{V}(m, h, \alpha) \cong L \times_w M$ for some w in A_h. To find that w we need the parameter module $\mathcal{P}(m, h, \alpha)$ so that Proposition 4.3 can be used. The parameter module contains A_h and lies inside R_h, but in our case these two coincide. Consequently $\mathcal{P}(m, h, \alpha) = A_h$. Since $A_h = \frac{z}{z-0} A_h$, Proposition 4.12 tells us that z itself is a suitable w. Therefore $\operatorname{End} \mathbb{V}(m, h, \alpha) \cong L \times_z M$, as desired for the second case. ∎

Realizing the Bridges K and $K \times K$

The bridges K and $K \times K$ are realizable but are not covered by the general construction of Theorem 4.14. Although it is not transparent, the realization of K using a quadratic regulator with distinct roots was carried out in [18, Proposition 3.6] and the remarks that follow it. In order to realize $K \times K$, take two disjoint, infinite subsets Δ and Γ of K. Use the height function h that gives the pole space $R_h = K \oplus \sum_{\theta \in \Delta} K X_\theta \oplus \sum_{\eta \in \Gamma} K X_\eta$. Take any functional α that equals 0 on the X_θ's and 1 on the X_ηs. The regulator of (h, α) becomes $(Y + 1)(Y + 0)$. The techniques used in the proof of Theorem 4.14 can be imitated to show that for $m = 1$, $\operatorname{End} \mathbb{V}(m, h, \alpha) \cong K \times K$.

A Realized Family of Affine Domains

We note that among the bridges realized through Theorem 4.14 are the domains $A \times_z K \cong K + zA$, where z is a non-zero, non-unit of a small pole algebra A. In particular, when $A = K[X]$ and $z \in K[X]$, the algebra $K + zK[X]$ is affine, as can be seen for example from [2]. When the degree of z is at least 3, these curves are singular, non-planar curves. For instance, as we learned[1], the affine rings $K + X^n K[X]$ must be generated with no fewer than n elements. This contrasts with the affine rings we realized in [16], which can be generated by three elements, as shown in [10].

Non-Realizable Bridges

The bridges not accounted for by Theorem 4.14 and the remarks that immediately follow it are $A \times_z K$ where A is big and $z \neq 0$. It turns out these are not realizable, and to prove it we focus on the ring isomorphism invariant of being *semi-local*, *i.e.,* having only finitely many maximal ideals. Using the reduction at the start of the proof of Theorem 4.14, it does no harm to suppose $K[X] \subseteq A$. From this it becomes clear that when A is big, it must be semi-local, having only the maximal ideals $(X - \theta)A$ where θ does not support A.

Proposition 4.15 *A bridge $L \times_z M$ across a pole algebra A is semi-local if and only if one of the following holds.*

(i) $z = 0$, *in which case* $L \times_z M \cong K$.
(ii) $z \neq 0$ *and* $L = M = A = K$, *which gives* $L \times_z M = K \times K$.

[1] Private communication, D. McKinnon.

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(iii) *$z \neq 0$, A is big and $L = A$, $M = K$ or vice versa.*

Proof The first two of the above possibilities clearly give a semi-local bridge. Suppose now that $L \times_z K$ satisfies condition (iii). Say $L = A$ and $M = K$. If z is a unit of A, the bridge becomes the product $A \times K$, which is clearly semi-local since A is semi-local. If z is not a unit of A, then $L \times_z M = A \times_z K \cong K + zA$. We shall check that this latter subring of A is semi-local. From here on the argument is likely to be well known.

First we observe that if $0 \neq r \in K + zA$ and $1/r \in A$, then $1/r$ is already in $K + zA$. Indeed, write $r = \lambda + zt$ where $\lambda \in K$ and $t \in A$. If $\lambda = 0$, then z is a unit of A which forces $K + zA = A$, leaving us with a vacuous observation. If $\lambda \neq 0$, there is the identity

$$\frac{1}{r} = \frac{1}{\lambda} - z\frac{t}{\lambda r},$$

which clearly puts $1/r$ in $K + zA$. Thus the units of $K + zA$ are those elements of $K + zA$ that are units in A.

If J_1, J_2 are distinct maximal ideals of $K + zA$, there exist x_1 in J_1 and x_2 in J_2 such that $x_1 + x_2 = 1$. As x_1, x_2 are non-units of $K + zA$, they remain non-units of A. Hence there exist maximal ideals I_1, I_2 of A such that $x_1 \in I_1$ and $x_2 \in I_2$. Since $x_1 + x_2 = 1$, these maximal ideals of A must be distinct. Thus distinct maximal ideals of $K + zA$ breed distinct maximal ideals of A. It follows that since A has only finitely many maximal ideals so also does $K + zA$.

For the converse, supposing that conditions (i)–(iii) all fail, we shall prove that $L \times_z M$ has infinitely many maximal ideals. In order for all conditions to fail it means that $z \neq 0$ and one of L or M is a small pole algebra properly containing K. Say L is the one. Because L has infinite dimension over K while zA has finite codimension in A, the intersection $zA \cap L$ remains infinite-dimensional. Hence we can pick a non-scalar r in $zA \cap L$. Then $(r, 0) \in L \times_z M$. Since L is small, there are infinitely many θ for which $X_\theta \notin L$. For such θ the scalar values $r(\theta)$ are defined. The reciprocals of the functions $r - r(\theta)$ have a pole at θ, and hence cannot lie in L. Thus every such $r - r(\theta)$ is a non-unit of L. Consequently every $(r - r(\theta), -r(\theta))$ is a non-unit of $L \times_z M$. These non-units cannot have infinite repetition because r is not a scalar function. The difference of any two of these infinitely many non-units is a unit, in fact lying in $K(1, 1)$. Thus maximal ideals respectively containing each of these infinitely many non-units cannot repeat themselves. ■

Proposition 4.16 *If A is a big pole algebra and z in A is non-zero, then the bridge $A \times_z K$ is not realizable.*

Proof By Proposition 4.15, $A \times_z K$ is semi-local, as it satisfies condition (iii). Assuming that $A \times_z K$ is realizable, it must be isomorphic to some other bridge $C_v \times_w C_u$ across a pole algebra A_h as specified in Proposition 4.12. The w comes from Proposition 4.3. This latter, isomorphic bridge $C_v \times_w C_u$ will be semi-local, and must itself satisfy one of the three conditions in Proposition 4.15. However we note that the situation of condition (iii), namely "A_h is big, and $C_v = A_h$, $C_u = K$ or vice versa," never happens. Indeed if this were the case, the definition of C_v just before (4.3) would yield that $\partial_\alpha + v$ has finite rank on A_h. Since A_h is big, the spur S_h would have to be finite-dimensional. Then the operator $\partial_\alpha + v$ would have finite rank on R_h, forcing $Y + v$ to be the regulator instead of $(Y + u)(Y + v)$. Since $w \neq 0$, condition (i) of Proposition 4.15 does not apply to $C_v \times_w C_u$ either, forcing condition (ii) to apply. However that would make $C_v \times_w C_u$, and thereby $A \times_z K$ isomorphic to $K \times K$, which certainly cannot be since $A \times_z K$ is infinite-dimensional over K. ∎

Realizable Domains and Their Radical

According to Proposition 4.9, a bridge across a pole algebra A is a domain if and only if it is isomorphic to a subalgebra of A of the form $K + zA$ for some non-unit z in A. For $z \neq 0$ we have seen through Theorem 4.14 and Proposition 4.16 that $K + zA$ is realizable if and only if A is small. Here we simply add the observation that A is small if and only if the Jacobson radical of $K + zA$ is zero.

Proposition 4.17 *For any non-zero, non-unit z in a pole algebra A, the radical of $K + zA$ is zero if and only if A is small.*

Proof If A is big, then $K + zA$ is semi-local by Proposition 4.15(iii). Since it is a domain, $K + zA$ is either a field or $\operatorname{rad}(K + zA)$ is not zero. However z, being a nonzero, non-unit of A, remains so in $K + zA$. Thus $\operatorname{rad}(K + zA)$ is not zero.

Suppose A is small. To show that $\operatorname{rad}(K + zA)$ is zero, it suffices to show that every non-zero r in $K + zA$ has a non-zero scalar λ such that $r - \lambda$ is a non-unit of $K + zA$. Well, if r is a scalar itself, then $\lambda = r$ does the job. If r is not a scalar, then one of the infinitely many θ for which $X_\theta \notin A$ will be such that $r(\theta) \neq 0$. Then $\lambda = r(\theta)$ does the job because $r - r(\theta)$ vanishes at θ. This makes $r - r(\theta)$ a non-unit of A as its reciprocal has a pole forbidden to functions in A. Then $r - r(\theta)$ remains a non-unit of $K + zA$. ∎

Corollary 4.18 *A domain isomorphic to a bridge over a pole algebra is realizable if and only if the domain has zero radical.*

5 When the Regulator Has No Root

In this final section we suppose that the regulator of (h, α) as in (1.14) is irreducible in $K(X)[Y]$. As noted in Section 2, $\operatorname{End}\mathbb{V}(m, h, \alpha)$ embeds into a quadratic field extension of $K(X)$. See the examples in [16] for illustrations of what can happen. In [20] we showed that a height function h admits non-trivial $\operatorname{End}\mathbb{V}(m, h, \alpha)$ with irreducible quadratic regulator if and only if h assumes the value ∞ at least once and finite values at least twice. Now we shall prove, as we obtained for realizable bridges in Corollary 4.18, that the Jacobson radical of $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is zero.

Proposition 5.1 *Suppose the generic matrix D is an endomorphism of $\mathbb{V}(m, h, \alpha)$.*

(i) *If $X_\theta \in R_h$ and $\lambda = \langle \alpha, X_\theta \rangle$, then the endomorphism $D - \lambda I$ is not a unit of $\mathbb{V}(m, h, \alpha)$.*
(ii) *For infinitely many scalars λ the endomorphism $D - \lambda I$ is a non-unit.*

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Proof (i) Suppose on the contrary that $D - \lambda I$ is a unit. Thus the inverse matrix

$$(D-\lambda I)^{-1} = \frac{1}{\lambda^2 - p\lambda + q}\begin{bmatrix} -\lambda & 1 \\ -q & p-\lambda \end{bmatrix},$$

leaves $V(m, h, \alpha)$ invariant. Using (1.5), $\partial_\alpha(X_\theta) = -\langle \alpha, X_\theta \rangle X_\theta = -\lambda X_\theta$, so that by (1.11) the vector $\binom{X_\theta}{-\partial_\alpha(X_\theta)} = \binom{X_\theta}{\lambda X_\theta}$ belongs to $V(m, h, \alpha)$. However

$$(D-\lambda I)^{-1}\begin{pmatrix} X_\theta \\ \lambda X_\theta \end{pmatrix} = \frac{1}{\lambda^2 - p\lambda + q}\begin{bmatrix} -\lambda & 1 \\ -q & p-\lambda \end{bmatrix}\begin{pmatrix} X_\theta \\ \lambda X_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -X_\theta \end{pmatrix},$$

a vector definitely not in $V(m, h, \alpha)$. We have a contradiction.

(ii) Suppose first that there are infinitely many X_θ in R_h. The values $\langle \alpha, X_\theta \rangle$ cannot have infinite repetition. Indeed, let J be an infinite set of θ's for which the functions X_θ belong to R_h and $\langle \alpha, X_\theta \rangle$ take one common value c, say. Let ℓ be the height function that takes the value 1 at each θ in J and is zero elsewhere. The pole space $R_\ell = K + \sum_{\theta \in J} KX_\theta$ is infinite-dimensional, since J is infinite. Using (1.5) we see that $\partial_\alpha(X_\theta) = -\langle \alpha, X_\theta \rangle X_\theta = -cX_\theta$, for every θ in J. Hence the operator $\partial_\alpha + c$ maps R_ℓ into K. This implies that $Y + c$ regulates (ℓ, α). The regulator $Y^2 + pY + q$ of (h, α) also maps R_ℓ to a finite-dimensional space because R_ℓ is inside R_h. By the nature of regulators $Y + c$ divides the irreducible polynomial $Y^2 + pY + q$, and that is a contradiction. Consequently, the scalar values $\langle \alpha, X_\theta \rangle$ go through an infinite set as X_θ runs through R_h. According to part (i) above, each of the infinitely many $\lambda = \langle \alpha, X_\theta \rangle$ will cause $D - \lambda I$ to be a non-unit.

The other alternative is that the pole space R_h has only finitely many X_θ in it. In this case we examine the curve defined by the polynomial $Y^2 - pY + q$, which is just as irreducible as the regulator. Let

$$L = \{\theta \in K : \theta \text{ is not a pole of } p \text{ or of } q \text{ and } X_\theta \notin R_h\}.$$

Clearly L is cofinite in K. For each θ in L there is a λ in K, in fact two at most, such that $\lambda^2 - p(\theta)\lambda + q(\theta) = 0$. As θ runs through L, infinitely many λ arise. Indeed if only finitely many λ result from solving the above equation we would have one λ for which infinitely many θ in L solve the above equation. This would imply that

$$\lambda^2 - p(X)\lambda + q(X) = 0$$

identically as a polynomial in X. This leads to the impossibility that the irreducible polynomial $Y^2 - p(X)Y + q(X)$ in $K(X)[Y]$ has root λ in K. Thus we have infinitely many λ in K for which there is θ in K where

- θ is not a pole of p nor of q;
- $X_\theta \notin R_h$;
- $\lambda^2 - p(\theta)\lambda + q(\theta) = 0$.

Hence for infinitely many λ, the rational function $\lambda^2 - p\lambda + q$ is not a unit of the pole algebra A_h. Moreover $\lambda^2 - p\lambda + q = \det(D - \lambda I)$. By Proposition 2.1 the determinant of a unit in $\operatorname{End} \mathbb{V}(m, h, \alpha)$ is a unit in A_h. Hence, we conclude that $D - \lambda I$ is not a unit for infinitely many scalars λ. ■

In the next result we do not assume that D is an endomorphism.

Theorem 5.2 *If the regulator of* (h,α) *is an irreducible quadratic, then the radical of* $\operatorname{End}\mathbb{V}(m,h,\alpha)$ *is zero.*

Proof It suffices to show that for any non-zero endomorphism φ there is a non-zero scalar λ such that $\varphi-\lambda I$ is a non-unit. According to Proposition 2.2,

$$\varphi = tD + \partial_\alpha(t)I + \mu I$$

for some t in the parameter module and some scalar μ. If $t=0$, then $\mu\neq 0$ and the scalar $\lambda=\mu$ gives $\varphi-\lambda I=0$, which is definitely a non-unit.

If $t\neq 0$, Proposition 2.4 ensures that the module $\mathbb{V}(m,h,\alpha)$ is isomorphic to the module $\mathbb{V}(m,h-\operatorname{ord}(t),\alpha*t)$. Their respective endomorphism algebras are conjugate as in Proposition 2.5. The generic matrix E for $(h-\operatorname{ord}(t),\alpha*t)$ is an endomorphism of $\mathbb{V}(m,h-\operatorname{ord}(t),\alpha*t)$ that conjugates to the endomorphism $tD+\partial_\alpha(t)I$ of $\mathbb{V}(m,h,\alpha)$. The regulator of $(h-\operatorname{ord}(t),\alpha*t)$ remains irreducible, as can be seen easily using Proposition 1.3. By Proposition 5.1 there are infinitely many λ such that $E-\lambda I$ is a non-unit of $\operatorname{End}\mathbb{V}(m,h-\operatorname{ord}(t),\alpha*t)$. The same applies to the endomorphism $tD+\partial_\alpha(t)I$ conjugate to E, and hence also to φ. Since there are infinitely many λ for which $\varphi-\lambda I$ is a non-unit, there certainly is a non-zero λ to do the job. ∎

5.1 Isomorphism Invariants for Endomorphism Algebras

One of the motivations of this paper was to capture isomorphism invariants for algebras that could be realized as $\operatorname{End}\mathbb{V}(m,h,\alpha)$. Our final result summarizes our findings in this regard. It makes no assumption regarding the regulator.

Theorem 5.3 *If* $\operatorname{End}\mathbb{V}(m,h,\alpha)$ *is a domain, then its radical is zero. Any semi-local* $\operatorname{End}\mathbb{V}(m,h,\alpha)$ *is isomorphic to* $K\times K$ *or a trivial extension* $K\ltimes S$ *for some vector space* S.

Proof Let $\operatorname{End}\mathbb{V}(m,h,\alpha)$ be a domain. If it is trivially just K, its radical is zero. Otherwise Theorem 3.2 shows that the regulator has either distinct roots or no roots. In the case of distinct roots we appeal to Theorem 4.13 and Corollary 4.18, and in the case of no roots we appeal to Theorem 5.2, to conclude that domains realized as endomorphism algebras must have zero radical.

Now let $\operatorname{End}\mathbb{V}(m,h,\alpha)$ be semi-local. If it is a domain, it must be a field. By [15, Theorem 3.3], the only field that $\operatorname{End}\mathbb{V}(m,h,\alpha)$ could be is K, which is $K\ltimes S$ with S being the zero space. If $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is not a domain, this algebra cannot embed in a quadratic field extension of $K(X)$. By the discussion following Proposition 2.2, the regulator must have roots in $K(X)$. If it has one repeated root, then $\operatorname{End}\mathbb{V}(m,h,\alpha)$ is $K\ltimes S$ by Theorem 3.2. If the regulator has two distinct roots, Propositions 4.15 and 4.16 apply making $\operatorname{End}\mathbb{V}(m,h,\alpha)$ isomorphic to $K\times K$. ∎

Pole Algebras as Endomorphism Algebras

Except for $K(X)$, a big pole algebra is a domain with non-zero radical, and therefore it is never isomorphic to End $\mathbb{V}(m, h, \alpha)$. The field $K(X)$ itself is never one of our endomorphism algebras, due to [15, Theorem 3.3]. Thus no big pole algebra comes up as one of our endomorphism algebras. In [18] small pole algebras have been realized using a different approach. Now we can see that by taking z to be a non-zero prime in a small pole algebra A we get $A = K + zA$, and Theorem 4.14 realizes this. This situation stands in contrast with the fact proven in [3] that the endomorphism algebras of torsion-free, rank-one Kronecker modules pick up exactly all pole algebras. For $K[X]$-modules the latter is a classical theorem of R. Baer.

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