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ON SKEW-COMMUTING MAPPINGS OF RINGS

MATEJ BREŠAR

A mapping f of a ring R into itself is called skew-commuting on a subset S of R if $f(s)s + sf(s) = 0$ for all $s \in S$. We prove two theorems which show that under rather mild assumptions a nonzero additive mapping cannot have this property. The first theorem asserts that if R is a prime ring of characteristic not 2, and $f\colon R \to R$ is an additive mapping which is skew-commuting on an ideal I of R, then $f(I) = 0$. The second theorem states that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime ring.

Let S be a subset of a ring R. A mapping f of R into itself is said to be *skew-commuting* on S if $f(s)s + sf(s) = 0$ for all $s \in S$. For results on skew-commuting mappings and their generalisations (such as semi-commuting, skew-centralising, semi-centralising mappings) we refer the reader to [4, 6, 7, 8]. In these papers the authors have showed that nonzero derivations and ring endomorphisms cannot be skew-commuting (semi-commuting, ...) on certain subsets of prime rings (for example, ideals). In the present paper we prove theorems of this kind for general additive mappings. Our first result is

THEOREM 1. *Let R be a prime ring of characteristic not 2. If an additive mapping $f\colon R \to R$ is skew-commuting on some ideal I of R, then $f(x) = 0$ for all $x \in I$.*

Clearly, the requirement that the characteristic of R is not 2 cannot be removed (consider, for instance, the identity on R). In fact, if the characteristic of a ring R is 2, then the notion of skew-commuting mappings coincides with the notion of commuting mappings, that is, the mappings f satisfying $f(x)x = xf(x)$. In [1] we showed that every additive commuting mapping of a prime ring R (of arbitrary characteristic) is of the form $x \to \lambda x + \zeta(x)$ where λ is an element in C, the extended centroid of R, and ζ is an additive mapping of R into C (see also [2, 3] for similar results). The fact that the structure of commuting mappings can be described has been one of the main motivations for this research.

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Suppose a ring R contains nonzero ideals I and J such that $IJ = 0 = JI$ (thus R is not prime). Any mapping f of R with range contained in J is certainly skew-commuting on I; however, it does not necessarily vanish on I. Thus Theorem 1 does not hold for semiprime rings in general. Nevertheless, the following is true:

THEOREM 2. *Let R be a 2-torsion free semiprime ring. If an additive mapping $f\colon R \to R$ is skew-commuting on R, then $f = 0$.*

Theorem 2 will follow easily from Theorem 1. In order to prove Theorem 1 we define $I_n = \{x^n \mid x \in I\}$ (n is a positive integer), and let us prove

LEMMA 1. *Let R be a prime ring, I be a nonzero ideal of R, and $a \in R$. If there exists a positive integer n such that $I_n a = 0$ (or $aI_n = 0$), then $a = 0$.*

PROOF: Suppose $a \neq 0$. Since R is prime there exists $w \in I$ such that $aw \neq 0$. For any $x \in R$, the element awx lies in I, hence $(awx)^n a = 0$ for all $x \in R$. But then $(awx)^{n+1} = 0$, $x \in R$, and so [5, Lemma 1.1] yields $aw = 0$, contrary to the assumption. Similarly one discusses the case when $aI_n = 0$. ∎

PROOF OF THEOREM 1: For the proof we need several steps. We begin with

LEMMA A. *For $x, y \in I$,*

(1) $f(x)y + yf(x) + f(y)x + xf(y) = 0$ *for all* $x, y \in I$.

(2) $x^4 f(x) = 0 = f(x)x^4$.

PROOF: Linearising $f(x)x + xf(x) = 0$ we obtain (1). Let us prove (2). From the initial hypothesis we see that for any $x \in I$, $f(x)$ commutes with x^2. Therefore, replacing y by x^2 in (1) we obtain

$$2x^2 f(x) + f(x^2)x + xf(x^2) = 0 \text{ for all } x \in I. \tag{3}$$

Multiply (3) from the right by x^2; since $f(x)x^2 = x^2 f(x)$ and since, by the initial hypothesis, $f(x^2)x^2 + x^2 f(x^2) = 0$, it follows that

$$2x^4 f(x) = x^2 f(x^2)x + x^3 f(x^2).$$

On the other hand, by (3) we see that

$$2x^4 f(x) = x^2\big(2x^2 f(x)\big) = -x^2 f(x^2)x - x^3 f(x^2)$$

Comparing the last two relations we arrive at $4x^4 f(x) = 0$. We have assumed that the characteristic of R is not 2, and so $x^4 f(x) = 0$. Since $f(x)x = -xf(x)$, we also have $f(x)x^4 = 0$. ∎

Lemma B. *For $u \in I_{10}$, $y \in I$, $uf(y)u = 0$.*

Proof: Multiply (1) from the left and from the right by x^4. According to (2) we obtain

$$x^4 f(y) x^5 + x^5 f(y) x^4 = 0 \quad \text{for all} \quad x, y \in I. \tag{4}$$

Taking x^2 for x in (4) we get

$$x^8 f(y) x^{10} + x^{10} f(y) x^8 = 0.$$

But from (4) if follows that

$$\begin{aligned} x^8 f(y) x^{10} &= x^4 (x^4 f(y) x^5) x^5 \\ &= -x^4 (x^5 f(y) x^4) x^5 \\ &= -x^5 (x^4 f(y) x^5) x^4 \\ &= x^5 (x^5 f(y) x^4) x^4 \\ &= x^{10} f(y) x^8. \end{aligned}$$

Comparing the last two identities one concludes that $x^8 f(y) x^{10} = 0$ for all $x, y \in I$. But then also $x^{10} f(y) x^{10} = 0$, which is the assertion of the lemma. ∎

There is nothing to prove if $I = 0$. Therefore, we assume henceforth that $I \neq 0$.

Lemma C. *There exists a nonzero left ideal L of R, contained in I, such that $f(L) = 0$.*

Proof: As a special case of (1) we have

$$f(x)u + uf(x) + f(u)x + xf(u) = 0 \text{ for all } x \in I,\ u \in I_{10}. \tag{5}$$

Multiplying (5) from the right by u, and then using Lemma B, we arrive at

$$f(x)u^2 + f(u)xu + xf(u)u = 0 \text{ for all } x \in I,\ u \in I_{10}. \tag{6}$$

Suppose $x \in I_{10}$. By Lemma B we then see that $xf(u)x = 0$, and also $x^2 f(x) = -xf(x)x = 0$. Therefore it follows from (6) that $x^3 f(u)u = 0$. That is, $vf(u)u = 0$ for all $v \in I_{30}$, $u \in I_{10}$. By Lemma 1 we then have $f(u)u = 0$. Thus (6) reduces to

$$f(x)u^2 + f(u)xu = 0 \text{ for all } x \in I,\ u \in I_{10}. \tag{7}$$

Substituting xu for x in (7) we obtain $f(xu)u^2 + f(u)xu^2 = 0$. On the other hand, $f(u)xu^2 = (f(u)xu)u = -f(x)u^3$. Consequently we have

$$f(xu)u^2 = f(x)u^3 \text{ for all } x \in I,\ u \in I_{10}. \tag{8}$$

Now, multiply (5) from the left by u. Since $uf(x)u = 0$ and $uf(u) = -f(u)u = 0$, it follows that $u^2 f(x) + uxf(u) = 0$, $x \in I$, $u \in I_{10}$. Replacing x by xu in this relation, and applying $uf(u) = 0$, we then get

$$u^2 f(xu) = 0 \text{ for all } x \in I,\ u \in I_{10}. \tag{9}$$

As a special case of (1) we have

$$f(x)yu + yuf(x) + f(yu)x + xf(yu) = 0$$

for all $x, y \in I$, $u \in I_{10}$. Multiply this identity from the left and from the right by u^2. In view of Lemma B, (9) and (8), we then get $u^2 f(x)yu^3 + u^2 xf(y)u^3 = 0$. Hence

$$vf(x)yv + vxf(y)v = 0$$

holds for all $v \in I_{30}$, $x, y \in I$. Replace in this relation y by $yvf(z)$ where $y, z \in I$, $v \in I_{30}$. Then the first term is zero by Lemma B, so we have $vxf(yvf(z))v = 0$. Since R prime it follows that

$$f(yvf(z))v = 0 \text{ for all } y, z \in I,\ v \in I_{30}. \tag{10}$$

Substituting $yvf(z)$ for y in (1) we obtain

$$f(x)yvf(z) + yvf(z)f(x) + f(yvf(z))x + xf(yvf(z)) = 0.$$

Multiplying from the right by v, and using Lemma B and (10), we then obtain

$$yvf(z)f(x)v + f(yvf(z))xv = 0 \text{ for all } x, y, z \in I,\ v \in I_{30}. \tag{11}$$

Taking ry for y, where $r \in R$ and $y \in I$, we get

$$ryvf(z)f(x)v + f(ryvf(z))xv = 0.$$

On the other hand we see from (11) that

$$ryvf(z)f(x)v = -rf(yvf(z))xv.$$

Comparing we obtain

$$\{f(ryvf(z)) - rf(yvf(z))\}xv = 0$$

for all $r \in R$, $x, y, z \in I$, $v \in I_{30}$. The primeness of R yields

$$f(ryvf(z)) = rf(yvf(z)) \text{ for all } r \in R,\ y, z \in I,\ v \in I_{30}. \tag{12}$$

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Multiply (12) from the left and from the right by $u \in I_{10}$. In view of Lemma B it follows that $urf(yv(z))u = 0$. Thus $f(yvf(z))u = 0$, and so, by Lemma 1,

$$f(yvf(z)) = 0 \text{ for all } y, z \in I,\ v \in I_{30}. \tag{13}$$

We may assume that $f(z) \neq 0$ for some $z \in I$. By Lemma 1, $vf(z) \neq 0$ for some $v \in I_{30}$. Hence $a = xvf(z) \neq 0$ for some $x \in I$. Thus $L = Ra$ is a nonzero left ideal of R, and since $a \in I$, L is contained in I. By (13), $f(L) = 0$. ∎

LEMMA D. $f(I) = 0$.

PROOF: From $f(L) = 0$ and (1) it follows at once that

$$f(x)t + tf(x) = 0 \text{ for all } t \in L,\ x \in I. \tag{14}$$

Replacing t by rt, where $r \in R$ and $t \in L$, it follows that $f(x)rt + rtf(x) = 0$. By (14), the second term is equal to $-rf(x)t$, therefore $(f(x)r - rf(x))t = 0$ for all $r \in R$, $x \in I$, $t \in L$. Since R is prime we then have $f(x)r - rf(x) = 0$ for all $r \in R$, $x \in I$. That is, $f(x)$ lies in the centre of R for every x in I. But then (14) implies that $f(x)L = 0$, $x \in I$, and therefore $f(x) = 0$. With this the theorem is proved. ∎

PROOF OF THEOREM 2: Since R is semiprime, the intersection of all prime ideals in R is zero.

Now pick a prime ideal P such that R/P is of characteristic not 2. We want to show that P is invariant under f. A linearisation of $f(x)x + xf(x) = 0$ gives $f(x)y + yf(x) + f(y)x + xf(y) = 0$, $x, y \in R$. Hence we see that

$$f(p)x + xf(p) \in P \text{ for all } p \in P,\ x \in R. \tag{15}$$

In particular, $f(p)xy + xyf(p) \in P$ for all $p \in P$, $x, y \in R$. That is, $(f(p)x + xf(p))y + x(yf(p) - f(p)y) \in P$. The first term is contained in P by (15), hence $x(yf(p) - f(p)y) \in P$, $p \in P$, $x, y \in R$. Since P is a prime ideal it follows that $yf(p) - f(p)y \in P$ for all $p \in P$, $y \in R$. Combining this statement with (15) we obtain $2f(p)x \in P$. Since the characteristic of R/P is not 2 it follows that $f(p)x \in P$ for all $p \in P$, $x \in R$. The ideal P is prime, therefore, $f(p) \in P$ for every $p \in P$.

Since $f(P) \subseteq P$, f induces an additive mapping F on R/P, defined by $F(x + P) = f(x) + P$. Of course, F is skew-commuting. Hence $F = 0$ by Theorem 1.

Thus we have proved that the range of f is contained in any prime ideal P such that R/P is of characteristic not 2. The theorem will be proved by showing that the intersection of all such ideals is equal to zero. There exist prime ideals $\{P_a \mid a \in A\}$ such that $\bigcap_a P_a = 0$. Let $B = \{b \in A \mid \text{ the characteristic of } R/P_b \text{ is not } 2\}$ and $C = \{c \in A \mid \text{ the characteristic of } R/P_c \text{ is } 2\}$. Thus $2x \in \bigcap_c P_c$ for every $x \in R$. Therefore, given $x \in \bigcap_b P_b$, we have $2x \in (\bigcap_c P_c) \cap (\bigcap_b P_b) = \bigcap_a P_a = 0$, and so $x = 0$ since R is 2-torsion free. Thus $\bigcap_b P_b = 0$. ∎

REMARK. A mapping f of a ring R is called *semi-commuting* on a subset S of R if for any $x \in S$, either $f(x)x + xf(x) = 0$ or $f(x)x - xf(x) = 0$. Suppose that R is 2-torsion free and 3-torsion free, and suppose that f is an additive mapping of R which is semi-commuting on some additive subgroup S of R. We claim that in this case f is either commuting on S or skew-commuting on S. Indeed, introducing biadditive mappings $A: S \times S \to R$ and $B: S \times S \to R$ by $A(x, y) = f(x)y + xf(y)$ and $B(x, y) = f(x)y - xf(y)$, we have $S = P \cup Q$ where $P = \{x \in S \mid A(x, x) = 0\}$, $Q = \{x \in S \mid B(x, x) = 0\}$. Suppose our assertion is not true, thus $P \neq S$ and $Q \neq S$. This means that $A(x, x) \neq 0$ and $B(y, y) \neq 0$ for some $x, y \in S$. Then, of course, $A(y, y) = 0$ and $B(x, x) = 0$. Now, consider the element $x + y$. If $x + y \in P$ then we have $A(x, x) + A(x, y) + A(y, x) = 0$, and if $x + y \in Q$ then $B(x, y) + B(y, x) + B(y, y) = 0$. Similarly we consider the elements $x - y$ and $x + 2y$. But then one can easily see that (since R is 2-torsion free and 3-torsion free) either $A(x, x) = 0$ or $B(y, y) = 0$, contrary to the assumption. This proves our assertion. According to Theorem 1 we then obtain the following result: Let f be an additive mapping of a prime ring of characteristic not 3. If f is semi-commuting on some ideal I of R, then f is commuting on I. Note that this result fairly generalises a theorem in [4].

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University of Maribor
PF, Koroška 160
62000 Maribor
Slovenia