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The de Rham Bundle on a Compactification of Moduli Space of Abelian Varieties

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Abstract. In this paper, we show that the Chern classes c_k of the de Rham bundle $\mathcal{H}_{\mathrm{dR}}$ defined on any 'good' toroidal compactification $\bar{\mathcal{A}}_g$ of the moduli space of Abelian varieties of dimension g are zero in the rational Chow ring of $\bar{\mathcal{A}}_g$, for $g = 4, 5$ and $k > 0$.

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1. Introduction

Let $\mathcal{X} \xrightarrow{\pi} T$ be a smooth projective morphism of nonsingular complex algebraic varieties. The de Rham bundle $\mathcal{H}_{\mathrm{dR}} = \mathbb{R}^1\pi_*(\mathcal{O}_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/T})$ is a complex vector bundle on T and is equipped with a flat connection ∇ called the Gauss–Manin connection. Suppose $\bar{T}$ is a smooth compactification of T such that $D = \bar{T} - T$ is a normal crossing divisor. Then there is a *canonical extension* of $\mathcal{H}_{\mathrm{dR}}$ to a vector bundle $\bar{\mathcal{H}}_{\mathrm{dR}}$ on $\bar{T}$, and the connection ∇ extends to a logarithmic connection $\bar{\nabla}$ on $\bar{\mathcal{H}}_{\mathrm{dR}}$ ([D]). H. Esnault observed that the Chern classes c_k of $\mathcal{H}_{\mathrm{dR}}$ in the de Rham cohomology of $\bar{T}$ are expressible in terms of cycles supported in D with coefficients depending on the residues of $\bar{\nabla}$ along D ([EV], Appendix B). In particular, when there is a semi-stable extension $\bar{\mathcal{X}} \longrightarrow \bar{T}$ of the family $\mathcal{X} \longrightarrow T$, the residues are nilpotent and $c_k = 0$. The following question is posed in [E], 3.6.

QUESTION: Are the Chern classes $c_k(\bar{\mathcal{H}}_{\mathrm{dR}}) = 0$ in the Chow groups $CH^k(\bar{T})_{\mathbb{Q}}$, with rational coefficients (resp. in $CH^k(T)_{\mathbb{Q}}$), for $k > 0$?

This is answered affirmatively by application of Grothendieck–Riemann–Roch theorem in the following cases:

(a) (Mumford). T is the moduli space $\mathcal{M}_g$ of smooth curves of genus $g \geqslant 1$ and $\bar{T}$ is a compactification of $\mathcal{M}_g$, ([M], Theorem 4.2).

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(b) (van der Geer). T is the moduli space $\mathcal{A}_g$, of principally polarized Abelian varieties (henceforth abbreviated to p.p.a.v) of dimension $g \geqslant 1$, ([vdG], Theorem 2.1).
(c) The variety $\bar{T}$ is any 'good' toroidal compactification $\bar{\mathcal{A}}_g$, of $\mathcal{A}_g$, when $g \leqslant 3$. (This follows, via the Torelli isomorphism and using **a**).

In the above cases we actually consider fine moduli spaces (with suitable level structures) and $\mathcal{X}$ is the universal family. We will consider fine moduli spaces and their compactifications in this paper. We show

THEOREM 1.1. *For* $g = 4, 5$, *the Chern classes* $c_k(\bar{\mathcal{H}}_{dR}) = 0$ *in* $CH^k(\bar{\mathcal{A}}_g)_{\mathbb{Q}}, k \geqslant 1$.

2. The Prym Morphism

In this paper, all the varieties are defined over the field of complex numbers and all Chow groups considered are taken with $\mathbb{Q}$-coefficients.

Let $\mathcal{R}_{g+1}$ be the coarse moduli space of smooth curves C' of genus $g+1$, together with a connected étale double cover $\eta: C \longrightarrow C'$. A natural compactification $\bar{\mathcal{R}}_{g+1}$ of $\mathcal{R}_{g+1}$ exists corresponding to the functor $\bar{R}_{g+1}$: if V is a scheme, $\bar{R}_{g+1}(V) =$ set of stable curves $\mathcal{C} \longrightarrow V$ of genus $2g+1$, together with a V-involution $i: \mathcal{C} \longrightarrow \mathcal{C}$, such that i is admissible ([FS], p. 617) on geometric fibres. Then $\bar{\mathcal{R}}_{g+1}$ is a compact irreducible algebraic variety of dimension $3g$ containing $\mathcal{R}_{g+1}$ as a dense open subset ([DS], 1.1.2). In fact, using the construction of Beauville ([B], 6.1) and with suitable level n structures we have ([DS], p. 32–33):

(1) There is a universal family $\epsilon: (\mathcal{C}, i) \longrightarrow S$ of stable curves of genus $2g+1$ (here i is an S-involution on $\mathcal{C}$).
(2) A quotient family of stable curves $\epsilon': \mathcal{C}' = \mathcal{C}/(i) \longrightarrow S$ of genus $g+1$.
(3) S is a complete variety of dimension $3g$ admitting a finite morphism $S \longrightarrow \bar{\mathcal{R}}_{g+1}$.

Then there exists an open subvariety $S_0 \subset S$ and a family $\mathcal{P} \longrightarrow S_0$ of principally polarized Prym varieties of dimension g which defines the *Prym morphism* $S_0 \longrightarrow \mathcal{A}_g$ ([B], p. 177). This morphism extends to $p_1: S \longrightarrow \mathcal{A}_g^*$, where $\mathcal{A}_g^*$ is the *Satake compactification* of $\mathcal{A}_g$ ([FS], Proposition 1.8).

Let $\bar{\mathcal{A}}_{g,n}$ denote any 'good' toroidal compactification ([FC], Chapter 6) of the nonsingular moduli space $\mathcal{A}_{g,n}$ of p.p.a.v.s with level n structure ($n \geqslant 4$ and even). This means that there is a universal semi-Abelian scheme $\mathcal{G} \longrightarrow \bar{\mathcal{A}}_{g,n}$ which restricts to the universal family $\mathcal{X}_n \longrightarrow \mathcal{A}_{g,n}$ and $\bar{\mathcal{A}}_{g,n} - \mathcal{A}_{g,n}$ is a normal crossing divisor. Moreover, there is a morphism $h: \bar{\mathcal{A}}_{g,n} \longrightarrow \mathcal{A}_g^*$, $h([G]) = A$, where G is a semi-Abelian variety: $1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$.

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Consider the following fibre product:

$$\begin{array}{ccc} S_n = S \times_{\mathcal{A}_g^*} \bar{\mathcal{A}}_{g,n} & \xrightarrow{\bar{p}} & \bar{\mathcal{A}}_{g,n} \\ \downarrow f' & & \downarrow h \\ S & \xrightarrow{p_1} & \mathcal{A}_g^* \end{array}$$

Consider a resolution of singularities $f: W \longrightarrow S_n$, where W is non-singular and complete, $W_0 = (f' \circ f)^{-1} S_0$ and $D = W - W_0$ is a normal crossing divisor (due to Hironaka). The composed morphism $p = \bar{p} \circ f : W \longrightarrow \bar{\mathcal{A}}_{g,n}$ is then a proper morphism. Consider the pullback families of curves $\epsilon: \mathcal{C}_W = (f' \circ f)^* \mathcal{C} \longrightarrow W$ and $\epsilon': \mathcal{C}'_W = (f' \circ f)^* \mathcal{C}' \longrightarrow W$ and denote the associated de Rham bundles by $\mathcal{H}_W$ and $\mathcal{H}_W^+$ on W, respectively. Let $\mathcal{H}_{\mathrm{dR}}$ (resp. $\bar{\mathcal{H}}_{dR}$) denote the de Rham bundle on $\mathcal{A}_{g,n}$ (resp. the *canonical extension* on $\bar{\mathcal{A}}_{g,n}$).

Remark 2.1. Since the de Rham bundle of a family of stable curves is canonical, $\mathcal{H}_W$ and $\mathcal{H}_W^+$ are the canonical extensions of their restrictions to W_0.

We denote by ω (resp. ω') for the invertible sheaf of relative differentials on $\mathcal{C}_W$ (resp. on $\mathcal{C}'_W$). Let

$$\omega^{\cdot}: 0 \longrightarrow \mathcal{O}_{\mathcal{C}_W} \xrightarrow{d} \omega \longrightarrow 0 \quad \text{and} \quad \omega'^{\cdot}: 0 \longrightarrow \mathcal{O}_{\mathcal{C}'_W} \xrightarrow{d} \omega \longrightarrow 0$$

be the relative de Rham complexes.

LEMMA 2.2. *If* $q: \mathcal{C}_W \longrightarrow \mathcal{C}'_W = \mathcal{C}_W/(i)$ *denotes the quotient morphism, then* $q^*\omega' \simeq \omega$.

Proof. There is a natural morphism $f: q^*\omega' \longrightarrow \omega$ and hence we obtain a section $t: \mathcal{O}_{\mathcal{C}_W} \longrightarrow q^*\omega'^{-1} \otimes \omega$. Since q is étale outside the locus R of singular points of the fibres of $\mathcal{C}_W \longrightarrow W$ ([B], p. 173), f is an isomorphism outside R. Hence, the section t of the line bundle $q^*\omega'^{-1} \otimes \omega$ is nonvanishing outside R. Since the generic fibre of ϵ is non-singular, R is of codimension 2. This implies that t is nonvanishing everywhere, i.e., f is an isomorphism. □

PROPOSITION 2.3. (1) *There is a vector bundle decomposition* $\mathcal{H}_W = \mathcal{H}_W^+ \oplus \mathcal{H}_W^-$, *where* $\mathcal{H}_W^-$ *is a rank* $2g$*-vector bundle on* W.

(2) $\mathcal{H}_W^- \simeq p^*\bar{\mathcal{H}}_{\mathrm{dR}}$.

(3) *The Chern classes* $c_k(\mathcal{H}_W) = c_k(\mathcal{H}_W^{\pm}) = 0$ in $CH^k(W)_{\mathbb{Q}}$, *for* $k \geqslant 1$.

Proof. (1) Since $q_*\mathcal{O}_{\mathcal{C}_W} = \mathcal{O}_{\mathcal{C}'_W} \oplus \tau$, for some rank 1, torsion free sheaf τ on $\mathcal{C}'_W$, by Lemma 2.2 and projection formula, $q_*\omega^{\cdot} = \omega'^{\cdot} \oplus (\omega'^{\cdot} \otimes \tau)$. Here $\omega'^{\cdot} \otimes \tau$ is the complex $0 \longrightarrow \mathcal{O}_{\mathcal{C}'_W} \otimes \tau \xrightarrow{d \otimes 1} \omega' \otimes \tau \longrightarrow 0$. Hence,

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$$\mathcal{H}_W = \mathbb{R}^1(\epsilon' \circ q)_*(\omega^{\cdot}) = \mathbb{R}^1\epsilon'_*(\omega'^{\cdot}) \oplus \mathbb{R}^1\epsilon'_*(\omega'^{\cdot} \otimes \tau).$$

Since $\mathcal{H}_W^+ = \mathbb{R}^1\epsilon'_*(\omega'^{\cdot})$ is a rank $2g+2$ subbundle, $\mathcal{H}_W^- = \mathbb{R}^1\epsilon'_*(\omega'^{\cdot} \otimes \tau)$ is a rank $2g$ vector bundle and $\mathcal{H}_W = \mathcal{H}_W^+ \oplus \mathcal{H}_W^-$.

(2) The pullback family $\mathcal{X} = (f' \circ f)^*\mathcal{P} \longrightarrow W_0$ is a family of principally polarized Prym varieties of dimension g. By the basic construction of Prym varieties, the de Rham bundle associated to the family $\mathcal{X} \longrightarrow W_0$ is canonically identified with $\mathcal{H}_{W_0}^-$. Now, the restricted morphism $p_{W_0}\colon W_0 \longrightarrow \mathcal{A}_{g,n}$ is defined by the family $\mathcal{X} \longrightarrow W_0$, i.e., $\mathcal{X} \simeq p_{W_0}^*\mathcal{X}_n$ over W_0 (here $\mathcal{X}_n \longrightarrow \mathcal{A}_{g,n}$ is the universal family). Hence there is an isomorphism $\mathcal{H}_{W_0}^- \simeq (p^*\bar{\mathcal{H}}_{\mathrm{dR}})_{|W_0}$. By (1) and Remark 2.1, the *canonical extension* of $\mathcal{H}_{W_0}^-$ is $\mathcal{H}_W^-$. Now by uniqueness, there is an isomorphism $\mathcal{H}_W^- \simeq p^*\bar{\mathcal{H}}_{\mathrm{dR}}$.

(3) Since the total Chern class is multiplicative, we obtain $c(\mathcal{H}_W) = c(\mathcal{H}_W^+).c(\mathcal{H}_W^-)$. By [M], Theorem 4.2, applied to the two families $\mathcal{C}_W$ and $\mathcal{C}'_W$ over W, we get $c(\mathcal{H}_W) = 1$ and $c(\mathcal{H}_W^+) = 1$, in $CH^*(W)_{\mathbb{Q}}$. This gives $c_k(\mathcal{H}_W^-) = 0$ in $CH^k(W)_{\mathbb{Q}}$, for all $k \geqslant 1$. □

Suppose $g = 4, 5$.

PROPOSITION 2.4. *There is a complete nonsingular subvariety W^o of W, such that the restriction p^o of p to W^o is generically finite, proper and surjective.*

Proof. When $g \leqslant 5$, the *Prym morphism* surjects onto the moduli space $\mathcal{A}_g$ ([B], 6.4). Since p is generically surjective and proper, the irreducibility of $\bar{\mathcal{A}}_{g,n}$ implies that p is surjective.

Suppose $g = 4$. Then $\dim(S) = 12$ and $\dim(\bar{\mathcal{A}}_4) = 10$ and the morphism p is generically of relative dimension 2. Since W is projective, choose two general hyperplane sections H_1 and H_2 which have their intersection with a general fibre of p at a finite set of points. Moreover, by Bertini's Theorem, we may assume that the intersection W^o of W with H_1 and H_2 is a nonsingular subvariety. Then the restriction p^o of p to W^o is generically finite, proper and surjective. Suppose $g = 5$. Then $\dim(S) = \dim(\bar{\mathcal{A}}_5) = 15$ and hence $W^o = W$. □

Proof of Theorem 1.1. Consider the isomorphism in Proposition 2.3 (2) and its restriction $\mathcal{H}_{W^o}^- \simeq (p^o)^*\bar{\mathcal{H}}_{\mathrm{dR}}$ to the subvariety W^o. By Proposition 2.3 (3), it follows that $c_k((p^o)^*\bar{\mathcal{H}}_{\mathrm{dR}}) = 0$, $k \geqslant 1$ in $CH^k(W^o)_{\mathbb{Q}}$. By Proposition 2.4 and projection formula, $c_k(\bar{\mathcal{H}}_{\mathrm{dR}}) = 0$ in $CH^k(\bar{\mathcal{A}}_{g,n})_{\mathbb{Q}}$, for $g = 4, 5$. □

Recall that the Hodge bundle $\mathbb{E}$ is the locally free sheaf $\pi_*(\Omega_{\mathcal{X}_n})$ on $\mathcal{A}_{g,n}$. Notice that the de Rham bundle $\mathcal{H}_{dR}$ fits in an exact sequence:

$$0 \longrightarrow \mathbb{E} \longrightarrow \mathcal{H}_{\mathrm{dR}} \longrightarrow R^1\pi_*(\mathcal{O}_{\mathcal{X}_n}) \longrightarrow 0.$$

Hence the total Chern classes of the above bundles satisfy the relation

$$c(\mathbb{E}).c(R^1\pi_*(\mathcal{O}_{\mathcal{X}_n})) = c(\mathcal{H}_{\mathrm{dR}}) \ldots (**)$$

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Now $R^1\pi_*(\mathcal{O}_{\mathcal{X}_n}) \simeq \mathbb{E}^*$, where $\mathbb{E}^*$ is the dual of $\mathbb{E}$. Indeed, $R^1\pi_*(\mathcal{O}_{\mathcal{X}_n}) \simeq e^*(\Omega^*_{\hat{\mathcal{X}}_n})$, where $\hat{\mathcal{X}}_n$ denotes the dual family $\hat{\mathcal{X}}_n \longrightarrow \mathcal{A}_{g,n}$ with zero section e and there is an isomorphism $\mathcal{X}_n \simeq \hat{\mathcal{X}}_n$ given by the principal polarization on $\mathcal{X}_n$. Again, consider the *canonical extension* $\bar{\mathbb{E}}$ of $\mathbb{E}$ to $\bar{\mathcal{A}}_{g,n}$ which is compatible with dual ([FC], p. 224). Denote the Chern classes of the extension $\bar{\mathbb{E}}$ by λ_l, $1 \leqslant l \leqslant g$, in $CH^*(\bar{\mathcal{A}}_{g,n})_{\mathbb{Q}}$.

Suppose $g = 4, 5$. Substituting $c_k(\bar{\mathcal{H}}_{\mathrm{dR}}) = 0$ in $(**)$, we obtain

COROLLARY 2.5. *For* $g = 4, 5$, *the cycle relation* $(1 + \lambda_1 + \cdots + \lambda_g).\ (1 - \lambda_1 + \cdots (-1)^g \lambda_g) = 1$ *holds in* $CH_*(\bar{\mathcal{A}}_g)_{\mathbb{Q}}$.

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