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CONVOLUTION KERNELS OF $(n+1)$-FOLD MARCINKIEWICZ MULTIPLIERS ON THE HEISENBERG GROUP

A.J. FRASER

We prove a characterisation, in terms of regularity and cancellation conditions, of the convolution kernels of Marcinkiewicz multiplier operators $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ on the Heisenberg group $\mathbf{H}_n$, where $\mathcal{L}_1, \dots, \mathcal{L}_n$ are the n partial sub-Laplacians. The necessity of these regularity and cancellation conditions was established by Fraser (2001); here, we prove their sufficiency.

1. INTRODUCTION

This paper deals with a class of convolution kernels on the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ which in a sense correspond to the product-type Calderón–Zygmund kernels on $\mathbb{R}^n$, and involve an underlying multi-parameter structure. These kernels were defined in [**3**], where they arose as the convolution kernels of spectral multiplier operators on the Heisenberg group analogous to the n-fold Marcinkiewicz multipliers on $\mathbb{R}^n$. We show here that they always arise as such.

The situation thus corresponds to that of Marcinkiewicz multipliers in $\mathbb{R}^n$, (see [**1, 5, 7**]) whose convolution kernels can be seen to be precisely the class of product-type Calderón-Zygmund kernels. Just as multipliers on $\mathbb{R}^n$ can be viewed as functions of $i\dfrac{\partial}{\partial x_1}, \dots, i\dfrac{\partial}{\partial x_n}$, we take for our operators on the Heisenberg group functions $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ of iT and the partial sub-Laplacians $\mathcal{L}_1, \dots, \mathcal{L}_n$. The $(n+1)$-fold Marcinkiewicz-type condition we require is that

$$(1) \qquad \left|(\xi_1\partial_{\xi_1})^{i_1} \cdots (\xi_n\partial_{\xi_n})^{i_n}(\eta\partial_\eta)^j m(\xi, \eta)\right| \leqslant C_{ij}.$$

Our work uses the methods of Müller, Ricci and Stein who in [**6**] studied the case of functions $m(\mathcal{L}, iT)$ of iT and the sub-Laplacian $\mathcal{L}$, where m satisfies a two-fold Marcinkiewicz-type condition,

$$\left|(\xi\partial_\xi)^i(\eta\partial_\eta)^j m(\xi, \eta)\right| \leqslant C_{ij}.$$

In [**3**], by lifting to a larger product group, where multi-parameter methods can be used, and then transferring the results obtained back down to $\mathbb{H}_n$, we established the

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boundedness on L^p, for $1 < p < \infty$, of these Marcinkiewicz multipliers and showed that their convolution kernels are polyradial distributions on $\mathbb{H}_n$, which are smooth away from the $z_i = 0$ planes and satisfy the regularity and cancellation conditions (4)–(7) given in Section 3 below. We now show that these conditions in fact characterise the convolution kernels of Marcinkiewicz multipliers $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$.

The proof is analogous to the proof in $\mathbb{R}^n$ of conditions on the multipliers of Calderón–Zygmund operators, starting from estimates on their convolutions kernels.

We do not have to work with the full group Fourier transform on the Heisenberg group, because our kernels are polyradial, and so their Fourier transforms are diagonalisable operators on the representation space. We are thus able to deal simply with their eigenvalues; that is, with the Gelfand transform of the kernels. Thus, making use of Geller's explicit expression ([**4**]) for the Gelfand transform of a polyradial function on $\mathbb{H}_n$, in terms of Laguerre functions (this and other facts about the Fourier transform on $\mathbb{H}_n$ are outlined below in Section 2), we begin in Section 4 by considering the case of a smooth, compactly supported kernel satisfying (4)–(7), and show that its Gelfand transform is bounded.

In Section 5, we show that a discrete Marcinkiewicz-type condition on the Gelfand transform of C_c^∞ kernels follows from the boundedness proved in Section 4, by the simple observation that certain differential and difference operators, applied to the Gelfand transforms of our kernels, still yield Gelfand transforms of the same kind of kernels. This corresponds to the fact in $\mathbb{R}^n$ that

$$\xi_j \partial_{\xi_j} \widehat{f} = (-f - x_j \partial_{x_j} f)^\frown , \tag{2}$$

and $-f - x_j \partial_{x_j} f$ is still a Calderón-Zygmund kernel if f is. The relations on $\mathbb{H}_n$, analogous to (2) are proved in Lemma 5.1. This discrete Marcinkiewicz-type condition on the Gelfand transform is then shown in Proposition 5.3 to extend to give the required Marcinkiewicz condition on the multiplier m. This is really a result about interpolation between integers to go from difference conditions on a function on $\mathbb{Z}^n$ to differential conditions on an extended smooth function on $\mathbb{R}^n$. (The argument is an adaptation to n-variables of that in [**6**]).

Thus, in Sections 4 and 5, the main result is established for smooth kernels of compact support. The general case is then proved in Section 6 by an approximation argument.

2. Preliminaries

We denote by $\mathbb{H}_n$ the $2n+1$-dimensional Heisenberg group, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication $(z,t) \cdot (w,s) = (z + w, t + s + 2 \operatorname{Im} z \cdot \overline{w})$. The identity under this multiplication is $(0,0)$, and the inverse $(z,t)^{-1}$ of (z,t) is $(-z,-t)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r > 0$, by $r(z,t) = (rz, r^2t)$. These dilations are group automorphisms. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|h| = |(z,t)| = \left(|z|^2 + |t|\right)^{1/2} .$$

Using coordinates $h = (z,t) = (x+iy, t)$, for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}_n$ equal to $\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial}{\partial y_j}$ and $\dfrac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} \qquad \text{and} \qquad T = \frac{\partial}{\partial t}$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T ,$$

for $j = 1, \ldots, n$, and all other commutators equal to 0.

A differential operator D on $\mathbb{H}_n$ is called homogeneous of degree d if

$$D\big(f(r\cdot)\big) = r^d(Df)(r\cdot) .$$

Thus, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are homogeneous of degree one, and T is homogeneous of degree two. The homogenous dimension of $\mathbb{H}_n$ is $2n+2$, the sum of the degrees of the homogeneous basis elements $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, and T.

The partial sub-Laplacians $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\mathbb{H}_n$ are given by

$$\mathcal{L}_j = -\frac{1}{4}(X_j^2 + Y_j^2) ,$$

for $j = 1, \ldots, n$. The operators $\mathcal{L}_1, \ldots, \mathcal{L}_n$, and iT form a family of commuting self-adjoint operators with commuting spectral measures, and so, by the spectral theorem, for $m \in L^\infty\big((\mathbb{R}^+)^n \times \mathbb{R}\big)$, we can define the joint spectral multiplier operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ which is then a bounded operator on $L^2(\mathbb{H}_n)$. Since $\mathcal{L}_1, \ldots, \mathcal{L}_n$, iT are left-invariant, $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ commutes with left-translations and is therefore given by convolution with a distribution $K \in S'(\mathbb{H}_n)$: $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)\varphi = \varphi * K$, for all $\varphi \in S(\mathbb{H}_n)$.

Given any $\tau \in \mathbb{T}^n$, the n-torus, define the operator ρ_τ on functions f on $\mathbb{H}_n$ by

$$\rho_\tau f(z,t) = f(\tau \cdot z, t) .$$

A function f on $\mathbb{H}_n$ will be called *polyradial* if $f = \rho_\tau f$ for all $\tau \in \mathbb{T}^n$. A distribution $K \in \mathcal{S}'(\mathbb{H}_n)$ is said to be *polyradial* if $K(\varphi) = K(\rho_\tau\varphi)$ for all $\tau \in \mathbb{T}^n$ and all $\varphi \in \mathcal{S}(\mathbb{H}_n)$. Since $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and iT commute with all ρ_τ, so does $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$. Therefore the convolution kernel K of $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ is polyradial.

We now set down some facts about the Fourier transform for polyradial functions on the Heisenberg group. These are covered in detail in [**4**].

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The Fock–Bargmann realisation of the (infinite-dimensional) irreducible unitary representations of $\mathbb{H}_n$ is as follows. For $\lambda > 0$, the Fock space $\mathcal{H}_\lambda$ is the set of all holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$\|F\|^2 = \left(\frac{2\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} |F(w)|^2 e^{-2\lambda|w|^2} dw < \infty.$$

$\mathcal{H}_\lambda$ is a Hilbert space, with orthonormal basis

$$E_k^\lambda(w) = \frac{\left(\sqrt{2\lambda}\right)^{|k|}}{\sqrt{k!}} w^k$$

for $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ (where $|k| = k_1 + \dots + k_n$, $k! = k_1! \cdots k_n!$, and $w^k = w_1^{k_1} \cdots w_n^{k_n}$ for $w \in \mathbb{C}^n$). For $\lambda < 0$, $\mathcal{H}_\lambda = \mathcal{H}_{-\lambda}$, $E_k^\lambda = E_k^{-\lambda}$. The irreducible unitary representations π_λ of $\mathbb{H}_n$ on $\mathcal{H}_\lambda$ are given by, for $\lambda > 0$,

$$\pi_\lambda(z,t)F(w) = e^{-i\lambda t} e^{-\lambda(|z|^2 + 2z\cdot w)} F(w + \bar{z})$$

for $F \in \mathcal{H}_\lambda$, and for $\lambda < 0$,

$$\pi_\lambda(z,t)F(w) = e^{-i\lambda t} e^{\lambda(|z|^2 - 2\bar{z}\cdot w)} F(w - z)$$

for $F \in \mathcal{H}_\lambda$. Then $d\pi_\lambda(iT) = \lambda I$, and for $j = 1, \dots, n$, $d\pi_\lambda(\mathcal{L}_j)$ is diagonal on the basis $\{E_k^\lambda\}$, with eigenvalues $(2k_j + 1)|\lambda|$ corresponding to E_k^λ.

A function $f \in L^1(\mathbb{H}_n)$ is polyradial if and only if $\pi_\lambda(f)$ is diagonal on the basis $\{E_k^\lambda\}$, for all $\lambda \neq 0$. In this case, let

$$\pi_\lambda(f)E_k^\lambda = \tilde{f}(k,\lambda)E_k^\lambda ,$$

for $k \in \mathbb{N}^n$, $\lambda \neq 0$, then $\tilde{f}$ is the Gelfand transform, also given by

$$\tilde{f}(k,\lambda) = \int_{\mathbb{H}_n} e^{-i\lambda t} \ell_k(2|\lambda| \, |\mathbf{z}|^2) f(z,t)\, dz\, dt \tag{3}$$

where $|\mathbf{z}|^2 = \left(|z_1|^2, \dots, |z_n|^2\right)$ for $z \in \mathbb{C}^n$, and $\ell_k = \ell_k^0$ is a Laguerre function, defined as follows. For $x > 0$, and $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, the Laguerre polynomials are

$$L_k^m(x) = \sum_{i=0}^{k} \binom{k+m}{k-i} \frac{(-x)^i}{i!} .$$

The Laguerre function ℓ_k^m on $\mathbb{R}^+$ is then

$$\ell_k^m(x) = \left(\frac{k!}{(k+m)!}\right)^{1/2} x^{m/2} L_k^m(x) e^{-x/2} .$$

And finally, for $k \in \mathbb{N}^n$, $m \in \left(\mathbb{N} \cup \{0\}\right)^n$, $x \in (\mathbb{R}^+)^n$, the Laguerre function ℓ_k^m is given by

$$\ell_k^m(x) = \ell_{k_1}^{m_1}(x_1) \cdots \ell_{k_n}^{m_n}(x_n) .$$

The inversion formula for polyradial f is

$$f(x,t) = c_n \int_{-\infty}^{\infty} \sum_{k\in\mathbb{N}^n} \widetilde{f}(k,\lambda) e^{i\lambda t} \ell_k\big(2|\lambda| \| z \|^2\big)\big(2|\lambda|\big)^n d\lambda .$$

Thus, the convolution kernel K of $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ for bounded m on $(\mathbb{R}^+)^n \times \mathbb{R}$ is given by

$$K(z,t) = c_n \int_{-\infty}^{\infty} \sum_{k\in\mathbb{N}^n} m\big((2k_1+1)|\lambda|, \dots, (2k_n+1)|\lambda|, \lambda\big) \\ e^{i\lambda t} \ell_k\big(2|\lambda| \| z \|^2\big)\big(2|\lambda|\big)^n d\lambda .$$

The joint spectrum of $\mathcal{L}_1, \dots, \mathcal{L}_n, iT$ is the closure in $(\mathbb{R}^+)^n \times \mathbb{R}$ of

$$\Big\{\big((2k_1+1)|\lambda|, \dots, (2k_n+1)|\lambda|, \lambda\big) : k \in \mathbb{N}^n \ , \ \lambda \neq 0\Big\} .$$

The multiplier $m(\xi, \eta)$ on this joint spectrum is then related to the Gelfand transform $\mu = \widetilde{K}$ by

$$\mu(k,\lambda) = m\big((2k_1+1)|\lambda|, \dots, (2k_n+1)|\lambda|, \lambda\big)$$

for $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, $\lambda \in \mathbb{R}\backslash\{0\}$.

The following properties of Laguerre functions $\ell_k = \ell_k^0$ on $\mathbb{R}^+$ (see for example [**2**]) will be used throughout this paper:

$$\text{(i)} \ \ x\ell_k' = k\big(\ell_k - \ell_{k-1}\big) - \frac{1}{2}x\ell_k \quad \text{(ii)} \ \ x\ell_k'' + \ell_k' - \frac{x}{4}\ell_k = -\Big(k + \frac{1}{2}\Big)\ell_k$$

$$\text{(iii)} \ \ |\ell_k| \leqslant 1 \qquad\qquad \text{(iv)} \ \ \big|\ell_k'\big| \leqslant C(k+1)$$

3. Kernels of Marcinkiewicz Multiplier Operators on the Heisenberg Group

Before stating our main result, we first define what will be referred to as a product-type kernel on a product group $G = G_1 \times \cdots \times G_N$. This definition applies in the generality of homogeneous groups; however, for our purposes in this paper we shall only consider products of the homogeneous groups $\mathbb{C}$ (where $Q = 2$), and $\mathbb{R}$ ($Q = 1$).

For $j \in \{1, \dots, N\}$, we let G_j be a homogeneous group of homogeneous dimension Q_j. Then G_j is equipped with an automorphic one-parameter dilation (which, for $r_j > 0$, we denote simply by $x_j \mapsto r_j x_j$, for $x_j \in G_j$) and a homogeneous norm $|\cdot|$. Given a basis $\{X_{j,1}, \dots, X_{j,n_j}\}$ of left-invariant vector-fields, for $I \in (\mathbb{Z}^+)^{n_j}$ (where $\mathbb{Z}^+$ denotes the set $\{0, 1, 2, \dots\}$), the degree of the left-invariant differential operator $X_j^I = X_{j,1}^{i_1} \dots X_{j,n_j}^{i_{n_j}}$ on G_j will be denoted by $d_j(I)$.

Notation: Throughout this paper, we shall adopt the following notational conventions for product groups. For x in a product group $G = G_1 \times \cdots \times G_N$, we let

$$\| x \| = \big(|x_1|, \dots, |x_N|\big),$$

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so that, given $J \in (\mathbb{Z}^+)^N$,

$$\| x \|^J = |x_1|^{j_1} \cdots |x_N|^{j_N}.$$

For $j \in \mathbb{Z}^+$, we denote by $\mathbf{j}$ the multi-index $(j, \dots, j) \in (\mathbb{Z}^+)^m$ for a dimension m which will always be clear from the context. We set $Q = (Q_1, \dots, Q_N)$, and for a multi-index $I = (I_1, \dots, I_N)$, with $I_j \in (\mathbb{Z}^+)^{n_j}, j = 1, \dots, N$, we also set

$$d(I) = \big(d_1(I_1), \dots, d_N(I_N)\big).$$

The differential operator $X^I = X_1^{I_1} \cdots X_N^{I_N}$ on G, with $X_j^{I_j}$ on G_j defined as above, then has degree $\left|d(I)\right|$.

We denote multi-parameter dilation, given $r = (r_1, \dots, r_N) \in (\mathbb{R}^+)^N$, by

$$\delta_r(x) = (r_1 x_1, \dots, r_N x_N)$$

for $x \in G$.

Frequently it will be necessary to split a variable x in a product group G into two component variables. In such cases we shall write $x = (x_{\underleftarrow{\ell}}, x_{\underrightarrow{\ell}})$, where

$$x_{\underleftarrow{\ell}} = (x_1, \dots, x_\ell), \quad x_{\underrightarrow{\ell}} = (x_{\ell+1}, \dots, x_N)$$

for $1 \leqslant \ell \leqslant n$, and

$$X_{\underleftarrow{\ell}}^J = X_1^{J_1} \cdots X_\ell^{J_\ell}, \quad X_{\underrightarrow{\ell}}^K = X_{\ell+1}^{K_{\ell+1}} \cdots X_N^{K_N}$$

for $J = (J_1, \dots, J_\ell) \in (\mathbb{Z}^+)^{n_1} \times \cdots \times (\mathbb{Z}^+)^{n_N}$ and $K = (K_{\ell+1}, \dots, K_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \cdots \times (\mathbb{Z}^+)^{n_N}$ We also set

$$G_{\underleftarrow{\ell}} = G_1 \times \cdots \times G_\ell, \quad G_{\underrightarrow{\ell}} = G_{\ell+1} \times \cdots \times G_N,$$

with corresponding definitions of $Q_{\underleftarrow{\ell}}$, $Q_{\underrightarrow{\ell}}$, $d_{\underleftarrow{\ell}}$, and $d_{\underrightarrow{\ell}}$.

We note that for $z \in \mathbb{C}^N$, $\| z \|^2 = |z_1|^2 \cdots |z_N|^2$, while $|z|^2$ is the square of the usual norm, $|z_1|^2 + \cdots + |z_N|^2$.

We shall define the product-type kernel conditions in terms of *normalised bump functions.* A C_c^∞ function φ is called a normalised bump function if φ is supported in the unit ball, and φ and all first order partial derivatives of φ are bounded by a fixed, pre-determined constant.

A function K on $G_1 \times \cdots \times G_N$ is said to be a kernel of product-type (or to satisfy product-type kernel conditions) if it satisfies the following conditions:

(a) the *regularity condition:*

$$\left|X^I K(x)\right| \leqslant C_I \| x \|^{-Q-d(I)}$$

for all $I = (I_1, \dots, I_N)$, $I_j \in (\mathbb{Z}^+)^{n_j}$, $j = 1, \dots, N$;

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(b) for each $\ell = 1, \dots, N$, the *cancellation condition in* $x_{\underleftarrow{\ell}}$:

$$\left| \int_{G_{\underleftarrow{\ell}}} X^I_{\underrightarrow{\ell}} K(x)\big(\delta_r(x_{\underleftarrow{\ell}})\big)\, dx_{\underleftarrow{\ell}} \right| \leqslant C_I \lvert x_{\underrightarrow{\ell}} \rvert^{-Q_{\underrightarrow{\ell}} - d_{\underrightarrow{\ell}}(I)}$$

for all $I = (I_{\ell+1}, \dots, I_N)$, $I_j \in (\mathbb{Z}^+)^{n_j}$, $j = \ell + 1, \dots, N$, all normalised bump functions φ on $G_{\underleftarrow{\ell}}$, and all $r \in (\mathbb{R}^+)^{\ell}$.

In addition, for each permutation $\sigma \in S_N$ K must satisfy the *cancellation condition in* $x_{\sigma(\underleftarrow{\ell})}$ obtained from (b) by permuting the indices $1, \dots, N$ by σ.

In the case where K is a tempered distribution, we assume that K is smooth away from the "planes" $\{x \in G : x_j = 0\}$, $j = 1, \dots, N$, and the cancellation conditions are to be understood as follows. Given φ in the Schwartz space $\mathcal{S}(G_{\underleftarrow{\ell}})$, we define the distribution K_φ by $K_\varphi(\psi) = K(\varphi \otimes \psi)$ for all $\psi \in \mathcal{S}(G_{\underrightarrow{\ell}})$, where $\varphi \otimes \psi(x_1, \dots, x_N) = \varphi(x_{\underleftarrow{\ell}})\psi(x_{\underrightarrow{\ell}})$.

The cancellation condition then states that for all normalised bump functions φ on $G_{\underleftarrow{\ell}}$, and for all $r \in (\mathbb{R}^+)^{\ell}$, the distributions $K_{\varphi\circ\delta_r} \in \mathcal{S}'(G_{\underrightarrow{\ell}})$, are smooth away from the planes $\{x_{\underrightarrow{\ell}} \in G_{\underrightarrow{\ell}} : x_j = 0\}$, $j = \ell + 1, \dots, N$, and uniformly satisfy

$$\left| X^I_{\underrightarrow{\ell}} K_{\varphi\circ\delta_r}(x_{\underrightarrow{\ell}}) \right| \leqslant C_I \lvert x_{\underrightarrow{\ell}} \rvert^{-Q_{\underrightarrow{\ell}} - d_{\underrightarrow{\ell}}(I)}$$

for all $I = (I_{\ell+1}, \dots, I_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \dots \times (\mathbb{Z}^+)^{n_N}$.

We now state the main result of this paper, the characterisation of kernels of Marcinkiewicz multipliers $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$, and observe that these kernels form a subset of the product-type kernels on $\mathbb{C}^n \times \mathbb{R}$. The cancellation conditions (5)–(7) below are to be interpreted for distribution kernels in the same manner as indicated above in the case of the product-type cancellation conditions.

Theorem 3.1. *Let m be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition* (1) *for all $i_1, \dots, i_n, j \in \mathbb{Z}^+$. Then, the convolution kernel K on $\mathbb{H}_n$ corresponding to the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ is polyradial, smooth away from the $z_i = 0$ planes, and satisfies the size condition*

$$\left|\partial_z^I \partial_t^k K(z,t)\right| \leqslant C_{I,k} \lvert z \rvert^{-2-I} \big(|z|^2 + |t|\big)^{-1-k} \tag{4}$$

for all $I \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$; as well as the following cancellation conditions: for all ℓ, $1 \leqslant \ell \leqslant n$,

$$\left| \int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} \partial_t^k K(z,t) \varphi\big(\delta_r(z_{\underleftarrow{\ell}})\big)\, dz_{\underleftarrow{\ell}} \right| \leqslant C_{I,k} \lvert z_{\underrightarrow{\ell}} \rvert^{-2-I} \big(|z_{\underrightarrow{\ell}}|^2 + |t|\big)^{-1-k} \tag{5}$$

for all normalised bump functions φ on $\mathbb{C}^\ell$, $r \in (\mathbb{R}^+)^\ell$, $I \in (\mathbb{Z}^+)^{n-\ell}$, and $k \in \mathbb{Z}^+$;

$$\left| \int_{\mathbb{C}^\ell} \int_{\mathbb{R}} \partial^I_{z_{\underrightarrow{\ell}}} K(z,t) \varphi\big(\delta_r(z_{\underleftarrow{\ell}}, t)\big)\, dz_{\underleftarrow{\ell}}\, dt \right| \leqslant C_I \lvert z_{\underrightarrow{\ell}} \rvert^{-2-I} \tag{6}$$

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for all normalised bump functions φ on $\mathbb{C}^\ell$, $r \in (\mathbb{R}^+)^{\ell+1}$, and $I \in (\mathbb{Z}^+)^{n-\ell}$; all conditions obtained from (5) *and* (6) *by permuting the indices* $1, \dots, n$; *and*

$$\left| \int_{\mathbb{R}} \partial_z^I K(z,t) \varphi(rt)\, dt \right| \leqslant C_I |z|^{-2-I} \tag{7}$$

for all normalised φ on $\mathbb{R}$, $r > 0$, and $I \in (\mathbb{Z}^+)^n$. The converse also holds: every polyradial distribution K on $\mathbb{H}_n$ which is smooth away from the $z_i = 0$ planes and satisfies (4)–(7) *is the convolution kernel of a multiplier operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ for some m on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition* (1) *for all $i_1, \dots, i_n, j \in \mathbb{Z}^+$.*

The forward direction was established in [**3**]. We prove the converse here.

4. The result for smooth compactly supported kernels

Proposition 4.1. *If $K \in C_c^\infty(\mathbb{H}_n)$ is polyradial and satisfies* (4)–(7), *then $\mu(k,\lambda) = \widetilde{K}(k,\lambda)$ (as given in* (3)*) is bounded by a constant that depends only on the constants in the conditions* (4)–(7).

Proof: By homogeneity, we may assume $\lambda = 1$. We have

$$\mu(k,1) = \int_{\mathbb{H}_n} e^{-it} \ell_k(2|z|^2) K(z,t)\, dz\, dt\,.$$

We break up the integral according to the size of the $|z_i|$'s by introducing a normalised bump function φ on $\mathbb{R}$, such that $\varphi \equiv 1$ on $\left[-(1/2),(1/2)\right]$. Observe that $\psi(w) = \varphi(|w|^2)$ is then a normalised bump function on $\mathbb{C}$. Thus,

$$\mu(k,1) = \sum_{\ell=0}^{n} \sum_{1 \leqslant i_1 < \dots < i_\ell \leqslant n} I_{\ell(i_1,\dots,i_\ell)}$$

with

$$\begin{aligned} I_{\ell(i_1,\dots,i_\ell)} = \int_{\mathbb{H}_n} & \Bigl(1 - \varphi(2k_{i_1}|z_{i_1}|^2)\Bigr) \cdots \Bigl(1 - \varphi(2k_{i_\ell}|z_{i_\ell}|^2)\Bigr) \\ & \cdot \varphi(2k_{j_1}|z_{j_1}|^2) \cdots \varphi(2k_{j_{n-\ell}}|z_{j_{n-\ell}}|^2) e^{-it} \ell_k(2|z|^2) K(z,t)\, dz\, dt \end{aligned}$$

where $j_1 < \dots < j_{n-\ell}$ are such that $\{i_1, \dots, i_\ell, j_1, \dots, j_{n-\ell}\} = \{1, \dots, n\}$.

Now, given ℓ, $0 \leqslant \ell \leqslant n$, by relabelling of variables, it suffices to consider only the term $I_{\ell(1,\dots,\ell)}$.

Since K is polyradial, we may define

$$f(2|z_1|^2, \dots, 2|z_\ell|^2, z_{\ell+1}, \dots, z_n, t) = K(z,t)\,.$$

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Then, changing variables $r_i = 2|z_i|^2$ for $i = 1, \dots, \ell$, we have,

$$\begin{aligned} I_{\ell(1,\dots,\ell)} = \Big(\frac{\pi}{2}\Big)^\ell \int_{\mathbb{C}^{n-\ell}\times\mathbb{R}} \int_{(\mathbb{R}^+)^\ell} & \big(1-\varphi(k_1 r_1)\big)\cdots\big(1-\varphi(k_\ell r_\ell)\big) \\ & \cdot \varphi\big(2k_{\ell+1}|z_{\ell+1}|^2\big)\cdots\varphi\big(2k_n|z_n|^2\big)\ell_{k_{\ell+1}}\big(2|z_{\ell+1}|^2\big)\cdots\ell_{k_n}\big(2|z_n|^2\big) \\ & \cdot \ell_{k_1}(r_1)\cdots\ell_{k_\ell}(r_\ell)e^{-it} f(r_1,\dots,r_\ell,z_{\ell+1},\dots,z_n,t)\,dr\,dz'\,dt \\ = \Big(\frac{\pi}{2}\Big)^\ell \int_{(\mathbb{R}^+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}} & \Phi_1(r)\ell(r)e^{-it} f(r,z',t)u(z')\,dr\,dz'\,dt \end{aligned}$$

where $r = (r_1, \dots, r_\ell)$, $z' = (z_{\ell+1}, \dots, z_n)$,

$$\Phi_1(r) = \big(1-\varphi(k_1 r_1)\big)\cdots\big(1-\varphi(k_\ell r_\ell)\big), \qquad \ell(r) = \ell_{k_1}(r_1)\cdots\ell_{k_\ell}(r_\ell),$$

and by properties (iii) and (iv) for Laguerre functions,

$$u(z') = \varphi\big(2k_{\ell+1}|z_{\ell+1}|^2\big)\cdots\varphi\big(2k_n|z_n|^2\big)\ell_{k_{\ell+1}}\big(2|z_{\ell+1}|^2\big)\cdots\ell_{k_n}\big(2|z_n|^2\big)$$

is a dilate of a normalised bump function on $\mathbb{C}^{n-\ell}$.

We remark that using (5) directly at this stage to estimate

$$\int_{\mathbb{C}^{n-\ell}} f(r,z',t)u(z')\,dz'$$

would yield $r_1^{-1}\cdots r_\ell^{-1}\big(r_1\cdots + r_\ell + |t|\big)^{-1}$, which is too large. There are two tricks at our disposal to improve matters. Writing e^{-it} as its own derivative and integrating by parts in t, we can move as many derivatives $\partial_t{}^N$ as we wish onto f, so that (5) then gives the estimate $r_1^{-1}\cdots r_\ell^{-1}\big(r_1 + \cdots + r_\ell + |t|\big)^{-N-1}$, which is integrable (for example, for $N = \ell$) on the support of $\big(1-\varphi(k_1 r_1)\big)\cdots\big(1-\varphi(k_\ell r_\ell)\big)$, but at a cost dependent on k:

$$\begin{aligned} \big|I_{\ell(1,\dots,\ell)}\big| &\leqslant C\int_{r_i \geqslant 1/(2k_i), i=1,\dots,\ell} r_1^{-1}\cdots r_\ell^{-1}\int_{\mathbb{R}}\big(r_1+\cdots+r_\ell+|t|\big)^{-\ell-1}\,dt\,dr \\ &\leqslant C\,k_1\cdots k_\ell\,. \end{aligned}$$

The second trick is the corresponding one in the $r_1, \dots, r_\ell$ variables, and we begin with this. By property (ii), we can write the Laguerre functions in terms of their derivatives:

$$\ell_{k_i}(r_i) = \frac{-1}{k_i + 1/2}\Big[r_i\Big(\frac{d}{dr_i}\Big)^2 + \frac{d}{dr_i} - \frac{1}{4}r_i\Big]\ell_{k_i}(r_i)\,.$$

Setting $L_i = L_i^2 + L_i^1 + L_i^0$, with $L_i^2 = r_i\Big(\dfrac{d}{dr_i}\Big)^2$, $L_i^1 = \dfrac{d}{dr_i}$, and $L_i^0 = -r_i/4$, then,

$$\ell = \ell_{k_1}\cdots\ell_{k_\ell} = \frac{(-1)^\ell}{(k_1+1/2)\cdots(k_\ell+1/2)}L_1\ell_{k_1}\cdots L_\ell\ell_{k_\ell}$$

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with $L_1\ell_{k_1}\cdots L_\ell\ell_{k_\ell}$ expanding to

$$(L_1^2\ell_{k_1}+L_1^1\ell_{k_1}+L_1^0\ell_{k_1})\cdots(L_\ell^2\ell_{k_\ell}+L_\ell^1\ell_{k_\ell}+L_\ell^0\ell_{k_\ell})$$
$$=\sum L^2_{a_{\ell_1+1}}\ell_{k_{a_{\ell_1+1}}}\cdots L^2_{a_{\ell_2}}\ell_{k_{a_{\ell_2}}}L^1_{a_1}\ell_{k_{a_1}}\cdots L^1_{a_{\ell_1}}\ell_{k_{a_{\ell_1}}}L^0_{a_{\ell_2+1}}\ell_{k_{a_{\ell_2+1}}}\cdots L^0_{a_\ell}\ell_{k_{a_\ell}}$$

where the summation is over all ℓ_1,ℓ_2, $0\leqslant\ell_1\leqslant\ell_2\leqslant\ell$, and all partitions a of $\{1,\dots,\ell\}$ into three sets, $\{a_{\ell_1+1}<\cdots<a_{\ell_2}\}$, $\{a_1<\cdots<a_{\ell_1}\}$, and $\{a_{\ell_2+1}<\cdots<a_\ell\}$ of cardinalities $\ell_2-\ell_1$, ℓ_1, and $\ell-\ell_2$. Thus, correspondingly,

$$I_{\ell(1,\dots,\ell)}=\sum_{\ell_1=0}^{\ell}\sum_{\ell_2=\ell_1}^{\ell}\sum_a A_a^{\ell_1,\ell_2}.$$

Given $0\leqslant\ell_1\leqslant\ell_2\leqslant\ell$, without loss of generality, we consider only the term $A^{\ell_1,\ell_2}=A_a^{\ell_1,\ell_2}$, with the partition a given by $a_i=i$, for $i=1,\dots,\ell$.

The proposition will be proved once we have shown that A^{ℓ_1,ℓ_2} is bounded. Letting

$$A^{\ell_1,\ell_2}=\left(\frac{\pi}{2}\right)^\ell\frac{(-1)^\ell}{(k_1+1/2)\cdots(k_\ell+1/2)}B^{\ell_1,\ell_2},$$

then $|A^{\ell_1,\ell_2}|\leqslant Ck_1^{-1}\cdots k_\ell^{-1}|B^{\ell_1,\ell_2}|$, and so it suffices to show $|B^{\ell_1,\ell_2}|\leqslant Ck_1\dots k_\ell$ (thus giving us lee-way to use the integration by parts in t trick if needed).

We thus consider

$$\begin{aligned}B^{\ell_1,\ell_2}=c\int_{(\mathbb{R}^+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}}&\Phi_1(r)f(r,z',t)e^{-it}u(z')\\&\cdot r_{\ell_1+1}\cdots r_{\ell_2}\left(\frac{d}{dr_{\ell_1+1}}\right)^2\ell_{k_{\ell_1+1}}(r_{\ell_1+1})\cdots\left(\frac{d}{dr_{\ell_2}}\right)^2\ell_{k_{\ell_2}}(r_{\ell_2})\\&\cdot\left(\frac{d}{dr_1}\right)\ell_{k_1}(r_1)\cdots\left(\frac{d}{dr_{\ell_1}}\right)\ell_{k_{\ell_1}}(r_{\ell_1})\\&\cdot r_{\ell_2+1}\cdots r_\ell\ell_{k_{\ell_2+1}}(r_{\ell_2+1})\cdots\ell_{k_\ell}(r_\ell)\,dr\,dz'\,dt\,.\end{aligned}$$

Since f has compact support, and because of the presence of the factor $\Phi_1(r)=\big(1-\varphi(k_1r_1)\big)\cdots\big(1-\varphi(k_\ell r_\ell)\big)$, we may integrate by parts in the r-variables. Thus, B^{ℓ_1,ℓ_2} becomes

$$\begin{aligned}c\int_{(\mathbb{R}^+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}}&\partial_{r_1}\cdots\partial_{r_{\ell_1}}\partial^2_{r_{\ell_1+1}}\cdots\partial^2_{r_{\ell_2}}\left[r_{\ell_1+1}\cdots r_{\ell_2}\Phi_1(r)f(r,z',t)\right]\\&\cdot\ell(r)e^{-it}u(z')r_{\ell_2+1}\cdots r_\ell\,dr\,dz'\,dt\,.\end{aligned}$$

In order to expand the derivative expression in the first line of the integrand, we observe that each second order derivative produces five kinds of terms:

$$\begin{aligned}\partial^2_{r_i}\Big\{r_i\big(1-\varphi(k_ir_i)\big)g\Big\}=\;&2\big(1-\varphi(k_ir_i)\big)\partial_{r_i}g-2r_ik_i\varphi'(k_ir_i)\partial_{r_i}g\\&-2k_i\varphi'(k_ir_i)g-r_ik_i^2\varphi''(k_ir_i)g+r_i\big(1-\varphi(k_ir_i)\big)\partial^2_{r_i}g\,.\end{aligned}$$

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Thus, $\partial^2_{r_{\ell_1+1}} \cdots \partial^2_{r_{\ell_2}}[r_{\ell_1+1}\cdots r_{\ell_2}\Phi_1 f]$ expands into a sum of $5^{\ell_2-\ell_1}$ terms given by, for $\ell_1 = j_1 \leqslant j_2 \leqslant j_3 \leqslant j_4 \leqslant j_5 \leqslant j_6 = \ell_2$, all terms obtained by permuting the variables $r_{\ell_1+1},\dots,r_{\ell_2}$ in the term

$$\Psi_1\ \Psi_2\ \Psi_3\ \Psi_4\ \Psi_5\ \Phi_2\ \partial_r^\beta f$$

where $\Phi_2(r)$ is just $\Phi_1(r)$ without the factors $\big(1-\varphi(k_{\ell_1+1}r_{\ell_1+1})\big)\cdots\big(1-\varphi(k_{\ell_2}r_{\ell_2})\big)$, and

$$ {}_1(r) = \prod_{i\in\mathcal{J}_1}\big(1-\varphi(k_ir_i)\big) \qquad \Psi_2(r) = \prod_{i\in\mathcal{J}_2} r_ik_i\varphi'(k_ir_i)$$

$$\Psi_3(r) = \prod_{i\in\mathcal{J}_3} k_i\varphi'(k_ir_i) \qquad \Psi_4(r) = \prod_{i\in\mathcal{J}_4} r_ik_i^2\varphi''(k_ir_i)$$

$$\Psi_5(r) = \prod_{i\in\mathcal{J}_5} r_i\big(1-\varphi(k_ir_i)\big)$$

where for $i=1,\dots,5$, $\mathcal{J}_i=\{j_i+1,\dots,j_{i+1}\}$, $\mathcal{J}_i=\emptyset$ if $j_i=j_{i+1}$, and $\beta=(b_1,\dots,b_\ell)\in(\mathbb{Z}^+)^\ell$ is the multi-index with all entries 0, except $b_i=1$, for $i\in\mathcal{J}_1\cup\mathcal{J}_2, b_i=2$, for $i\in\mathcal{J}_5$, so that

$$\partial_r^\beta = \partial_{r_{\ell_1+1}}\cdots\partial_{r_{j_3}}\partial^2_{r_{j_5+1}}\cdots\partial^2_{r_{\ell_2}}\ .$$

Next, each first order derivative produces two kinds of terms:

$$\partial_{r_i}\Big[\big(1-\varphi(k_ir_i)\big)g\Big] = -k_i\varphi'(k_ir_i)g + \big(1-\varphi(k_ir_i)\big)\partial_{r_i}g$$

Thus, the full expansion of

$$\partial_{r_1}\cdots\partial_{r_{\ell_1}}\partial^2_{r_{\ell_1+1}}\cdots\partial^2_{r_{\ell_2}}\big[r_{\ell_1+1}\cdots r_{\ell_2}\Phi(r)f(r,z',t)\big]$$

has $2^{\ell_1}5^{\ell_2-\ell_1}$ terms given by, for each $0\leqslant j_0\leqslant j_1\leqslant j_2\leqslant j_3\leqslant j_4\leqslant j_5\leqslant j_6$, the $\binom{\ell_1}{j_0}\binom{\ell_2}{j_2j_3j_4j_5}$ different terms obtained by permuting the variables $r_1,\dots,r_{\ell_1}$ and $r_{\ell_1+1},\dots,r_{\ell_2}$ in the term

$$\Psi_{-1}\Psi_0\ \Psi_1\ \Psi_2\ \Psi_3\ \Psi_4\ \Psi_5\ \Phi\ \partial_r^\alpha f\ ,$$

with $\Phi(r)=\big(1-\varphi(k_{\ell_2+1}r_{\ell_2+1})\big)\cdots\big(1-\varphi(k_\ell r_\ell)\big)$, and

$$\Psi_{-1} = \prod_{i\in\mathcal{J}_{-1}} k_i\varphi'(k_ir_i) \qquad \Psi_0 = \prod_{i\in\mathcal{J}_0}\big(1-\varphi(k_ir_i)\big),$$

where for $i=-1,0,(j_{-1}=0)$, $\mathcal{J}_i$ is defined as above, and $\alpha=\beta+\gamma$, with $\gamma=(c_1,\dots,c_\ell), c_i=1$ for $i\in\mathcal{J}_0$, and 0 otherwise, so that

$$\partial_r^\alpha = \partial_{r_{j_0+1}}\cdots\partial_{r_{j_3}}\partial^2_{r_{j_5+1}}\cdots\partial^2_{r_{\ell_2}}\ .$$

We also set $\Psi_6=r_{\ell_2+1}\cdots r_\ell\Phi$, and $\mathcal{J}_6=\{\ell_2+1,\dots,\ell\}$.

The integral B^{ℓ_1,ℓ_2} therefore splits up accordingly into $2^{\ell_1}5^{\ell_2-\ell_1}$ terms. Given j_0, j_2, j_3, j_4, j_5 as above, it suffices to consider the term

$$B_j^{\ell_1,\ell_2} = \int_{(\mathbb{R}+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}} \partial_r^\alpha f(r,z',t)\Psi(r)\ell(r)e^{-it}u(z')\,dr\,dz'\,dt$$

with $\Psi(r) = \Psi_{-1}\Psi_0\ \Psi_1\ \Psi_2\ \Psi_3\ \Psi_4\ \Psi_5\ \Psi_6$. The Proposition will be proved if we show that $\left|B_j^{\ell_1,\ell_2}\right| \leqslant Ck_1\cdots k_\ell$. We shall establish this estimate below, in Case 1 (if $j_5 < \ell_2$ or $\ell_2 < \ell$) and Case 2 (otherwise).

Case 1. $\mathcal{J}_5 \cup \mathcal{J}_6 \neq \emptyset$. In this case, we use the integration by parts in t trick to improve the estimate (5) by an extra r_i^{-1} for each $i \in \mathcal{J}_5$ and by r_i^{-2} for each $i \in \mathcal{J}_6$. That is, we let $N = |\mathcal{J}_5| + 2|\mathcal{J}_6| = \ell_2 - j_5 + 2(\ell - \ell_2)$. Then, using

$$e^{-it} = (i)^N \left(\frac{d}{dt}\right)^N e^{-it}\ ,$$

and integrating by parts N times in t, we obtain,

$$\begin{aligned}\left|B_j^{\ell_1,\ell_2}\right| &= C\left|\int_{(\mathbb{R}+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}} \partial_t^N\partial_r^\alpha f(r,z',t)\Psi(r)e^{-it}u(z')\,dr\,dz'\,dt\right| \\ &\leqslant C\int_\Omega\int_{\mathbb{R}}\left|\int_{\mathbb{C}^{n-\ell}} \partial_t^N\partial_r^\alpha f(r,z',t)u(z')\,dz'\right|\left|\Psi(r)\right|\,dt\,dr\ ,\end{aligned}$$

where Ω is the region

$$\Omega = \Big\{r \in \mathbb{R}^\ell :\ \frac{1}{2k_i} \leqslant r_i \leqslant \frac{1}{k_i},\quad i \in \mathcal{J}_{-1}\cup\mathcal{J}_2\cup\mathcal{J}_3\cup\mathcal{J}_4; \\ \qquad\qquad \frac{1}{2k_i} \leqslant r_i,\qquad i \in \mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_5\cup\mathcal{J}_6\Big\}$$

containing the support of Ψ. Estimating the inner integral by the cancellation condition (5) in z', then

$$\begin{aligned}|B_j^{\ell_1,\ell_2}| &\leqslant C\int_{\Omega\times\mathbb{R}} \prod_{i\in\mathcal{J}_{-1}\cup\mathcal{J}_3\cup\mathcal{J}_4\cup\mathcal{J}_6} r_i^{-1} \prod_{i\in\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_2} r_i^{-3/2} \prod_{i\in\mathcal{J}_5} r_i^{-2} \\ &\qquad\qquad \left(r_1+\cdots+r_\ell+|t|\right)^{-1-N}\left|\Psi(r)\right|\,dt\,dr \\ &\leqslant C\int_{\Omega\times\mathbb{R}} \prod_{i\in\mathcal{J}_{-1}\cup\mathcal{J}_3} k_i r_i^{-1} \prod_{i\in\mathcal{J}_2} k_i r_i^{-1/2} \prod_{i\in\mathcal{J}_4} k_i^2 \prod_{i\in\mathcal{J}_0\cup\mathcal{J}_1} r_i^{-3/2} \prod_{i\in\mathcal{J}_5} r_i^{-1} \\ &\qquad\qquad \left(r_1+\cdots+r_\ell+|t|\right)^{-1-N}\,dt\,dr\end{aligned}$$

But

$$\int_{\mathbb{R}}\left(r_1+\cdots+r_\ell+|t|\right)^{-1-N}dt \leqslant C(r_1+\cdots+r_\ell)^{-N} \leqslant C\prod_{i\in\mathcal{J}_5} r_i^{-1}\prod_{i\in\mathcal{J}_6} r_i^{-2}$$

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since $N = |\mathcal{J}_5| + 2|\mathcal{J}_6|$, whence $|B_j^{\ell_1,\ell_2}| \leqslant Ck_1 \cdots k_\ell$, as required.

CASE 2. $\mathcal{J}_5 \cup \mathcal{J}_6 = \emptyset$.

In this case, the cancellation condition (5) in z' gives too large an estimate, which cannot be improved by integrating by parts in t. We must therefore take into account the cancellation in t. Recall that φ is a normalised bump function on $\mathbb{R}$, with $\varphi \equiv 1$ on $[-(1/2), (1/2)]$. Including the factor 1,

$$1 = \varphi(t) + \big(1 - \varphi(t)\big)$$

in the integrand, we then break B^{ℓ_1,ℓ_2} accordingly into two integrals:

$$B_j^{\ell_1,\ell_2} = B_{j,1}^{\ell_1,\ell_2} + B_{j,2}^{\ell_1,\ell_2} .$$

We first observe that $|B_{j,1}^{\ell_1,\ell_2}|$ is bounded by

$$\int_\Omega \left| \int_{\mathbb{C}^{n-\ell}\times\mathbb{R}} \partial_r^\alpha f(r,z',t) e^{-it} u(z')\varphi(t)\, dz'\, dt \right| \big|\Psi(r)\big|\, dr ,$$

with Ω as above. By the cancellation condition (6) in z' and t, since $e^{-it}u(z')\varphi(t)$ is a dilate of a normalised bump function in z' and t,

$$\begin{aligned}
\big|B_{j,1}^{\ell_1,\ell_2}\big| &\leqslant C \int_\Omega \prod_{i\in\mathcal{J}_{-1}\cup\mathcal{J}_3\cup\mathcal{J}_4} r_i^{-1} \prod_{i\in\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_2} r_i^{-3/2}\ \big|\Psi(r)\big|\, dr \\
&\leqslant C \int_\Omega \prod_{i\in\mathcal{J}_{-1}\cup\mathcal{J}_3} k_i r_i^{-1} \prod_{i\in\mathcal{J}_2} k_i r_i^{-1/2} \prod_{i\in\mathcal{J}_4} k_i^2 \prod_{i\in\mathcal{J}_0\cup\mathcal{J}_1} r_i^{-3/2}\, dr \\
&\leqslant\ C\ k_1 \cdots k_\ell .
\end{aligned}$$

Next, for $B_{j,2}^{\ell_1,\ell_2}$, writing $e^{-it} = i\dfrac{d}{dt}e^{-it}$, and integrating by parts in t, we obtain,

$$\begin{aligned}
\big|B_{j,2}^{\ell_1,\ell_2}\big| &= C\left| \int_{(\mathbb{R}^+)^\ell\times\mathbb{C}^{n-\ell}\times\mathbb{R}} \Psi(r)e^{-it}u(z')\big(1-\varphi(t)\big)\partial_r^\alpha f(r,z',t)\, dr\, dz'\, dt \right| \\
&= C\left| \int_{(\mathbb{R}^+)^\ell\times\mathbb{C}^{n-\ell}} \int_{\mathbb{R}} \Big[\varphi'(t)\partial_r^\alpha f(r,z',t) + \big(1-\varphi(t)\big)\partial_t\partial_r^\alpha f(r,z',t)\Big] \right. \\
&\qquad\qquad \left. \Psi(r)e^{-it}u(z')\, dt\, dr\, dz' \right| \\
&\leqslant C\int_\Omega \left[\int_{(1/2)\leqslant|t|\leqslant 1} \left| \int_{\mathbb{C}^{n-\ell}} \partial_r^\alpha f(r,z',t)u(z')\, dz' \right| dt \right. \\
&\qquad \left. + \int_{|t|\geqslant 1/2} \left| \int_{\mathbb{C}^{n-\ell}} \partial_t\partial_r^\alpha f(r,z',t)u(z')\, dz' \right| dt \right] \big|\Psi(r)\big|\, dr
\end{aligned}$$

Then, by the cancellation condition (5) in z', $\big|B_2^{\ell_1,\ell_2}\big|$ is bounded by

$$\begin{aligned}
C\int_\Omega \prod_{i\in\mathcal{J}_{-1}\cup\mathcal{J}_3\cup\mathcal{J}_4} r_i^{-1} \prod_{i\in\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_2} r_i^{-3/2} & \left[\int_{(1/2)\leqslant|t|\leqslant 1} (r_1 + \cdots + r_\ell + |t|)^{-1}\, dt \right. \\
& \left. + \int_{|t|\geqslant(1/2)} \big(r_1 + \cdots + r_\ell + |t|\big)^{-2}\, dt \right] \big|\Psi(r)\big|\, dr .
\end{aligned}$$

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Since the integrals over t are bounded, we are left with the same integral obtained for $B_{j,1}^{\ell_1,\ell_2}$, which concludes the proof of Proposition 4.1. $\square$

REMARK. We notice from the proof that in Proposition 4.1, conditions (4)–(7) were in fact not required for *all* derivatives on the kernel K, but only as follows: (4) for all $\partial_z^I \partial_t^k$ where $I \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$ such that $i_j \leqslant 2$, $j = 1, \ldots, n$ and $k \leqslant \max\Big(1, \sum_{j=0}^{n}(\delta_{i_j 2} + 2\delta_{i_j 0})\Big)$; (5) for all $\partial_{z_{\underset{\rightarrow}{\ell}}}^I \partial_t^k$ for I, k as above, with $n - \ell$ replacing n; (6) for all $\partial_{z_{\underset{\rightarrow}{\ell}}}^I$ with I as for (5); and (7) for all ∂_z^I with I as for (4).

5. Conditions on the Gelfand Transform

Define the difference operators Δ_i, $i = 1, \ldots, n$ by

$$\Delta_i \mu(k, \lambda) = \mu(k, \lambda) - \mu(k - 1_i, \lambda)$$

where $1_i = (j_1, \ldots, j_n) \in \mathbb{N}^n$ with $j_\ell = \delta_{\ell i}$.

Lemma 5.1. *If $f \in C_c^\infty(\mathbb{H}_n)$ is polyradial, with $\mu(k, \lambda) = \widetilde{f}(k, \lambda)$ as in (3), then*

$$\lambda \frac{\partial}{\partial \lambda} \mu(k, \lambda) = -\Big[(n+1)f + t\frac{\partial}{\partial t} f + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} f\Big]^{\sim}(k, \lambda)\ , \tag{8}$$

$$k_j \Delta_j \mu(k, \lambda) = -\Big[f + z_j \frac{\partial}{\partial z_j} f + i(\operatorname{sgn}\lambda)|z_j|^2 \frac{\partial}{\partial t} f\Big]^{\sim}(k, \lambda)\ . \tag{9}$$

PROOF: Since f is polyradial, we may set $f_0\big(2|z_1|^2, \ldots, 2|z_n|^2, t\big) = f(z, t)$. Then by the change of variables $r_i = 2|z_i|^2$, $i = 1, \ldots, n$, we have

$$\begin{aligned}\mu(k, \lambda) &= \int_{\mathbb{H}_n} e^{-i\lambda t} \ell_k(2|\lambda| \| z \|^2) f(z, t)\, dz\, dt \\ &= \Big(\frac{\pi}{2}\Big)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(2|\lambda| \| z \|^2) f_0(r_1, \ldots, r_n, t)\, dr\, dt\ .\end{aligned}$$

We first prove (8). We notice that $\lambda\partial/\partial\lambda\Big[e^{-i\lambda t}\ell_k\big(|\lambda| r\big)\Big]$ equals

$$\begin{aligned}&\lambda \frac{\partial e^{-i\lambda t}}{\partial \lambda} \ell_k\big(|\lambda| r\big) + \sum_{j=1}^{n} e^{-i\lambda t} \ell_{k_1}\big(|\lambda| r_1\big) \cdot\cdot \widehat{\jmath} \cdot\cdot\, \ell_{k_n}\big(|\lambda| r_n\big) \lambda \frac{\partial}{\partial \lambda} \ell_{k_j}\big(|\lambda| r_j\big) \\ &\qquad = t \frac{\partial e^{-i\lambda t}}{\partial t} \ell_k\big(|\lambda| r\big) + \sum_{j=1}^{n} e^{-i\lambda t} \ell_{k_1}\big(|\lambda| r_1\big) \cdot\cdot \widehat{\jmath} \cdot\cdot\, \ell_{k_n}\big(|\lambda| r_n\big) r_j \frac{\partial}{\partial r_j} \ell_{k_j}\big(|\lambda| r_j\big)\end{aligned}$$

where the shorthand $a_1 \cdot\cdot \widehat{\jmath} \cdot\cdot\, a_n$ will be used to denote the product $a_1 \cdots a_{j-1} a_{j+1} \cdots a_n$, (that is, without the j^{th} factor a_j). Observing the presence of the

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factor r_j, and the fact that f_0 has compact support, we integrate by parts in t and the r_j variables, to obtain

$$\begin{aligned}
\lambda\frac{\partial}{\partial\lambda}\mu(k,\lambda) &= -\left(\frac{\pi}{2}\right)^n\Bigg(\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big)\frac{\partial}{\partial t}\big(tf_0(r,t)\big)\,dr\,dt \\
&\qquad+\sum_{j=1}^{n}\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big)\frac{\partial}{\partial r_j}\big(r_jf_0(r,t)\big)\,dr\,dt\Bigg) \\
&= -\left(\frac{\pi}{2}\right)^n\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big) \\
&\qquad\qquad\left[(n+1)f_0(r,t)+t\frac{\partial}{\partial t}f_0(r,t)+\sum_{j=1}^{n}r_j\frac{\partial}{\partial r_j}f_0(r,t)\right]dr\,dt\,,
\end{aligned}$$

which proves (8), since $r_j\dfrac{\partial}{\partial r_j}f_0(r,t) = z_j\dfrac{\partial}{\partial z_j}f(z,t)$.

Now, for (9), using property (i) for the Laguerre function ℓ_{k_j},

$$k_j\Delta_j\ell_{k_j}\big(|\lambda|r_j\big) = |\lambda|r_j\ell'_{k_j}\big(|\lambda|r_j\big)+\frac{1}{2}|\lambda|r_j\ell_{k_j}\big(|\lambda|r_j\big)$$

and thus $k_j\Delta_j\mu(k,\lambda)$ equals

$$\begin{aligned}
&\left(\frac{\pi}{2}\right)^n\int_{(\mathbb{R}^+)^n\times\mathbb{R}} e^{-i\lambda t}\ell_{k_1}\big(|\lambda|r_1\big)\cdot\cdot\widehat{j}\cdot\cdot\,\ell_{k_n}\big(|\lambda|r_n\big)r_j\frac{\partial}{\partial r_j}\ell_{k_j}\big(|\lambda|r_j\big)f_0(r,t)\,dr\,dt \\
&\qquad\qquad+\left(\frac{\pi}{2}\right)^n\int_{(\mathbf{R}^+)^n\times\mathbf{R}}\ell_k\big(|\lambda|r\big)\frac{1}{2}|\lambda|r_je^{-i\lambda t}f_0(r,t)\,dr\,dt\,.
\end{aligned}$$

Integrating by parts in r_j for the first term, and in t for the second, where we write $|\lambda|e^{-i\lambda t} = i(\operatorname{sgn}\lambda)\dfrac{\partial}{\partial t}e^{-i\lambda t}$, then the above is equal to

$$\begin{aligned}
&-\left(\frac{\pi}{2}\right)^n\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big)\frac{\partial}{\partial r_j}\,\big(r_jf_0(r,t)\big)\,dr\,dt \\
&-\left(\frac{\pi}{2}\right)^n\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big)(i\operatorname{sgn}\lambda)\frac{1}{2}r_j\frac{\partial}{\partial t}f_0(r,t)\,dr\,dt \\
&= -\int_{(\mathbf{R}^+)^n\times\mathbf{R}} e^{-i\lambda t}\ell_k\big(|\lambda|r\big)\left[f_0(r,t)+r_j\frac{\partial}{\partial r_j}f_0(r,t)+i\operatorname{sgn}\lambda\frac{r_j}{2}\frac{\partial}{\partial t}f_0(r,t)\right]dr\,dt
\end{aligned}$$

which proves (9). ∎

With this lemma and Proposition 4.1 in hand, the Gelfand transform of our kernel is easily seen to satisfy the discrete Marcinkiewicz condition

$$\left|(k_1\Delta_1)^{\alpha_1}\cdots(k_n\Delta_n)^{\alpha_n}(\lambda\partial_\lambda)^b\mu(k,\lambda)\right| \leqslant C_{\alpha b} \tag{10}$$

for all α_i, $b\in\mathbb{N}\cup\{0\}$:

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COROLLARY 5.2. *If $K \in C_c^\infty(\mathbb{H}_n)$ is polyradial, and satisfies* (4)–(7), *then* $\mu(k,\lambda) = \widetilde{K}(k,\lambda)$ *satisfies* (10), *with constants $C_{\alpha,b}$ that depend only on the constants in* (4)–(7).

PROOF: Since $t\dfrac{\partial}{\partial t}K$, $z_j\dfrac{\partial}{\partial z_j}K$ and $|z_j|^2\dfrac{\partial}{\partial t}K$ also satisfy (4)–(7) if K does, then by Lemma 5.1, applying $\lambda\dfrac{\partial}{\partial\lambda}$ or $k_j\Delta_j$ to the ~-transform of a C_c^∞ function satisfying (4)–(7) still yields a ~-transform of a C_c^∞ function satisfying (4)–(7). Thus the result follows from Proposition 4.1. ∎

REMARK. Further to the remark at the end of Section 4, we observe that it is of course here in Corollary 5.2 that conditions (4)–(7) are required for *all* derivatives on the kernel K.

The Marcinkiewicz condition (1) on the multiplier m then follows by the following argument of interpolation between integers.

PROPOSITION 5.3. *If $\mu(k,\lambda) = m\big((2k_1+1)|\lambda|,\dots,(2k_n+1)|\lambda|,\lambda\big)$ satisfies* (10) *then m satisfies* (1) *with constants $C_{\alpha b}$ depending only on the constants $C_{\alpha b}$ in the condition* (10) *on μ.*

PROOF: Let $\varphi \in C_c^\infty(\mathbb{R})$ be supported in $\big[-(3\pi/2),(3\pi/2)\big]$, with $\varphi \equiv 1$ on $\big[-(\pi/2),(\pi/2)\big]$, and

$$\sum_{k\in\mathbf{Z}} \varphi(x+2k\pi) = 1$$

for all $x \in \mathbb{R}$. Then, setting $\Phi(x) = \varphi(x_1)\cdots\varphi(x_n)$ for $x \in \mathbb{R}^n$, we have

$$\widehat{\Phi}(k) = \delta_{k0} \quad \text{for } k \in \mathbb{Z}^n\,.$$

For all $\lambda \neq 0$, we set $a_k(\lambda) = \mu(k,\lambda)$ for $k \in \mathbb{N}^n$, and $a_k(\lambda) = -a_{k'}(\lambda)$ for k, $k' \in \mathbb{Z}^n$, if k and k' differ only in one component, say the j^{th}, where $k_j = -k'_j$. For $\nu \in \mathbb{R}^n$, $\lambda \neq 0$, define

$$\widetilde{\mu}(\nu,\lambda) = \sum_{k\in\mathbf{Z}^n} a_k(\lambda)\widehat{\Phi}(\nu-k)\,.$$

Then $\partial_\lambda^b\widetilde{\mu}(k,\lambda) = \partial_\lambda^b\mu(k,\lambda)$ for $k \in \mathbb{N}^n$, for all $b \in \mathbb{N}\cup\{0\}$, so that $\widetilde{u}$ is a smooth extension of μ.

For fixed $\lambda \neq 0$, and $b \in \mathbb{N}\cup\{0\}$, let $\widetilde{a}_k = (\lambda\partial_\lambda)^b a_k(\lambda)$. Then by definition, the $\widetilde{a}_k$ are odd in each k_j-component. By (10),

$$\left|(k_1\Delta_1)^{\alpha_1}\cdots(k_n\Delta_n)^{\alpha_n}\widetilde{a}_k\right| \leqslant C_{\alpha b}$$

for all $\alpha \in \big(\mathbb{N}\cup\{0\}\big)^n$. And consequently

$$\left|\Delta_1^{\alpha_1}\cdots\Delta_n^{\alpha_n}k_1^{\alpha_1}\cdots k_n^{\alpha_n}\widetilde{a}_k\right| \leqslant C_{\alpha b} \tag{11}$$

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holds for all $\alpha \in (\mathbb{N} \in \{0\})^n$. Since in particular, $|\widetilde{a}_k| \leqslant C_b$, then the series

$$\sum_{k \in \mathbf{Z}^n} \widetilde{a}_k e^{ik\cdot x}$$

converges in the sense of distributions to a distribution H on the n–torus $\mathbb{T}^n (= [-\pi, \pi)^n)$. We observe that, viewing H as a periodic distribution on $\mathbb{R}^n$, then the compactly supported distribution ΦH has Fourier transform

$$(\Phi H)\hat{\ }(\nu) = \sum_{k \in \mathbf{Z}^n} \widetilde{a}_k \widehat{\Phi}(\nu - k) = (\lambda \partial_\lambda)^b \widetilde{\mu}(\nu, \lambda) .$$

We define

$$h_N(x) = \sum_{k_1=-N}^{N} \cdots \sum_{k_n=-N}^{N} \widetilde{a}_k e^{ik\cdot x} .$$

Given $\alpha \in (\mathbb{N} \cup \{0\})^n$, if there are only finitely many non-zero $\widetilde{a}_k's$, then fixing x, and for $i = 1, \dots, n$, splitting the summation in k_i according to the size of $|k_i|$ with respect to $|x_i|^{-1}$, it follows from summation by parts and (11) that

$$\left| \sum_{\text{finite}} (\Delta_1^{\alpha_1} \dots \Delta_n^{\alpha_n} k_1^{\alpha_1} \dots k_n^{\alpha_n} \widetilde{a}_k)\, e^{ik\cdot x} \right| \leqslant C_{\alpha,b} |x_1|^{-1} \dots |x_n|^{-1}$$

independently of the number of nonzero $\widetilde{a}_k's$. But the sum in absolute value on the left-hand side above equals

$$(1 - e^{ix_1})^{\alpha_1} \dots (1 - e^{x_n})^{\alpha_n} (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \sum_{\text{finite}} \widetilde{a}_k e^{ik\cdot x}$$

and therefore

$$\left| \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} h_N(x) \right| \leqslant C_{\alpha,b} |x_1|^{-1-\alpha_1} \dots |x_n|^{-1-\alpha_n} \tag{12}$$

holds uniformly in N, for all $\alpha \in (\mathbb{N} \cup \{0\})^n$.

Viewing h_N as periodic functions on $\mathbb{R}^n$, then for $\beta, \gamma \in (\mathbb{N} \cup \{0\})^n$, the functions

$$g_N = x^\gamma (\partial_{x_1} x_1)^{\beta_1} \dots (\partial_{x_n} x_n)^{\beta_n} (\Phi h_N)$$

also satisfy the same condition (12) uniformly in N for all $\alpha \in (\mathbb{N} \cup \{0\})^n$.

Since the $\widetilde{a}_k$ are odd in each k_i, then the periodic functions h_N are odd in each x_i. Therefore, given $1 \leqslant i \leqslant n$, if $\gamma_i \in \mathbb{N} \cup \{0\}$ is even, then for fixed $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, the functions

$$(\Phi h_N)(x_1, \dots, x_{i-1}, \cdot\, , x_{i+1}, \dots, x_n)$$

are odd on $[-(\pi/2), (\pi/2)]$, where $\varphi \equiv 1$. Consequently, by (12), if $\beta, \gamma \in (\mathbb{N} \cup \{0\})^n$ with γ_i even for all $i = 1, \dots, n$, then the functions g_N satisfy the product-type cancellation conditions on $\mathbb{R}^\ell$ (see Section 3), uniformly in N.

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Since g_N converge in the sense of distributions to the compactly supported distribution $x^\gamma(\partial_{x_1}x_1)^{\beta_1}\dots(\partial_{x_n}x_n)^{\beta_n}(\Phi H)$, then by Lemma 5.5 below,

$$\left|\partial_\nu^\gamma(\nu_1\partial_{\nu_1})^{\beta_1}\dots(\nu_n\partial_{\nu_n})^{\beta_n}(\Phi H)\hat{}(\nu)\right| \leqslant C_{\beta,\gamma,b}$$

holds for all $\gamma,\beta \in \left(\mathbb{N}\cup\{0\}\right)^n$ with each γ_i even ($i = 1,\dots,n$), and consequently by Lemma 5.4, for all $\gamma,\beta \in \left(\mathbb{N}\cup\{0\}\right)^n$. Therefore

$$\left|\left(\left(\nu_1+\frac{1}{2}\right)\partial_{\nu_1}\right)^{\alpha_1}\dots\left(\left(\nu_n+\frac{1}{2}\right)\partial_{\nu_n}\right)^{\alpha_n}(\Phi H)\hat{}(\nu)\right| \leqslant C_{\alpha,b}$$

for all $\alpha \in \left(\mathbb{N}\cup\{0\}\right)^n$, and since $(\Phi H)\hat{}(\nu) = (\lambda\partial_\lambda)^b\widetilde{\mu}(\nu,\lambda)$, then

$$\left|\left(\left(\nu_1+\frac{1}{2}\right)\partial_{\nu_1}\right)^{\alpha_1}\dots\left(\left(\nu_n+\frac{1}{2}\right)\partial_{\nu_n}\right)^{\alpha_n}(\lambda\partial_\lambda)^b\widetilde{\mu}(\nu,\lambda)\right| \leqslant C_{\alpha,b}$$

for all $\alpha \in \left(\mathbb{N}\cup\{0\}\right)^n, b \in \mathbb{N}\cup\{0\}$. Hence,

$$m(\xi,\eta) = \widetilde{\mu}\left(\frac{1}{2}\left(\frac{\xi_1}{|\eta|}-1\right),\dots,\frac{1}{2}\left(\frac{\xi_n}{|\eta|}-1\right),\eta\right)$$

satisfies (1), concluding the proof of Proposition 5.3. ∎

Lemma 5.4. *Let f be a smooth function on $\mathbb{R}$ such that*

$$\left|(d/dt)^i\,(td/dt)^j\,f(t)\right| \leqslant C_{i,j} \tag{13}$$

for all $j \in \mathbb{Z}^+$ and all $i \in 2\mathbb{Z}^+$. Then (13) holds for all $i,j \in \mathbb{Z}^+$.

Proof: Given $i,j \in \mathbb{Z}^+$ with i even, we let $g(t) = (td/dt)^j f(t)$, and show that

$$\left|(d/dt)^{i+1}g(t)\right| \leqslant C_{i,j}\,. \tag{14}$$

From (13), we have

$$\left|(d/dt)^i(td/dt)g(t)\right| \leqslant C_{i,j}\,, \text{ and } \quad \left|(d/dt)^i g(t)\right| \leqslant C_{i,j}\,.$$

Using

$$(d/dt)^i\,(td/dt) = (td/dt)\,(d/dt)^i + i\,(d/dt)^i\,,$$

then $\left|(d/dt)^{(i+1)}g(t)\right| \leqslant C_{i,j}\,|t|^{-1}$; proving (14) for $|t| \geqslant 1$. Now, by (13), $\left|(d/dt)^{(i+2)}g(t)\right| \leqslant C_{i,j}$, whence

$$\left|(d/dt)^{(i+1)}g(t) - (d/dt)^{(i+1)}g(1)\right| \leqslant C_{i,j}\,|t-1|\,.$$

Thus for $|t| < 1$

$$\left|(d/dt)^{(i+1)}g(t)\right| \leqslant 2C_{i,j} + \left|(d/dt)^{(i+1)}g(1)\right|\,,$$

establishing (14) for all $t \in \mathbb{R}$. ∎

LEMMA 5.5. *Let K be a distribution on $\mathbb{R}^n$ satisfying product-type kernel conditions for all multi-indices I with $i_j \leqslant 1$ for all j. Then $\widehat{K}$ is bounded (by a constant depending only on those in the regularity and cancellation conditions K satisfies).*

The proof in the case of K a smooth function of compact support (which is in fact all that we require in Proposition 5.3) uses a similar (but much simpler) argument to that in Proposition 4.1 in Section 4. The result for general K can then be obtained by approximation, as in Theorem 3.1 in Section 6.

6. Approximation of a distribution kernel

We now conclude the proof of Theorem 3.1. We reduce to the case of K smooth, with compact support as follows. Let φ be a polyradial normalised bump function on $\mathbb{H}_n$ of the product form given in Lemma 6.1 below, with $\int \varphi = 1$, and $\varphi(0) = 1$. Denote by φ_{ε_1}, the normalised dilated function

$$\varphi_{\varepsilon_1}(z,t) = {\varepsilon_1}^{-2n-2}\varphi(z/\varepsilon_1, t/\varepsilon_1^2)$$

for $\varepsilon_1 > 0$, and by $\varphi(\varepsilon_2\cdot)$, the dilated function

$$\varphi\big(\varepsilon_2\cdot(z,t)\big) = \varphi(\varepsilon_2 z, \varepsilon_2^2 t)$$

for $\varepsilon_2 > 0$. Then the functions $K_{\varepsilon_1,\varepsilon_2} = \varphi(\varepsilon_2\cdot)(K*\varphi_{\varepsilon_1})$ are smooth, compactly supported, polyradial, converge to K in the sense of distributions, as $\varepsilon_1, \varepsilon_2 \to 0$, and by Lemmas 6.1 and 6.2 below, satisfy (4)–(7) uniformly in ε_1 and ε_2.

By Corollary 5.2 and Proposition 5.3, the multipliers $m_{\varepsilon_1,\varepsilon_2}$, such that

$$m_{\varepsilon_1,\varepsilon_2}(\mathcal{L}_1,\dots,\mathcal{L}_n, iT)f = f * K_{\varepsilon_1,\varepsilon_2}$$

for $f \in S(\mathbb{H}_n)$, satisfy the Marcinkiewicz condition (1) for all $i_1,\dots,i_n, j \in \mathbb{Z}^+$, uniformly in ε_1, $\varepsilon_2 > 0$.

It follows by Ascoli's Theorem that a subsequence $m_{\varepsilon_{1j},\varepsilon_{2j}}$, with $\varepsilon_{1j} \to 0$, $\varepsilon_{2j} \to 0$ as $j \to \infty$ is uniformly bounded, and converges pointwise to a function m on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying (1). But since $K_{\varepsilon_{1j},\varepsilon_{2j}}$ converge to K in $S'(\mathbb{H}_n)$, then K is the convolution kernel of $m(\mathcal{L}_1,\dots,\mathcal{L}_n, iT)$.

LEMMA 6.1. *Let ψ be a radial normalised bump function on $\mathbb{C}$, ψ_0 a normalised bump function on $\mathbb{R}$, and define φ on $\mathbb{H}_n$ by*

$$\varphi(z,t) = \psi(z_1)\cdots\psi(z_n)\psi_0(t)\ .$$

If φ_ε denotes the normalised dilate,

$$\varphi_\varepsilon(z,t) = \varepsilon^{-2n-2}\varphi(\varepsilon^{-1}z, \varepsilon^{-2}t)$$

of φ, for $\varepsilon > 0$, then the smooth functions $K *_{\mathbb{R}^{2n+1}} \varphi_\varepsilon$ obtained by convolving K as a distribution on $\mathbb{R}^{2n+1}$ with φ_ε, are polyradial, and satisfy (4)–(7) uniformly in $\varepsilon > 0$.

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PROOF: All convolutions in the proof of this lemma are Euclidean, rather than Heisenberg, and thus henceforth we shall drop the subscripts to $*$.

Let $\varepsilon > 0$ be given. The function $K * \varphi_\varepsilon$ is easily seen to be polyradial. For f on $\mathbb{R}^N$, and $u \in \mathbb{R}^N$, we let $\tilde{f}(x) = f(-x)$ and denote by $f^{[u]}$ the Euclidean translate by u of $\tilde{f}$. That is, $f^{[u]}(v) = \tilde{f}(v-u) = f(u_1 - v_1, \dots, u_N - v_N)$. Thus, $K * \varphi_\varepsilon(z,t) = K(\varphi_\varepsilon^{[z,t]})$. Given $\tau \in \mathbb{T}^n$, since by construction φ is polyradial,

$$\varphi_\varepsilon^{[(\tau z,t)]}(\zeta, s) = \varphi_\varepsilon^{[z,t]}(\tau^{-1}\zeta, s) = \rho_{\tau^{-1}}\varphi_\varepsilon^{[z,t]}(\zeta, s) .$$

But as K is a polyradial distribution, $K(\rho_{\tau^{-1}}\varphi_\varepsilon^{[z,t]}) = K(\varphi_\varepsilon^{[z,t]})$, from which it follows that $K * \varphi_\varepsilon$ is polyradial.

We now show that $K * \varphi_\varepsilon$ satisfies the regularity condition (4). Let $(z,t) \in \mathbb{H}_n$, with $z_i \neq 0$ for $i = 1, \dots, n$. For any $I \in (\mathbb{Z}^+)^n$, and $k \in \mathbb{Z}^+$,

$$\partial_z^I \partial_t^k (K * \varphi_\varepsilon)(z,t) = K * (\partial_\zeta^I \partial_s^k \varphi_\varepsilon)(z,t) = K\big((\partial_\zeta^I \partial_s^k \varphi_\varepsilon)^{[z,t]}\big) \tag{15}$$

with

$$\begin{aligned}(\partial_\zeta^I \partial_s^k \varphi_\varepsilon)^{[z,t]}(\zeta, s) &= (\partial_\zeta^I \partial_s^k \varphi_\varepsilon)(z-\zeta, t-s) \\ &= (\partial_{\zeta_1}^{i_1}\psi_\varepsilon)^{[z_1]}(\zeta_1) \cdots (\partial_{\zeta_n}^{i_n}\psi_\varepsilon)^{[z_n]}(\zeta_n)(\partial_s^k(\psi_0)_{\varepsilon^2})^{[t]}(s)\end{aligned}$$

where ψ_δ and $(\psi_0)_{\varepsilon^2}$ are the usual (Euclidean) normalised dilates of ψ on $\mathbb{C}^n$ and of ψ_0 on $\mathbb{R}$.

We shall consider separately the cases corresponding to the sises of the $|z_j|$, $j = 1, \dots, n$, with respect to ε. When all $|z_j|$ are large with respect to ε, then (15) is just the convolution of a smooth function, $\partial_\zeta^I \partial_s^i K$, with φ_ε, and we shall use the regularity condition (4) on K. (Case 1 below.) When some of the $|z_j|$ are small compared to ε, then the corresponding $(\partial_{\zeta_j}^{i_j}\psi_\varepsilon)^{[z_j]}$ are essentially normalised bump functions, and so we shall use the relevant cancellation condition (5) or (6) on K. (Cases 2 and 3 below.)

CASE 1. If $|z_j| > 2\varepsilon$ for all $j = 1, \dots, n$, then since ψ_ε is supported in the ε-disc, $\{w \in \mathbb{C} : |w| \leqslant \varepsilon\}$ in $\mathbb{C}$, $(\partial_\zeta^I \partial_s^k \varphi_\varepsilon)^{[z,t]}$ is supported in the region

$$\{(\zeta, s) \in \mathbb{H}_n : |\zeta_i| > \varepsilon \ , \ i = 1, \dots, n\} \ ,$$

in which K is a smooth function. Therefore

$$\begin{aligned}\big|\partial_z^I \partial_t^k (K * \varphi_\varepsilon)(z,t)\big| &= \left| \int_{\mathbb{R}^{2n+1}} \partial_\zeta^I \partial_s^k K(\zeta, s)\varphi_\varepsilon(z-\zeta, t-s)d\zeta\, ds \right| \\ &\leqslant C_{I,k} \int_{\mathbb{R}^{2n+1}} \boldsymbol{|}\zeta\boldsymbol{|}^{-2-I}\big(|\zeta|^2 + |s|\big)^{-1-k}\big|\varphi_\varepsilon(z-\zeta, t-s)\big|d\zeta\, ds\end{aligned}$$

by (4). But for (ζ, s) in the support of $\varphi_\varepsilon(z-\zeta, t-s)$, we have $|\zeta_i| > |z_i|/2$ for $i = 1, \dots, n$. If $|t| > 2\varepsilon^2$, then for (ζ, s) in the support of $\varphi_\varepsilon(z-\zeta, t-s)$ we also have $|s| > |t|/2$. Thus

$$\begin{aligned}\big|\partial_z^I \partial_t^k (K * \varphi_\varepsilon)(z,t)\big| &\leqslant C_{I,k}\boldsymbol{|}z\boldsymbol{|}^{-2-I}\big(|z|^2 + |t|\big)^{-1-k} \int \big|\varphi_\varepsilon^{[z,t]}(\zeta, s)\big|d\zeta\, ds \\ &\leqslant C_{I,k}\boldsymbol{|}z\boldsymbol{|}^{-2-I}\big(|z|^2 + |t|\big)^{-1-k}.\end{aligned}$$

If $|t| \leqslant 2\varepsilon^2$, then $|z|^2 \geqslant \left(|z|^2+|t|\right)/2$, and therefore

$$\begin{aligned}\left|\partial_z^I \partial_t^k (K * \varphi_\varepsilon)(z,t)\right| &\leqslant C_{I,k} \boldsymbol{|} z \boldsymbol{|}^{-2-I} |z|^{-2-2k} \int \left|\varphi_\varepsilon^{[z,t]}(\zeta,s)\right| d\zeta\, ds \\ &\leqslant C_{I,k} \boldsymbol{|} z \boldsymbol{|}^{-2-I} \left(|z|^2+|t|\right)^{-1-k}\end{aligned}$$

CASE 2. If only ℓ of the $|z_i| \leqslant 2\varepsilon$, $0<\ell<n$, we may assume, by relabelling variables if necessary, that $|z_1|,\dots,|z_\ell| \leqslant 2\varepsilon$ and $|z_{\ell+1}|,\dots,|z_n| > 2\varepsilon$. We split φ accordingly: $\varphi(\zeta,s) = \varphi_1(\zeta_{\underleftarrow{\ell}})\varphi_2(\zeta_{\underrightarrow{\ell}},s)$. Now,

$$\partial_w^i \psi_\varepsilon = \varepsilon^{-2-i}(\partial_w^i \psi)\circ \varepsilon^{-1}$$

for $i \in \mathbb{Z}^+$. For $j=1,\dots,\ell$, since $(\partial_{\zeta_j}^{i_j}\psi)\circ\varepsilon^{-1}$ is supported in the ε-disc in $\mathbb{C}$, and $|z_j| \leqslant 2\varepsilon$, then $\left((\partial_{\zeta_j}^{i_j}\psi)\circ\varepsilon^{-1}\right)^{[z_j]}$ is supported in the 3ε-disc in $\mathbb{C}$, and is thus a dilate by $(3\varepsilon)^{-1}$ of the normalised bump function $h_j(w) = (\partial_w^{i_j}\psi)(\varepsilon^{-1}z_j - 3w)$ on $\mathbb{C}$. Therefore

$$\Phi(\zeta_{\underleftarrow{\ell}}) = \left((\partial_{\zeta_{\underleftarrow{\ell}}}^{I_{\underleftarrow{\ell}}}\varphi_1)\circ\delta_e\right)^{[z_{\underleftarrow{\ell}}]}(\zeta_{\underleftarrow{\ell}}) = h_1\left((3\varepsilon)^{-1}\zeta_1\right)\cdots h_\ell\left((3\varepsilon)^{-1}\zeta_\ell\right)$$

is a dilate of a normalised bump function on $\mathbb{C}^\ell$. Here, $e=(\varepsilon^{-1},\dots,\varepsilon^{-1}) \in (\mathbb{R}^+)^\ell$.

Consequently, by the cancellation condition (5), the distribution K_Φ on $\mathbb{H}^{n-\ell}$, defined as in Section 4, is smooth away from the $\zeta_{\ell+1}=0,\dots,\zeta_n=0$ planes and satisfies

$$\left|\partial_{\zeta_{\underrightarrow{\ell}}}^J \partial_s^j K_\Phi(\zeta_{\underrightarrow{\ell}},s)\right| \leqslant C_{I_{\underleftarrow{\ell}},J,j} \boldsymbol{|}\zeta_{\underrightarrow{\ell}}\boldsymbol{|}^{-2-J}\left(|\zeta_{\underrightarrow{\ell}}|^2+|s|\right)^{-1-j}$$

for all $J \in (\mathbb{Z}^+)^{n-\ell}$, $j \in \mathbb{Z}^+$, where the constant does not depend on ε. But

$$\partial_z^I \partial_t^k (K*\varphi_\varepsilon)(z,t) = \varepsilon^{-2\ell-|I_{\underleftarrow{\ell}}|} K_\Phi\left((\partial_{\zeta_{\underrightarrow{\ell}}}^{I_{\underrightarrow{\ell}}}\partial_s^k(\varphi_2)_\varepsilon)^{[z_{\underrightarrow{\ell}},t]}\right)\ ,$$

and since $|z_{\ell+1}|,\dots,|z_n| > 2\varepsilon$, we are therefore reduced to the situation of Case 1 on $\mathbb{H}^{n-\ell}$. Thus,

$$\begin{aligned}\left|\partial_z^I\partial_t^k(K*\varphi_\varepsilon)(z,t)\right| &= \varepsilon^{-2\ell-|I_{\underleftarrow{\ell}}|}\left|\partial_{z_{\underrightarrow{\ell}}}^{I_{\underrightarrow{\ell}}}\partial_t^k\left(K_\Phi*(\varphi_2)_\varepsilon\right)(z_{\underrightarrow{\ell}},t)\right| \\ &\leqslant C_{I,k}\,\varepsilon^{-2\ell-|I_{\underleftarrow{\ell}}|}\boldsymbol{|}z_{\underrightarrow{\ell}}\boldsymbol{|}^{-2-I_{\underrightarrow{\ell}}}\left(|z_{\underrightarrow{\ell}}|^2+|t|\right)^{-1-k} \\ &\leqslant C_{I,k}\boldsymbol{|}z\boldsymbol{|}^{-2-I}\left(|z|^2+|t|\right)^{-1-k}\ ,\end{aligned}$$

using the fact that $|z_1|,\dots,|z_\ell| < 2\varepsilon < |z_{\ell+1}|,\dots,|z_n|$.

CASE 3(a). If $|z_i| \leqslant 2\varepsilon$ for all $i=1,\dots,n$, and $|t|>2\varepsilon^2$, we define Φ on $\mathbb{C}^n$ as in Case 2, with $\ell=n$.

Then Φ is a dilate of a normalised bump function on $\mathbb{C}^n$, and so by the cancellation condition (5) on K, the distribution K_Φ on $\mathbb{R}$ is smooth away from 0, and satisfies

$$\left|\partial_s^j K_\Phi(s)\right| \leqslant C_{I,j}|s|^{-1-j}$$

for all $j \in \mathbb{Z}^+$, where the constant does not depend on ε.

Since $|t| > 2\varepsilon^2$, and $\partial_s^k(\psi_0)_{\varepsilon^2}$ is supported in $[-\varepsilon^2, \varepsilon^2]$, then $\left(\partial^k(\psi_0)_{\varepsilon^2}\right)^{[t]}$ is supported outside $[-\varepsilon^2, \varepsilon^2]$. Therefore

$$\begin{aligned}
\left|\partial_z^I\partial_t^k(K * \varphi_\varepsilon)(z,t)\right| &= \varepsilon^{-2n-|I|}\left|K_\Phi\left((\partial_s^k(\psi_0)_{\varepsilon^2})^{[t]}\right)\right| \\
&= \varepsilon^{-2n-|I|}\left|\int_{\mathbb{R}} \partial_s^k K_\Phi(s)(\psi_0)_{\varepsilon^2}(t-s)\,ds\right| \\
&\leqslant C_{I,k}\varepsilon^{-2n-|I|}\int_{\mathbb{R}} |s|^{-1-k}\left|(\psi_0)_{\varepsilon^2}(t-s)\right|ds\,.
\end{aligned}$$

But for s in the support of $(\psi_0)_{\varepsilon^2}(t-s)$, $|s| \geqslant |t|/2$, and since $|t| > 2\varepsilon^2$, and $|z_i| \leqslant 2\varepsilon$ for $i = 1, \ldots, n$, then $|t| \geqslant c\left(|z|^2 + |t|\right)$. Thus, the result follows.

CASE 3(b). If all $|z_j| \leqslant 2\varepsilon$, and $|t| \leqslant 2\varepsilon^2$, then $\left(\partial_\zeta^I\partial_s^k\varphi_\varepsilon\right)^{[z,t]} = \varepsilon^{-2n-|I|-2-2k}\ \Phi$ where

$$\Phi = \left((\partial_{\zeta_1}^{i_1}\psi)\circ\varepsilon^{-1}\right)^{[z_1]}\cdots\left((\partial_{\zeta_n}^{i_n}\psi)\circ\varepsilon^{-1}\right)^{[z_n]}\left((\partial_s^k\psi_0)\circ\varepsilon^{-2}\right)^{[t]}$$

is a dilate of a normalised bump function on $\mathbb{H}_n$, and so the cancellation condition (6) on K yields

$$\begin{aligned}
\left|\partial_z^I\partial_t^k\left(K *_{\mathbb{R}^{2n+1}} \varphi_\varepsilon\right)(z,t)\right| = \varepsilon^{-2n-|I|-2-2k}\left|K(\Phi)\right| &\leqslant C\ \varepsilon^{2n-|I|-2-2k} \\
&\leqslant C|z|^{-2-I}\left(|z|^2 + |t|\right)^{-1-k}\,,
\end{aligned}$$

concluding the proof of (4).

Before proceeding with the proof of the cancellation conditions (5)–(7), we recall that for a distribution K on $\mathbb{R}^{N_1+N_2}$ and $f_1 \in \mathcal{S}(\mathbb{R}^{N_1})$, the distribution K_{f_1} on $\mathbb{R}^{N_2}$ is defined by

$$K_{f_1}(f_2) = K(f_1 \otimes f_2) \text{ for all } f_2 \in \mathcal{S}(\mathbb{R}^{N_2}). \tag{16}$$

We observe that if $g_1,\ f_1 \in \mathcal{S}(\mathbb{R}^{N_1})$, $g_2 \in \mathcal{S}(\mathbb{R}^{N_2})$, then letting $g = g_1 \otimes g_2$,

$$(K * g)_{f_1} = K_{f_1 * \widetilde{g}_1} * g_2\,. \tag{17}$$

The cancellation conditions (5)–(7) are defined using (16). To consider these cancellation conditions, we write $\mathbb{H}_n = H_0 \times \cdots \times H_n$, where exactly one $H_i = \mathbb{R}$ and all others are $\mathbb{C}$. Abusing terminology somewhat, we shall refer to a kernel on $\mathbb{H}_n$ satisfying (4)–(7) as a *product-type* kernel on $H_0 \times \cdots \times H_n$. A product-type kernel on $\mathbb{C}^n$ or on $\mathbb{R}$, however, will mean the usual product-type kernel on $\mathbb{C} \times \cdots \times \mathbb{C}$ or respectively on $\mathbb{R}$ (one-fold product) defined in Section 3.

We let $G_1 = H_{\underset{\leftarrow}{\ell}}$, $G_2 = H_{\underset{\rightarrow}{\ell}}$, and split φ into $\varphi_1 \otimes \varphi_2$, with φ_i on G_i, $i = 1, 2$. For the cancellation in the G_1-variables on $K * \varphi_\varepsilon$, given a normalised bump function η on G_1, and $r \in (\mathbb{R}^+)^{\ell+1}$, we must show that $(K * \varphi_\varepsilon)_{\eta\circ\delta_r}$ satisfies the product-type regularity condition on G_2 independently of η or r.

But by (17),

$$(K * \varphi_\varepsilon)_{\eta\circ\delta_r} = K_{(\eta\circ\delta_r)*\widetilde{\varphi}_{1_\varepsilon}} * \varphi_{2_\varepsilon},$$

and so if we show that $K_{(\eta\circ\delta_r)*\widetilde{\varphi}_{1_\varepsilon}}$ is a product-type kernel on G_2, then it will suffice to prove that the regularity condition on G_2 is satisfied by $K * \varphi_{2_\varepsilon}$ whenever K is a product-type kernel on G_2. But we proved this above for the case where $G_2 = \mathbb{H}_n$. For $G_2 = \mathbb{C}^{n-\ell}$, the result follows as in Cases 1 and 2 above; for $G_2 = \mathbb{R}$, as in Cases 1 and 3(b).

To see that $K_{(\eta\circ\delta_r)*\widetilde{\varphi}_{1_\varepsilon}}$ is a product-type kernel on G_2, we let $\Phi = (\eta\circ\delta_r) * \widetilde{\varphi}_{1_\varepsilon}$. Then

$$\Phi(x_{\underleftarrow{\ell}}) = \int_{G_1} \eta\big(\delta_r(x - \varepsilon y)\big)\widetilde{\varphi}_1(y)dy = \int_{G_1} \eta\big(\delta_r(\varepsilon y)\big)\widetilde{\varphi}_1\Big(\frac{x}{\varepsilon} - y\Big)\,dy,$$

and we see that $\left|\partial_{x_i}\Phi(x_{\underleftarrow{\ell}})\right| \leqslant Cs_i$, where $s_i = \min(r_i, \varepsilon^{-1})$, while Φ is supported in the region where $|x_i| \leqslant \varepsilon + r_i^{-1} \leqslant 2s_i^{-1}$. Thus $\Phi(x_{\underleftarrow{\ell}})$ is a dilate by $s = (s_0, \dots, s_\ell)$ of a normalised bump function on G_1. It follows easily from the product-type cancellation conditions on K that K_Φ is thus of product-type on G_2, which concludes the proof of Lemma 6.1. ∎

LEMMA 6.2. *Let φ be as in Lemma 6.1, and denote by $\varphi(\varepsilon\cdot)$ the dilate*

$$\varphi\big(\varepsilon\cdot(z,t)\big) = \varphi(\varepsilon z, \varepsilon^2 t)$$

of φ, for $\varepsilon > 0$. If K is a polyradial distribution on $\mathbb{H}_n$ satisfying (4)–(7), *then $K_\varepsilon = \varphi(\varepsilon\cdot)K$ are polyradial, and satisfy* (4)–(7) *uniformly in $\varepsilon > 0$.*

PROOF: We first prove (4). This follows directly from (4) on K. Since K is smooth away from the $z_i = 0$ planes, then so is K_ε. We set $e = (\varepsilon, \dots, \varepsilon, \varepsilon^2) \in (\mathbb{R}^+)^{n+1}$, so that $\varphi(\varepsilon\cdot) = \varphi\circ\delta_e$, where δ denotes multi-parameter dilation on $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, and let Ω be the region in $\mathbb{H}^n$, containing the support of $\varphi\circ\delta_e$, where $|z_1|, \dots, |z_n| \leqslant \varepsilon^{-1}$, and $|t| \leqslant \varepsilon^{-2}$. Then given any $(z,t) \in \mathbb{H}_n$ with $z_i \neq 0$ for $i = 1, \dots, n$,

$$\begin{aligned}
\left|\partial_z^I\partial_t^j K_\varepsilon(z,t)\right| &\leqslant \sum_{U\leqslant I}\sum_{v=0}^{j} c_{U,v}\left|\partial_z^U\partial_t^v(\varphi\circ\delta_e)(z,t)\partial_z^{I-U}\partial_t^{j-v}K(z,t)\right| \\
&\leqslant C_{I,j}\sum_{U\leqslant I}\sum_{v=0}^{j} \varepsilon^{|U|+2v}\,\chi_\Omega(z,t)\left|\partial_z^{I-U}\partial^{j-v}K(z,t)\right| \\
&\leqslant C_{I,j}\sum_{U\leqslant I}\sum_{v=0}^{j} \mathbf{|}z\mathbf{|}^{-U}\big(|z|^2 + |t|\big)^{-v}\left|\partial_z^{I-U}\partial_t^{j-v}K(z,t)\right| \\
&\leqslant C_{I,j}\,\mathbf{|}z\mathbf{|}^{-2-I}\big(|z|^2 + |t|\big)^{-1-j}
\end{aligned}$$

by (4) on K. The summations in U above are over multi-indices $U \in (\mathbb{Z}^+)^n$ such that $u_j \leqslant i_j$ for $j = 1, \dots, n$.

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For the cancellation conditions (5)–(7) on K_ε, we write $\mathbb{H}_n = H_0 \times \cdots \times H_n$, as in the proof of the previous lemma. Given ℓ, $0 \leqslant \ell \leqslant n$, we let $G_1 = H_{\underleftarrow{\ell}}$, $G_2 = H_{\underrightarrow{\ell}}$, and split φ and e accordingly into $\varphi_1 \otimes \varphi_2$ and $e_1 \otimes e_2$ respectively. Given a normalised bump function η on G_1, and $r \in (\mathbb{R}^+)^{\ell+1}$, we observe that

$$(K_\varepsilon)_{\eta \circ \delta_r} = (\varphi_2 \circ \delta_{e_2})\, K_{(\varphi_1 \circ \delta_{e_1})(\eta \circ \delta_r)}$$

as distributions on G_2. But $(\varphi_1 \circ \delta_{e_1})(\eta \circ \delta_r)$ is a dilate of a normalised bump function on G_1. Consequently, by the cancellation condition in the G_1-variables on K, $K_{(\varphi_1 \circ \delta_{e_1})(\eta \circ \delta_r)}$ satisfies the regularity condition on G_2. We are therefore reduced to the situation of estimate (4) proved above for the case of $G_2 = \mathbb{H}^n$. The cases $G_2 = \mathbb{C}^{n-\ell}$ and $G_2 = \mathbb{R}$ follow in the same manner. This concludes the proof of Lemma 6.2. $\square$

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University of New South Wales
Sydney NSW 2052
Australia