FORMS OF THE RINGS $R[X]$ AND $R[X, Y]$

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Let R be a ring and let $S = \operatorname{Spec} R$. Let us consider the *étale fini* topology on S [**5**]. By a form of a given S-scheme T we mean any affine S-scheme W that is locally (in the *étale fini* topology) isomorphic to T. We shall consider forms of the R-schemes $T = \operatorname{Spec} R[X]$ and $T = \operatorname{Spec} R[X, Y]$.

In the case where $R = k$ is a field, the above definition gives the classical definition of forms of k-algebras [**2**]. The problem of determining the forms of $k[X]$ is easy. If A is a k-algebra such that, for some separable extension K of k, there exists a K-isomorphism between $K \otimes A$ and $K[X]$, then A and $k[X]$ are isomorphic as k-algebras.

The following result is due to Šafarevič [**13**]. If k is a field, then there are no non-trivial forms of the affine plane. It means that, if A is a k-algebra such that, for some separable extension K of k, there exists a K-isomorphism between $K \otimes A$ and $K[X, Y]$, then A and $k[X, Y]$ are k-isomorphic.

The main results of this paper are the following theorems.

THEOREM 1. *Let R be a noetherian local ring. Then any form of* $\operatorname{Spec} R[X]$ *is trivial.*

THEOREM 2. *Let R be a discrete valuation ring for which the residue field is algebraically closed. Then any form of* $\operatorname{Spec} R[X, Y]$ *is trivial.*

1. Forms of a ring. Let S be an affine scheme, $S = \operatorname{Spec} R$, and let T be a given affine S-scheme. We say that an affine S-scheme W is a form of T if W is locally (in the *étale fini* topology) isomorphic to T. This means that, if $T = \operatorname{Spec} A$ and $W = \operatorname{Spec} B$, where A, B are R-algebras, then there exists a finite collection $\{R_{f_i}\}_{i \in I}$ of rings such that each R_{f_i} is a localisation of R with respect to the multiplicative system generated by f_i, $\operatorname{Spec} R = \bigcup_{i \in I} \operatorname{Spec} R_{f_i}$, and there exists a collection $\{R_i'\}_{i \in I}$ of rings such that R_i' is a projective and separable extension of R_{f_i} for every $i \in I$, such that the R_i'-algebras $A \otimes_R R_i'$ and $B \otimes_R R_i'$ are isomorphic for every $i \in I$. Because [**14**] every projective and separable extension $R_{f_i} \subset R_i'$ can be imbedded in a Galois extension of R_{f_i} (in sense of [**4**]), it follows that we can assume that the R_i' are Galois extensions of R_{f_i}.

We say that the R-algebra B is a form of the R-algebra A (or shortly that the ring B is a form of the ring A) if the S-scheme $W = \operatorname{Spec} B$ is a form of the S-scheme $T = \operatorname{Spec} A$. We say that the R-algebra B is a trivial form of the R-algebra A if A and B are isomorphic.

If R is a local ring, then the above definition of a form of a given R-algebra simplifies as follows. The R-algebra B is a form of the R-algebra A if there exists a Galois extension R' of R such that the R'-algebras $A \otimes_R R'$ and $B \otimes_R R'$ are isomorphic. We say then that B is a form of A split by R'.

This definition is identical to the classical definition of forms of a variety over a field [**10**]. Similarly, as in the classical situation, the distinct (up to R-isomorphism) forms of the R-

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algebra A are in one-to-one correspondence with the elements of a suitable first cohomology set. Indeed, the classes of R-isomorphic forms split by a given Galois extension R' of R are in one-to-one correspondence with the elements of $H^1(R'/R, F)$, the first cohomology set of the complex obtained from the sequence $R' \rightrightarrows R' \otimes_R R' \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots$ by the action of a functor F, where $F(X) = \mathrm{Aut}(X \otimes A)$ (see [7]). In the situation considered, the set $H^1(R'/R, F)$ is isomorphic to the first Galois cohomology set $H^1(G(R'/R), \mathrm{Aut}(R' \otimes A))$ (see [4], Theorem 5.4).

2. Forms of $R[X]$. Let R be a ring with no nilpotent elements, S any Galois extension of R with Galois group G. Then S has no nilpotent elements. Indeed, there exist elements $x_1, \ldots, x_n$; $y_1, \ldots, y_n \in S$ such that

$$(*) \qquad s = \sum_{i=1}^{n} \mathrm{tr}(sy_i)x_i$$

for all $s \in S$ [4, Theorem 1.3.b]. Since $\mathrm{tr}(t) = \sum_{\sigma \in G} \sigma(t)$ and $R \subset S$ is Galois, $\mathrm{tr}(sy_i) \in R$. Suppose that $s \in S$ is nilpotent. Then sy_i is nilpotent, $\sigma(sy_i)$ is nilpotent for every $\sigma \in G$ and therefore $\mathrm{tr}(sy_i)$ is nilpotent or $\mathrm{tr}(sy_i) = 0$; but R has no nilpotent elements, so $\mathrm{tr}(sy_i) = 0$ and by (*) we have $s = 0$.

It is easy to see that the image of X under any S-automorphism of $S[X]$ is of the form $aX + b$, where $a \in S^*$, $b \in S$. Therefore there exists an exact sequence of groups

$$0 \to S^+ \to \mathrm{Aut}(S[X]) \to S^* \to 1$$

(where S^+ is the additive group of S) which gives (in virtue of [2]) an exact sequence of cohomology sets

$$H^1(G, S^+) \to H^1(G, \mathrm{Aut}(S[X])) \to H^1(G, S^*).$$

If R is a local ring, we have $H^1(G, S^+) = 0$, $H^1(G, S^*) = 0$ [1, Theorem A9]. Consequently $H^1(G, \mathrm{Aut}(S[X])) = 0$. S is an arbitrary Galois extension of R; hence we have

PROPOSITION 2.1. *If R is a local ring with no nilpotent elements, then there are no nontrivial forms of $R[X]$.*

If there are nilpotent elements in R, then the group $\mathrm{Aut}(R[X])$ is not so simple as in the previous case. The following fact is proved in [6]. Let $\mathfrak{n}$ be a nilpotent ideal in R. An endomorphism of $R[X]$ that maps X into $f(X)$ is an automorphism if and only if the endomorphism of $R/\mathfrak{n}[X]$ that maps X into $\bar{f}(X)$ (where $\bar{f}$ is the reduction of f modulo $\mathfrak{n}$) is an automorphism.

LEMMA 2.2. *Let R be a noetherian local ring, S any Galois extension of R with the Galois group G. If $\mathfrak{n}$ is the nilradical of R, then $\mathfrak{n}S$ is the nilradical of S and $H^1(G, (\mathfrak{n}^k S)^+) = 0$ for every positive integer k.*

Proof. $\mathfrak{n}S$ is contained in the nilradical of S. On the other hand $S/\mathfrak{n}S$ is a Galois extension of the ring $R/\mathfrak{n}$ without nilpotent elements; hence $S/\mathfrak{n}S$ has no nilpotent elements. Therefore $\mathfrak{n}S$ is the nilradical of S. Since the extension $R \subset S$ is faithfully flat, we have $\mathfrak{n}^k S \cap R = \mathfrak{n}^k$ [3, Ch. I, §3, no 5]. The ring $S/\mathfrak{n}^k S$ is a Galois extension of $R/\mathfrak{n}^k$ with the Galois group G [4, Lemma 1.7]. Therefore the exact sequence of additive groups

$$0 \to \mathfrak{n}^k S \to S \to S/\mathfrak{n}^k S \to 0$$

gives rise to the exact sequence of cohomology groups

$$H^0(G, \mathfrak{n}^k S) \to H^0(G, S) \to H^0(G, S/\mathfrak{n}^k S) \to H^1(G, \mathfrak{n}^k S) \to H^1(G, S) \to \dots,$$

i.e., to the exact sequence

$$\mathfrak{n}^k \to R \xrightarrow{\varphi} R/\mathfrak{n}^k \xrightarrow{\psi} H^1(G, \mathfrak{n}^k S) \to H^1(G, S) \to \dots$$

In the last sequence φ is an epimorphism; so $\ker \psi = R/\mathfrak{n}^k$. Since $H^1(G, S) = 0$, we have $H^1(G, \mathfrak{n}^k S) = 0$.

LEMMA 2.3. *If S is a Galois extension of the noetherian local ring R with the Galois group G, then* $H^1(G, \operatorname{Aut}(S[X])) = 0$.

Proof. Let $\mathfrak{n}$ be the nilradical of R and let k be the least natural number such that $\mathfrak{n}^k = 0$. If $k = 1$, then R has no nilpotent elements and $H^1(G, \operatorname{Aut}(S[X])) = 0$ by Proposition 2.1.

Suppose that the proposition holds for rings in which the nilpotence degree of the nilradical is less than k. Let us consider the subgroup $N \subset \operatorname{Aut}(S[X])$ of all the automorphisms that map X into $X + f(X)$, where all coefficients of f belong to $\mathfrak{n}^{k-1}S$. It is easy to see that N is isomorphic to the countable direct sum $\oplus(\mathfrak{n}^{k-1}S)^+$ of the additive group $(\mathfrak{n}^{k-1}S)^+$, N is a normal subgroup of $\operatorname{Aut}(S[X])$ and the factor group is isomorphic to $\operatorname{Aut}(S/\mathfrak{n}^{k-1}S[X])$. Therefore we have the exact sequence

$$H^1(G, \oplus(\mathfrak{n}^{k-1}S)^+) \to H^1(G, \operatorname{Aut}(S[X])) \to H^1(G, \operatorname{Aut}(S/\mathfrak{n}^{k-1}S[X]))$$

in which the first term is trivial by Lemma 2.2 and the last term is trivial by the assumption. Hence $H^1(G, \operatorname{Aut}(S[X])) = 0$.

Theorem 1 is now an immediate consequence of this lemma.

REMARK. All forms considered in this paper are forms in the *étale fini* topology. If we consider a more general topology, e.g. the faithfully flat, quasi compact topology, then Theorem 1 is not true. Let, for example, k be a nonperfect field of characteristic p, $\xi \notin k$, $\xi^p \in k$. It is easy to see that the ring $k[X, Y]/(X^p - Y - \xi^p Y^p)$ is a nontrivial form of the ring $k[X]$.

Let us consider forms of $R[X]$ in the case when R is a principal ideal domain.

PROPOSITION 2.4. *If R is a principal ideal domain, then there are no nontrivial forms of* $R[X]$.

Proof. Suppose that the R-algebra S is a form of $R[X]$. Let $\mathfrak{m}$ be any maximal ideal of R. Since the localisation $R_\mathfrak{m}$ is a local ring, the $R_\mathfrak{m}$-algebra $R_\mathfrak{m} \otimes S$ is a trivial form of $R_\mathfrak{m}[X]$ (by Theorem 1). $R_\mathfrak{m}$ is a faithfully flat R-module [3, Ch. II, §2, no 4] and $R_\mathfrak{m} \otimes S$ is an $R_\mathfrak{m}$-algebra of finite presentation; hence, by [8, exp. 8, no 3], S is an R-algebra of finite presentation. Let $t_1, \dots, t_n$ be a set of generators of S: $S = R[t_1, \dots, t_n]$. If $f: R_\mathfrak{m}[X] \to R_\mathfrak{m} \otimes S$ is an isomorphism and $g: R_\mathfrak{m} \otimes S \to R_\mathfrak{m}[X]$ is the inverse isomorphism, then we have $f(X) = F(t_1, \dots, t_n)$ and $g(t_i) = G_i(X)$ $(i = 1, \dots, n)$, where $F \in R_\mathfrak{m}[T_1, \dots, T_n]$ and $G_i \in R_\mathfrak{m}[X]$

$(i = 1, \ldots, n)$. Let us consider all coefficients of polynomials $F, G_1, \ldots, G_n$. Any of these coefficients is of the form a_i/b_i, where $a_i \in R, b_i \in R - \mathbf{m}$. Let M be the multiplicative system of R generated by all the denominators b_i of these coefficients. We have an isomorphism of $R_M[X]$ onto $R_M \otimes S$ described by the formula $f(X) = F(t_1, \ldots, t_n)$.

Therefore there exists a covering of the space $\operatorname{Spec} R$ by open sets $\operatorname{Spec} R_{M_i}$ such that the R_{M_i}-algebras S_{M_i} and $R_{M_i}[X]$ are isomorphic for every i. The collection of these isomorphisms gives an element from the Čech cohomology set $\check{H}^1(\operatorname{Spec} R, \operatorname{Aut}(R[X]))$ on $\operatorname{Spec} R$ with the Zariski topology. Hence the forms of $R[X]$ are in one-to-one correspondence with the elements of $\check{H}^1(\operatorname{Spec} R, \operatorname{Aut}(R[X]))$ (see [7]). A simple induction makes it possible to consider only coverings of $\operatorname{Spec} R$ by two open sets. Let S be trivial over $\operatorname{Spec} R_{M_1}$ and over $\operatorname{Spec} R_{M_2}$, where $\operatorname{Spec} R = \operatorname{Spec} R_{M_1} \cup \operatorname{Spec} R_{M_2}$. Suppose that $M_1 = R - (p)$, $M_2 = R - (q)$, where $(p, q) = 1$. It is obvious that any automorphism of

$$\operatorname{Spec} R_{M_1 \cup M_2}[X] = \operatorname{Spec} R_{M_1}[X] \cap \operatorname{Spec} R_{M_2}[X]$$

can be represented as a composition $a \circ b$, where $a \in \operatorname{Aut}(R_{M_1}[X])$, $b \in \operatorname{Aut}(R_{M_2}[X])$. This shows that $\check{H}^1(\operatorname{Spec} R, \operatorname{Aut}(R[X]))$ is trivial. Therefore any form of $R[X]$ is trivial.

3. Forms of $R[X, Y]$. The following description of $\operatorname{Aut}(K[X, Y])$ in the case where K is an algebraically closed field is due to Šafarevič [13]. The group $\operatorname{Aut}(K[X, Y])$ is a free product of groups B_K and L_K with amalgamated subgroup $T_K = B_K \cap L_K$, where B_K is the group of automorphisms that map X into $aX + b$ and Y into $f(X) + cY$, $a, c \in K^*$, $b \in K$, $f(X) \in K[X]$ and L_K is the group of linear automorphisms with translations. We shall now prove this proposition for any field.

LEMMA 3.1. *Let k be a field. Then the group* $\operatorname{Aut}(k[X, Y])$ *is a free product of groups B_k and L_k with amalgamated subgroup $T_k = B_k \cap L_k$, where B_k is the group of automorphisms that map X into $aX + b$ and Y into $f(X) + cY$, $a, c \in k^*$, $b \in k$, $f(X) \in k[X]$, and L_k is the group of linear automorphisms with translations over k.*

Proof. Let g be an automorphism of $k[X, Y]$. We can consider g as an element of $\operatorname{Aut}(K[X, Y])$, where K is an algebraic closure of k, so that $g \in B_K *_{T_K} L_K$. On the other hand, g can be represented as a finite product of linear automorphisms with translations over k and automorphisms that map X into X and Y into $f(X) + Y$, where $f(X) \in k[X]$ [11]. Therefore the group $\operatorname{Aut}(k[X, Y])$ is generated by B_k and L_k. But $B_K \cap \operatorname{Aut}(k[X, Y]) = B_k$, $L_K \cap \operatorname{Aut}(k[X, Y]) = L_k$, $T_k = B_k \cap L_k$; so, by [12, §D.8, no 1], the group $\operatorname{Aut}(k[X, Y])$ is a free product of B_k and L_k with amalgamated subgroup T_k.

The following example shows that this proposition is not true in the case of rings.

Example 3.2. Let R be an integral domain, k its field of fractions, p any nonzero non-invertible element in R. Let us consider the following automorphisms of $k[X, Y]$:

a_1, which maps X into X and Y into $(1/p)(X^2 - Y)$;
a_2, which maps X into $X + pY$ and Y into Y; and
a_3, which maps X into X and Y into $X^2 + pY$.

It is easy to check that $a_1 \circ a_2 \circ a_3$ restricted to $R[X, Y]$ is an automorphism of $R[X, Y]$.

Suppose that $a_1 \circ a_2 \circ a_3$ can be represented as a product of suitable automorphisms belonging to B_R and L_R. Then there exists an automorphism $t \in T_k$ such that $t \circ a_3 \in B_R$. If t maps X into $aX+b$ and Y into $cX+dY+e$, then $t \circ a_3$ maps X into $aX+b$ and Y into $cX+dX^2+dpY+e$. Therefore we have $d \in R$, $dp \in R^*$; but this is impossible.

Lemma 3.3. *Let R be a discrete valuation ring,* $\mathbf{m}$ *its maximal ideal, $\hat{R}$ the completion of R in the* $\mathbf{m}$*-adic topology. Let K be the field of fractions of R, $\hat{K}$ the field of fractions of $\hat{R}$. Then* $\mathrm{Aut}(\hat{K}[X, Y]) = \mathrm{Aut}(\hat{R}[X, Y]) . \mathrm{Aut}(K[X, Y])$.

Proof. Let a be any automorphism of $\hat{K}[X, Y]$. We can represent a in the form $a = a_1 \circ \ldots \circ a_n$, where $a_i \in L_{\hat{K}}$ or $a_i \in B_{\hat{K}}$ (Lemma 3.1). Since the set of elements of K is dense in $\hat{K}$ in the $\mathbf{m}$-adic topology and the set of polynomials $K[X, Y]$ is dense in $\hat{K}[X, Y]$, the composition $a_1 \circ \ldots \circ a_n$ is a continuous operation. Let $\bar{a}_i \in L_K$ or $\bar{a}_i \in B_K$ respectively be such that all coefficients of $\bar{a}_i$ are sufficiently near to corresponding coefficients of a_i. Then $(id_{\hat{R}} \otimes \bar{a}) \circ a^{-1}$ and $a \circ (id_{\hat{R}} \otimes \bar{a})^{-1}$ are automorphisms arbitrarily near to the identity automorphism of $\hat{K}[X, Y]$, i.e., each of them maps X into a polynomial of the form $X + F(X, Y)$ and Y into a polynomial of the form $Y + G(X, Y)$, where all coefficients of $F(X, Y)$ and $G(X, Y)$ belong to the given power of $\mathbf{m}$. This means that $a \circ (id_{\hat{R}} \otimes \bar{a})^{-1}$ can be represented in the form $id_{\hat{R}} \otimes b$, where b is an automorphism of $\hat{R}[X, Y]$. Thus we have, for any $a \in \mathrm{Aut}(\hat{K}[X, Y]$, that $a = (id_{\hat{R}} \otimes b) \circ (id_{\hat{R}} \otimes \bar{a})$, where $b \in \mathrm{Aut}(\hat{R}[X, Y])$, $\bar{a} \in \mathrm{Aut}(K[X, Y])$.

Lemma 3.4. *Let R be a discrete valuation ring. If an R-algebra S is a form of $R[X, Y]$ such that $\hat{R} \otimes S$ is a trivial form of $\hat{R}[X, Y]$, then S is trivial.*

Proof. By assumption, there exists an isomorphism $g : \hat{R}[X, Y] \to \hat{R} \otimes S$ and by [**13**] an isomorphism $f : K \otimes S \to K[X, Y]$. Let $f' = id_{\hat{R}} \otimes f$, $g' = id_{\hat{R}} \otimes g$. The composition $g' \circ f'$ is an automorphism of $\hat{K}[X, Y]$ and by Lemma 3.3 $g' \circ f' = (id_{\hat{R}} \otimes a) \circ (id_{\hat{R}} \otimes b)$, where $a \in \mathrm{Aut}(\hat{R}[X, Y])$, $b \in \mathrm{Aut}(K[X, Y])$. Therefore we have an isomorphism h' of $\hat{K} \otimes S$ onto $\hat{K}(X, Y)$, which can be defined in two ways: $h' = (id_{\hat{R}} \otimes b) \circ f'^{-1} = (id_{\hat{R}} \otimes a)^{-1} \circ g'$. Let us observe that $h' = id_{\hat{R}} \otimes (b \circ f^{-1})$; so $h' |_{K \otimes S}$ is an isomorphism of $K \otimes S$ onto $K[X, Y]$. Similarly $h' = id_{\hat{R}} \otimes (a^{-1} \circ g)$; so $h' |_{\hat{R} \otimes S}$ is an isomorphism $\hat{R} \otimes S$ onto $\hat{R}[X, Y]$. Since $\hat{R} \otimes S \simeq \hat{R}[X, Y]$ is a flat $\hat{R}$-algebra, S is a flat R-algebra [**8**, exp. IV, Cor. 5.8]. We can now apply Proposition 6 of [**3**, Ch. I, §2, no 6] and we have $(K \otimes S) \cap (\hat{R} \otimes S) = (K \cap \hat{R}) \otimes S = R \otimes S = S$. Therefore $h = h' |_S$ is an R-isomorphism of S onto $R[X, Y]$.

Proof of Theorem 2. Since R is a discrete valuation ring for which the residue field is algebraically closed, the residue field of $\hat{R}$ is algebraically closed and, by [**9**, Ch. IV, Proposition 18.8.1], $\hat{R}$ has no nontrivial Galois extensions. Therefore every form of $\hat{R}[X, Y]$ is trivial and, by Lemma 3.4, every form of $R[X, Y]$ is trivial.

The following proposition provides a generalisation of Theorem 2.

Proposition 3.5. *Let R be a local noetherian ring with the nilradical* $\mathbf{n}$, *let S be a Galois extension of R with the Galois group G. If* $H^1(G, \mathrm{Aut}(S/\mathbf{n}S[X, Y])) = 0$, *then* $H^1(G, \mathrm{Aut}(S[X, Y])) = 0$.

First we must establish the following lemma.

LEMMA 3.6. *Let $\mathfrak{n}$ be a nilpotent ideal of R. An endomorphism of $R[X, Y]$ that maps X into $f(X, Y)$ and Y into $g(X, Y)$ is an automorphism if and only if the endomorphism of $R/\mathfrak{n}[X, Y]$ that maps X into $\bar{f}(X, Y)$ and Y into $\bar{g}(X, Y)$, where $\bar{f}, \bar{g}$ are the reductions of f, g modulo $\mathfrak{n}$ respectively, is an automorphism.*

Proof. The necessity is obvious. Suppose that $f(X, Y), g(X, Y) \in R[X, Y]$ are such that there is an automorphism of $R/\mathfrak{n}[X, Y]$ that maps X into $\bar{f}(X, Y)$ and Y into $\bar{g}(X, Y)$. Then $X, Y \in R/\mathfrak{n}[X, Y]$ can be represented in the forms $X = \sum \bar{a}_{ij} . \bar{f}(X, Y)^i . \bar{g}(X, Y)^j$, $Y = \sum \bar{b}_{ij} . \bar{f}(X, Y)^i . \bar{g}(X, Y)^j$, where $\bar{a}_{ij}$, $\bar{b}_{ij}$ are suitable elements of $R/\mathfrak{n}$. Let $a_{ij}, b_{ij} \in R$ be arbitrary inverse images of $\bar{a}_{ij}$, $\bar{b}_{ij}$, respectively. We have

$$\sum a_{ij} . f(X, Y)^i . g(X, Y)^j = X + F(X, Y),$$

$$\sum b_{ij} . f(X, Y)^i . g(X, Y)^j = Y + G(X, Y),$$

where all coefficients of F and G belong to $\mathfrak{n}$. Let A be the ring generated over R by $X+F(X, Y)$ and $Y+G(X, Y)$. We must show that $X \in A$, $Y \in A$. Let $\mathfrak{n}^k = 0$. Let us assume that every monomial aX^iY^j, for which $a \in \mathfrak{n}^{k-r}$, belongs to A. Suppose that $b \in \mathfrak{n}^{k-r-1}$. Since $bX = b(X+F(X, Y)) - b . F(X, Y)$, we have $bX \in A$, because $X+F(X, Y) \in A$ and $b . F(X, Y) \in A$ by assumption. Similarly $bY \in A$. If $bX^sY^t \in A$, then

$$bX^{s+1}Y^t = bX^sY^t(X+F(X, Y)) - bX^sY^tF(X, Y) \in A$$

and similarly $bX^sY^{t+1} \in A$. Therefore every monomial bX^iY^j for which $b \in \mathfrak{n}^{k-r-1}$, belongs to A. Thus our assumption is true for every $r \leqq k$ and in particular $X, Y \in A$. Therefore we have the sufficiency.

Proof of Proposition 3.5. Let k be the nilpotence degree of the nilradical $\mathfrak{n}$. If $k = 1$, then $\mathfrak{n} = 0$ and $\text{Aut}(S/\mathfrak{n}S[X, Y]) = \text{Aut}(S[X, Y])$. Suppose that $k > 1$ and that our proposition is true for any ring such that the nilpotence degree of its nilradical is less than k. Let N be the subgroup of $\text{Aut}(S[X, Y])$ composed of the automorphisms that map X into $X+f(X, Y)$ and Y into $Y+g(X, Y)$, where $f, g \in \mathfrak{n}^{k-1}S[X, Y]$. It is easy to see that N is isomorphic to the countable direct sum $\oplus(\mathfrak{n}^{k-1}S)^+$ of the additive group $(\mathfrak{n}^{k-1}S)^+$. Moreover N is a normal subgroup and the factor group is isomorphic to $\text{Aut}(S/\mathfrak{n}^{k-1}S[X, Y])$. Therefore we have the following exact sequence:

$$H^1(G, \oplus(\mathfrak{n}^{k-1}S)^+) \to H^1(G, \text{Aut}(S[X, Y])) \to H^1(G, \text{Aut}(S/\mathfrak{n}^{k-1}S[X, Y]))$$

in which the first term is trivial by Lemma 2.2 and the last term is trivial by the assumption. Therefore $H^1(G, \text{Aut}(S[X, Y])) = 0$.

Corollary 3.7. *Let R be a local noetherian ring with nilradical $\mathfrak{n}$. If there are no nontrivial forms of $R/\mathfrak{n}[X, Y]$, then there are no nontrivial forms of $R[X, Y]$.*

THEOREM 2′. *Let R be a local noetherian ring with nilradical $\mathfrak{n}$, such that $R/\mathfrak{n}$ is a discrete valuation ring for which the residue field is algebraically closed. Then there are no nontrivial forms of $R[X, Y]$.*

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Problems.

1. Is the thesis of Theorem 2 true without the assumption that the residue field is algebraically closed?

2. Is the thesis of Theorem 2 true only under the assumption that R is a regular local ring?

3. Are there nontrivial forms of $R[X, Y]$ if R is a principal ideal domain?

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