VARIETIES AND SIMPLE GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

In her book on varieties of groups, Hanna Neumann posed the following problem [13, p. 166]:

"*Can a variety other than $\mathfrak{O}$ contain an infinite number of non-isomorphic non-abelian finite simple groups?*"

The answer to this question does not seem to be known at present. However, in [7], Heineken and Neumann described an algorithm for determining whether or not there are any non-abelian finite simple groups satisfying a given law. They also outlined a way in which their algorithm could be used to show that "only finitely many of the known non-abelian finite simple groups can satisfy a given non-trivial law"; in this paper, we shall follow their suggestions, and prove the

THEOREM. *Let $\mathscr{S}$ be a set of mutually non-isomorphic non-abelian finite simple groups, each of which is either an alternating group or a group of Lie type, and let $\mathscr{S}$ generate a proper subvariety of $\mathfrak{O}$. Then $\mathscr{S}$ is finite.*

Apart from the alternating groups and the groups of Lie type, there are (up to isomorphism) only finitely many non-abelian finite simple groups known at present, so it follows from the Theorem that any infinite set of non-isomorphic non-abelian finite simple groups currently known generates $\mathfrak{O}$. Moreover, provided new finite simple groups continue to be discovered in finite families, as has been happening ever since the discovery of the Ree groups, it will remain possible to make this generalisation from the alternating groups and the groups of Lie type to all known non-abelian finite simple groups; it is only when an infinite set of new finite simple groups is discovered that a reassessment will be needed. There is some speculation at present that there may indeed be such a set — the groups of

Ree type, possessing many of the properties of the Ree groups of type ${}^2G_2$; we will show later that at most finitely many non-isomorphic groups of Ree type can satisfy a given non-trivial law.

It would be interesting to know whether there is any integer e for which there are infinitely many non-isomorphic non-abelian finite simple groups of exponent e. For certain values of e, there are none: for instance, a theorem of Burnside [3, p. 343] proves this when e is divisible by at most two primes, and the Feit-Thompson Theorem [6] proves it when e is odd. If $e \equiv 2 \bmod (4)$, or if e is co-prime to 3, then there are at most finitely many non-isomorphic simple groups of exponent e: in the first case, as pointed out by Kovács in [11], one uses Walter's result [18] that a finite simple group with abelian Sylow 2-subgroups is isomorphic to a known group or a group of Ree type, while in the second case, one uses J. G. Thompson's recent (and unpublished) result that a non-abelian finite simple group of order co-prime to 3 is isomorphic to a Suzuki group. Thus if the Burnside variety $\mathfrak{B}_e$ is to contain infinitely many non-isomorphic non-abelian finite simple groups, then e must be divisible by $12p$, where p is a prime greater than 3.

Another interesting problem is to determine which infinite simple groups satisfy non-trivial laws. In section 4, we shall show that many of the more familiar infinite simple groups generate $\mathfrak{O}$, though we shall also show that for each sufficiently large prime p, there is an infinite simple group of exponent p. In [11], Kovács showed that any infinite, simple, locally finite group G involves infinitely many non-isomorphic non-abelian finite simple groups; hence, if G satisfies non-trivial laws, then we have a positive answer to Neumann's question. However, no such group G seems to be known at present.

The results given here were included in a dissertation [9] written at the University of Oxford under the supervision of Dr. P. M. Neumann; the methods of proof are substantially those used in [9]—and outlined in [7]—though some of the proofs have been shortened by quoting recent results of Tits [17] on free subgroups of linear groups. I am grateful to Dr. Neumann for suggesting this problem to me, and for giving me so much valuable guidance on how to solve it. I would also like to thank the Science Research Council of the United Kingdom, with whose financial support much of this work was done.

2. Preliminaries

The term "variety" will always refer to a variety of groups, except that in the proof of Lemma 5 we will need to discuss algebraic varieties (in the sense of algebraic geometry). The notation and definitions for varieties of groups are based on those in [13]; in particular, $\mathfrak{O}$ is the variety of all groups, $\mathfrak{A}$ the variety of abelian groups, and $\mathfrak{B}_e$ the variety of groups of exponent dividing e.

By a group of Lie type, we mean any Chevalley group, Steinberg group, Suzuki group, or Ree group. For the definitions and basic properties of these groups, we use Carter's survey article [4]; however, our notation differs from his in that we shall distinguish families of "twisted" groups from the corresponding families of Chevalley groups by attaching a symbol denoting the order of the relevant group of "twisted" automorphisms—thus the Suzuki groups are represented by the symbol $^2B_2$, since they are derived from the Chevalley groups of type B_2 by means of an automorphism group of order 2.

S_n and A_n denote the symmetric and alternating groups of degree n, F_n the Galois field of order n, I_n the $n \times n$ identity matrix (with entries in some suitable ring with an identity element), and Z the ring of rational integers.

Two recent results of Tits will play an important part in our proofs, so for convenience we state them here.

PROPOSITION 1. *Over a field of characteristic* 0, *a linear group either has a non-abelian free subgroup or has a soluble subgroup of finite index.* [17, *Theorem* 1]

PROPOSITION 2. *Let V be a vector space over a field k of characteristic different from* 0, *and let G be a subgroup of $GL(V)$. Then the following three properties are equivalent:*

i) G contains no non-abelian free groups;
ii) G has a soluble normal subgroup R such that G/R is locally finite;
iii) G has a subgroup H of finite index such that if V' is any composition factor of the kH-module V and if k' is the endomorphism ring of V' (i.e. the centraliser of H in $\mathrm{End}_k V'$), then k' is a field and V' has a k'-basis with respect to which the matrices representing the elements of H are scalar multiples (by elements of k') of matrices whose entries are algebraic over the prime field of k. [17, *Theorem* 2]

3. The proof of the theorem

We suppose that $\mathscr{S}$ satisfies the conditions of the Theorem and generates a proper subvariety $\mathfrak{V}$ of $\mathfrak{O}$.

LEMMA 1. *$\mathscr{S}$ contains only finitely many alternating groups.*

PROOF. Let G be any finite group of order m. Then by Cayley's Theorem there is a faithful permutation representation ρ of G on the elements of G by right multiplication. Let $\Omega = G \cup \{a, b\}$ where a and b are distinct objects not contained in G. We define a permutation representation σ of G on Ω as follows: each element g of G induces $g\rho$ on the elements of G, and either fixes or transposes a and b according to whether $g\rho$ is even or odd. It is easily seen that σ is a faithful representation of G by even permutations, so that G is isomorphic to a subgroup of A_{m+2}, and hence of A_n for all $n \geqq m+2$. Hence if $\mathscr{S}$ contains A_n for unbounded values of n, then $\mathfrak{V}$ contains all finite groups, so that by the residual finiteness of free groups [8], $\mathfrak{B}$ contains non-abelian free groups. Thus $\mathfrak{B} = \mathfrak{O}$, against our assumption, so the result is proved.

LEMMA 2. *The groups of Lie type in $\mathscr{S}$ have bounded rank.*

PROOF. The families of simple groups of Lie type for which the rank parameter is unbounded are the families $A, B, C, D, {}^2A$, and ${}^2D$. Now for each positive integer n, the alternating group A_n is involved in the Weyl groups of these simple groups (and hence in the simple groups themselves) for all sufficiently high values of the rank parameter: specifically, A_n is involved in the Weyl groups of $A_l(q)$ for all $l \geqq n-1$, of $B_l(q)$, $C_l(q)$ and $D_l(q)$ for all $l \geqq n$, of ${}^2A_l(q)$ for all $l \geqq 2n-1$, and of ${}^2D_l(q)$ for all $l \geqq n+1$ [2VI §4, and 4]. Hence, if the groups of Lie type in $\mathscr{S}$ have unbounded rank, then $\mathfrak{B}$ contains alternating groups of unbounded finite degrees, contradicting Lemma 1, so the result follows.

Having bounded the rank of a group of Lie type in $\mathscr{S}$, we now bound the order of its underlying field. We do this first for the groups $A_1(q)$, then show that the corresponding problem for the remaining families of Chevalley groups, Steinberg groups, and Ree groups may be reduced to this case, and finally deal with the family ${}^2B_2$ of the Suzuki groups by reducing the problem to the corresponding problem for the family B_2.

For convenience, in dealing with the family A_1, we will work with the special linear groups $SL_2(q)$, and later with the general linear groups $GL_2(q)$, rather than with the simple groups $PSL_2(q)$; we shall show that there is no loss of generality in doing this.

LEMMA 3. *$\mathscr{S}$ contains $PSL_2(q)$ for only finitely many values of q.*

PROOF. CASE (i). Suppose $\mathscr{S}$ contains $PSL_2(q) = (A_1(q))$ where the characteristic p of F_q ranges over an infinite set P of primes. Then since $\mathfrak{B}$ is subgroup-closed, $\mathfrak{B}$ contains $PSL_2(p)$ for all $p \in P$. Now let $\mathfrak{U}$ be the variety $[\mathfrak{B}, \mathfrak{E}]$ consisting of those groups G such that the central quotient $G/Z(G)$ lies in $\mathfrak{B}$. Then $\mathfrak{U} \neq \mathfrak{O}$ since if v is a non-trivial law of $\mathfrak{B}$ involving variables $x_1, \cdots, x_r$, then the commutator $[v, x_{r+1}]$ is a non-trivial law of $\mathfrak{U}$. Clearly, $\mathfrak{U}$ contains $SL_2(p)$ if and only if $\mathfrak{B}$ contains $PSL_2(p)$, so we have $SL_2(p) \in \mathfrak{U}$ for all $p \in P$.

For each prime p, there is an epimorphism ϕ_p from $SL_2(Z)$ to $SL_2(p)$, obtained by reducing the matrix entries modulo (p). Since P is infinite, we have $\cap_{p \in P} \ker(\phi_p) = \{I_2\}$, so that $SL_2(Z)$ is residually $SL_2(p)$ as p ranges over P, and hence $SL_2(Z) \in \mathfrak{U}$. However, $SL_2(Z)$ has non-abelian free subgroups (Sanov described such a subgroup explicitly in [14]), so that $\mathfrak{U} = \mathfrak{O}$, a contradiction.

Thus the characteristic p of a group $PSL_2(q)$ in $\mathscr{S}$ is bounded, so it is sufficient to show that for each prime p, $\mathscr{S}$ contains $PSL_2(p^m)$ for only finitely many values of m.

CASE (ii). Suppose $\mathscr{S}$ contains $PSL_2(p^m)$ where p is a fixed prime and m ranges over an infinite set M of positive integers.

Let $\mathfrak{W}$ be the variety $\mathfrak{U}\mathfrak{A}$, where $\mathfrak{U}$ is the variety $[\mathfrak{B}, \mathfrak{E}]$ defined above. Then $\mathfrak{W}$ is a proper subvariety of $\mathfrak{O}$, consisting of those groups G such that the commutator subgroup G' lies in $\mathfrak{U}$; thus $\mathfrak{W}$ contains $GL_2(p^m)$ for all $m \in M$.

Suppose a non-trivial reduced word $w = w(x_1, \cdots, x_r)$ is a law of $\mathfrak{W}$. If $\mathscr{X} = \{x_{ijk} : 1 \leqq i \leqq r,\ 1 \leqq j \leqq 2,\ 1 \leqq k \leqq 2\}$ is a set of $4r$ independent commuting indeterminates over F_p, and if for each $i\,(1 \leqq i \leqq r)$, X_i is the 2×2 matrix with (j,k)-entry x_{ijk}, then we may replace each variable x_i in w by X_i, giving a 2×2 matrix $W = w(X_1, \cdots, X_r)$ with (j,k)—entry w_{jk} lying in the extension-field $F_p(\mathscr{X})$.

For each $i = 1, \cdots, r$, let n_i be the sum of the negative exponents of x_i appearing in w, and let a be the polynomial in $F_p[\mathscr{X}]$ defined by $a = \Pi_{i=1}^r \det(X_i)^{-n_i}$. Then since the entries w_{jk} of W are obtained by performing ring-operations on the elements of $\mathscr{X}$, and by dividing by $\det(X_i)$ whenever x_i^{-1} appears in w, we have $aw_{jk} \in F_p[\mathscr{X}]$ for all j, k.

Now for each $m \in M$, w is a law of $GL_2(p^m)$, so that if the Kronecker symbol δ_{jk} denotes the (j,k)-entry of I_2, then $w_{jk} - \delta_{jk}$ vanishes whenever the variables are chosen from F_{p^m} so that the polynomial $b = \Pi_{i=1}^r \det(X_i)$ is non-zero. Thus the four functions $(w_{jk} - \delta_{jk})b$ vanish identically over F_{p^m}, and hence so do the four polynomials $w'_{jk} = (w_{jk} - \delta_{jk})ab$. These polynomials depend only on w, and hence have fixed degrees, while m may take arbitrarily high values in M, so each w'_{jk} must be the zero element of $F_p[\mathscr{X}]$. Thus w is a law of $GL_2(k)$ for every field k of characteristic p, since whenever $b \neq 0$ over k, we have $a \neq 0$ and hence $w_{jk} = \delta_{jk}$. If we take $k = F_p(t)$, where t is transcendental over F_p, then the matrix $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ represents an element of infinite order in $PSL_2(k)$, so that $GL_2(k)$ is not soluble-by-locally-finite; hence, by Proposition 2, $GL_2(k)$ has a non-abelian free subgroup, and can satisfy no non-trivial laws. This contradiction proves the result.

LEMMA 4. *$\mathscr{S}$ contains only finitely many Chevalley groups, Steinberg groups, or Ree groups.*

PROOF. The finite simple groups of Lie type may be divided into fourteen families, denoted by the symbols $A, B, \cdots, G$ (the Chevalley groups), ${}^2A, {}^2D, {}^3D, {}^2E$ (the Steinberg groups), ${}^2F, {}^2G$ (the Ree groups), and ${}^2B$ (the Suzuki groups). A typical member of the family denoted by such a symbol X is a group $X_l(q)$, where l is the rank parameter and $q = p^m$ is the order of the underlying field. By Lemma 2, the groups in $\mathscr{S}$ have bounded rank, so it is sufficient to show that for each family X (except the family of Suzuki groups, which are dealt with in Lemma

5) and for each rank l, $\mathscr{S}$ contains only finitely many groups $X_l(q)$. We do this by showing that $X_l(q)$ involves $PSL_2(q')$ where q' tends to infinity with q. Thus if $\mathscr{S}$ contains infinitely many groups $X_l(q)$, $\mathfrak{B}$ must contain infinitely many groups $PSL_2(q')$, contradicting Lemma 3. (This method fails with the Suzuki groups since they involve non-abelian simple groups of no other type.)

We now find q' as a function of q for each family X. If X is a family of Chevalley groups, then as shown in [5], $X_l(q)$ is generated by elements $x_r(t)$, where r ranges over all roots of the complex Lie algebra of type X_l, and t ranges over F_q. For each positive root r there is a homomorphism $\phi_r : SL_2(q) \to X_l(q)$ given by $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \mapsto x_r(t)$ and $\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} \mapsto x_{-r}(t)$, such that $|\ker(\phi_r)| \leqq 2$ [5, pp. 35 and 47]. Thus $X_l(q)$ involves $PSL_2(q)$, so we may take $q' = q$ for the Chevalley groups.

The Steinberg groups ${}^2A_l(q), {}^2D_l(q), {}^3D_4(q)$, and ${}^2E_6(q)$ are the subgroups of the Chevalley groups $A_l(q)$, $D_l(q)$, $D_4(q)$, and $E_6(q)$ fixed by an automorphism σ which maps each generator $x_r(t)$ to $x_{r'}(t')$, where $r \mapsto r'$ is a symmetry of the root-system and $t \mapsto t'$ is an automorphism of F_q; σ has order 2, 2, 3, and 2 respectively. In each case, inspection of the Dynkin diagram shows that there is a positive root s fixed by the symmetry: in all cases except ${}^2A_l(q)$ where l is even, s may be taken to be a fundamental root corresponding to a node fixed by the graph-symmetry; in the remaining case, the Dynkin diagram of type A_l (l even) has two adjacent nodes transposed by the graph-symmetry, and s may be taken to be the sum of the corrresponding fundamental roots. In all cases, σ fixes the elements $x_s(t)$ for which $t = t'$, so these lie in the Steinberg group. However, by considering Chevalley's homomorphisms ϕ_s, as above, we see that these elements generate a homomorphic image of $SL_2(q')$, where $F_{q'}$ is the subfield of F_q fixed by the field-automorphism, and this image involves $PSL_2(q')$. Thus for the four families of Steinberg groups, we may take $q' = q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{3}}$ and $q^{\frac{1}{2}}$ respectively.

The automorphisms defining the Ree groups are too "non-algebraic" to allow us to use Chevalley's homomorphisms; however, it is known that ${}^2F_4(2^m)$ and ${}^2G_2(3^m)$ involve $PSL_2(2^m)$ and $PSL_2(3^m)$ respectively [16, 12], so here we may take $q' = q$ and the result follows.

Digression. A group G of Ree type, if it exists, has order $q^3(q^3+1)(q-1)$, where $q = 3^m$ and m is an odd integer greater than 2. Like the Ree group ${}^2G_2(q)$, G has an involution i such that $C_G(i)$ involves $PSL_2(q)$, so that we may apply the method of proof of Lemma 4 to show that at most finitely many non-isomorphic groups of Ree type can satisfy a given non-trivial law.

Lemma 5. *$\mathscr{S}$ contains only finitely many Suzuki groups.*

Proof. If $q = 2^{2n+1}$ where n is a positive integer, then F_q has an automorphism $\theta : x \mapsto x^{2^{n+1}}$ such that θ^2 is the Frobenius automorphism $x \mapsto x^2$. If $u(\alpha, a, \beta, b)$, $h(\gamma, c)$, and τ are the 4×4 matrices

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$$\begin{bmatrix} 1 & & & \\ \alpha & 1 & & \\ \alpha a+\beta & a & 1 & \\ \alpha^2 a+\alpha\beta+b & \beta & \alpha & 1 \end{bmatrix},$$

$$\begin{bmatrix} \gamma c & & & \\ & \gamma & & \\ & & \gamma^{-1} & \\ & & & \gamma^{-1}c^{-1} \end{bmatrix}, \text{ and } \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},$$

where the entries not explicitly given are all zero, then every element of the Suzuki group $Sz(q) = {}^2B_2(q)$ has a unique expression as $u(\alpha, \alpha^\theta, \beta, \beta^\theta)h(\gamma, \gamma^\theta)$ or $u(\alpha, \alpha^\theta, \beta, \beta^\theta)h(\gamma, \gamma^\theta)\tau u(\delta, \delta^\theta, \varepsilon, \varepsilon^\theta)$ with $\alpha, \beta, \gamma, \delta, \varepsilon \in F_q$ and $\gamma \neq 0$ [15].

Suppose $v = v(x_1, \cdots, x_r)$ is a non-trivial reduced word which is a law of $\mathfrak{V}$. Then let

$$\mathscr{Y} = \{\alpha_i, \beta_i, \gamma_i, \delta_i, \varepsilon_i, a_i, b_i, c_i, d_i, e_i : 1 \leqq i \leqq r\}$$

be a set of 10r independent commuting indeterminates over F_2, and let

$$Y_i = Y(\alpha_i, \cdots, e_i) = u(\alpha_i, a_i, \beta_i, b_i)h(\gamma_i, c_i)\tau u(\delta_i, d_i, \varepsilon_i, e_i)$$

for each $i = 1, \cdots, r$. Then Y_i is a 4×4 matrix over $F_2(\mathscr{Y})$, and the entries of both $\gamma_i c_i Y_i$ and $\gamma_i c_i Y_i^{-1}$ are polynomials of degree at most 2 in the variables in $\mathscr{Y}$. Suppose n_i is the sum of the moduli of the exponents of x_i appearing in v (so that, for instance, if $v = x_1^{-1}x_2^{-1}x_1x_2$, then $n_1 = n_2 = 2$); then the matrix

$$V = (v(Y_1, \cdots, Y_r) - I_4) \cdot \textstyle\prod_{i=1}^{r} (\gamma_i c_i)^{n_i}$$

has entries v_{jk} in $F_2[\mathscr{Y}]$ $(1 \leqq j, k \leqq 4)$, each polynomial v_{jk} having degree at most $2n_i$ in each of the variables $\alpha_i, \cdots, e_i$.

Two possibilities now arise: either each v_{jk} is the zero polynomial, or else some v_{jk} is non-zero.

CASE (i). Suppose each of the sixteen polynomials v_{jk} is formally zero. Then if we take any field K of characteristic 2 and replace the variables $\alpha_i, \cdots, e_i$ $(1 \leqq i \leqq r)$ by elements $\alpha_i', \cdots, e_i'$ of K with $\gamma_i', c_i' \neq 0$, the resulting matrices $Y_i' = Y(\alpha_i', \cdots, e_i')$ must satisfy $v(Y_1', \cdots, Y_r') = I_4$.

From now on, we take K to be an algebraically closed field of characteristic 2, and we put $G = Sp_4(K)$. Thus if A^t denotes the transpose of a matrix A, then G is the group consisting of those 4×4 matrices A with entries in K such that $A^t\tau A = \tau$. This condition is equivalent to the vanishing of sixteen polynomials in the entries of A (the polynomial coefficients lying in the prime field of K), so G is an algebraic variety in the affine space K^{16}. Moreover, group multiplication and inversion in G are K-morphisms, so G is an algebraic group, and if we endow G with the Zariski topology then G is a topological group. Now G has a Borel subgroup (that is, a maximal connected soluble subgroup) B consisting of the lower triangular matrices in G; if $\alpha_i', \cdots, e_i'$ range over K subject only to the conditions $\gamma_i', c_i' \neq 0$, then each matrix Y_i' ranges over all elements of the double coset $C = B\tau B$ of B in G. We may think of v as being a "law" of C in the sense that $v(Y_1', \cdots, Y_r') = I_4$ whenever $Y_1', \cdots, Y_r' \in C$; we now show, by means of a density argument, that v is a law of G.

G is the disjoint union of eight double cosets $B\sigma B$ of B, where the representatives σ may be chosen from the subgroup W consisting of the permutation matrices in G (W is a dihedral group of order 8). For each $\sigma \in W$, the sixteen entries of a typical element of $B\sigma B$ are rational functions of a set of $6 + N_\sigma$ parameters lying in K, so that the K-closure $\overline{B\sigma B}$ of $B\sigma B$ has dimension at most $6 + N_\sigma$. Chevalley [5] showed that N_σ takes the values 0 (for $\sigma = I_4$), 1, 1, 2, 2, 3, 3, and 4 (for $\sigma = \tau$); for instance, when $\sigma = \tau$, we may take the parameters for Y_i' to be $\alpha_i', \cdots, e_i'$. Thus $\bar{C}$ has dimension at most 10, and for $\sigma \neq \tau$, $\overline{B\sigma B}$ has dimension at most 9. Since G is 10-dimensional, this implies that $\bar{C}$ is 10-dimensional.

If H is the cartesian product of r copies of G, then H forms, in the natural way, an algebraic group defined over K. In any algebraic group, the connected component of the identity is a normal subgroup of finite index; since G is an infinite simple group, this implies that H is connected. Now H has dimension 10r, and being a connected group, H is irreducible as an algebraic variety, so H has no proper subvarieties of dimension $10r$. However, if $D = C \times C \times \cdots \times C$ (to r terms), then $\bar{D}$ has dimension 10r, so we have $\bar{D} = H$.

If $(A_1, \cdots, A_r)$ is a typical element of H, then the sixteen entries m_{jk} of the matrix $M = v(A_1, \cdots, A_r) - I_4$ are polynomial functions of the entries of $A_1, \cdots, A_r$, since $\det(A_i) = 1$ for $i = 1, \cdots, r$. We have shown that each $m_{jk}(1 \leqq j, k \leqq 4)$ vanishes whenever $(A_1, \cdots, A_r)$ is an element $(Y_1', \cdots, Y_r')$ of D; since D is dense in H, it follows that each m_{jk} vanishes for all $(A_1, \cdots, A_r) \in H$, so that v is a law of G.

Since K is algebraically closed, G has subgroups isomorphic to $Sp_4(2^m)$ for all positive integers m. Hence v is a law of infinitely many non-isomorphic finite Chevalley groups, contradicting Lemma 4. Thus case (i) cannot arise.

Case (ii). Suppose one of the sixteen polynomials v_{jk} is non-zero. Then v_{jk} is a sum of monomials of the form

$$m = \prod_{i=1}^{r} \alpha_i^{m(\alpha_i)} \cdots e_i^{m(e_i)},$$

where the exponents $m(\alpha_i), \cdots, m(e_i)$ are non-negative integers. If we replace each variable $a_i, \cdots, e_i$ by $\alpha_i^{2^{n+1}}, \cdots, \varepsilon_i^{2^{n+1}}$ respectively, we obtain a polynomial $v_{jk}^{(n)}$ in the variables $\{\alpha_i, \cdots, \varepsilon_i : 1 \leqq i \leqq r\}$, each monomial m being replaced by

$$m^{(n)} = \prod_{i=1}^{r} \alpha_i^{m(\alpha_i) + 2^{n+1} m(a_i)} \cdots \varepsilon_i^{m(\varepsilon_i) + 2^{n+1} m(e_i)}.$$

We now show that for all sufficiently large n, $v_{jk}^{(n)}$ is non-trivial. If $v_{jk}^{(n)}$ is formally zero, then there are two distinct monomials m_1 and m_2 appearing in v_{jk} such that $m_1^{(n)} = m_2^{(n)}$. However, since $m_1 \neq m_2$, at least one of the integers $m_1(\alpha_i) - m_2(\alpha_i), \cdots, m_1(e_i) - m_2(e_i)$ is non-zero for some i. Now if $m_1(a_i) \neq m_2(a_i)$, or if $m_1(a_i) = m_2(a_i)$ and $m_1(\alpha_i) \neq m_2(\alpha_i)$, then we can ensure that $m_1^{(n)}$ and $m_2^{(n)}$ are divisible by distinct powers of α_i, and are therefore different, by taking n to be any integer such that $2^{n+1}|(m_1(a_i) - m_2(a_i))| > |m_1(\alpha_i) - m_2(\alpha_i)|$ or $n > 0$ respectively. Similarly, we can find suitable lower bounds for n when some other integer $m_1(\beta_i) - m_2(\beta_i), \cdots, m_1(e_i) - m_2(e_i)$ is non-zero, so for all sufficiently large n we have $m_1^{(n)} \neq m_2^{(n)}$. Since v_{jk} has only finitely many pairs of distinct monomials, we may ensure that no pair are replaced by equal (and thus cancellable) monomials in $v_{jk}^{(n)}$ by taking n sufficiently large.

Now v_{jk} has degree at most $2n_i$ in each of the variables $\alpha_i, \cdots, e_i$, so $v_{jk}^{(n)}$ has degree at most $2n_i(1 + 2^{n+1})$ in each of the variables $\alpha_i, \cdots, \varepsilon_i$. For all sufficiently large n, we have $q = 2^{2n+1} > 2n_i(1 + 2^{n+1})$ for all i, so that $v_{jk}^{(n)}$ takes non-zero values on F_q. Since $v_{jk}^{(n)}$ is divisible by γ_i for each i, this means that there are elements $\tilde{\alpha}_i, \cdots, \tilde{\varepsilon}_i$ of F_q with $\tilde{\gamma}_i \neq 0$ $(i = 1, \cdots, r)$, such that the matrices $\tilde{Y}_i = Y(\tilde{\alpha}_i, \tilde{\alpha}_i^\theta, \cdots, \tilde{\varepsilon}_i, \tilde{\varepsilon}_i^\theta)$ in ${}^2B_2(q)$ satisfy $v(\tilde{Y}_1, \cdots, \tilde{Y}_r) \neq I_4$. Thus we have shown that for all sufficiently large n, v cannot be a law of ${}^2B_2(2^{2n+1})$, so the result is proved.

The main theorem now follows immediately from Lemmas 1, 4 and 5, since these deal with all finite simple groups of alternating or Lie type.

4. Infinite simple groups

We conclude by considering the laws of some infinite simple groups. No originality is claimed for the results in this section: most of them are to be found in Kovács' survey article [11] or else have become established as mathematical "folk-lore", and they are included here purely for comparison with the results on finite simple groups.

We can form an infinite analogue of the finite alternating groups as follows: let Ω be a countable infinite set, and let G be the group consisting of those permutations of Ω which move only finitely many elements and which induce an even permutation on those elements; then G is an infinite simple group, and is locally a finite alternating group. In fact, for every positive integer n, G has subgroups isomorphic to A_n, so that by the Theorem, G generates $\mathfrak{O}$.

If K is any infinite field, then there exist infinite simple Chevalley groups $A_1(K), \cdots, G_2(K)$; moreover, if K has suitable groups of field-automorphisms, we may form infinite simple Steinberg groups, Ree groups, and Suzuki groups over K. We now show that each of these groups generates $\mathfrak{O}$.

If G is any infinite Chevalley group or Steinberg group over K, then as in the proof of Lemma 4, G involves $PSL_2(k)$ where k is a subfield of K such that $|K:k|$ is finite. If k has characteristic 0, then G involves $PSL_2(Z)$ and hence has non-

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abelian free subgroups, so that G generates $\mathfrak{O}$. If k has non-zero characteristic p, then let k_a be the algebraic closure of the prime field in k. If k_a is infinite, then k_a has finite subfields of unbounded orders, so G involves infinitely many non-isomorphic groups $PSL_2(p^m)$, and the result follows from the Theorem. If, on the other hand, k_a is finite, then k has an element t which is transcendental over the prime field; as in case (ii) of the proof of Lemma 3, $SL_2(k)$ is not soluble-by-locally-finite, so Proposition 2 implies that $SL_2(k)$ has non-abelian free subgroups, and hence G generates $\mathfrak{O}$.

If G is an infinite Ree group or Suzuki group over K, then let K_a be the algebraic closure of the prime field in K. If K_a is infinite, then as above, the Theorem shows that G generates $\mathfrak{O}$; if K_a is finite, then in the natural representation of G over K, condition (iii) of Proposition 2 cannot hold, so G has non-abelian free subgroups and hence generates $\mathfrak{O}$.

Thus each of these infinite simple analogues of the finite alternating and Lie-type groups generates $\mathfrak{O}$. However, there are examples of infinite simple groups satisfying non-trivial laws. By results of Adjan and Novikov [1], there exists, for each sufficiently large odd integer p, a finitely generated infinite group H of exponent p. Now if p is prime, then by a theorem of Kostrikin [10] there is a finite bound on the indices of the subgroups of finite index in H. Hence H has only finitely many subgroups of finite index, and their intersection N also has finite index in H. Thus N is finitely generated, so N has a maximal proper normal subgroup M. By definition of N, $|N : M|$ is infinite, so N/M is an infinite simple group of exponent p.

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