Can. J. Math., Vol. XXX, No. 3, 1978, pp. 449–454

THE EMBEDDING OF COMPACT CONVEX SETS IN LOCALLY CONVEX SPACES

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1. Introduction. In studying compact convex sets it is usually assumed that the compact convex set X is contained in a Hausdorff topological vector space L where the topology on X is the relative topology. Usually one assumes that L is locally convex. The reason for this is that most of the major theorems such as the Krein-Milman, Choquet-Bishop-de Leeuw, and most of the fixed point theorems require that there be enough continuous affine functions on X to separate points. A natural question then is the following: When can a compact convex set X be embedded in a locally convex space L? One result has been obtained by Jamison, O'Brien, and Taylor in [**3**]. If X is a convex subset of a real vector space, η: $X \times X \times [0, 1] \rightarrow X$ is defined by $\eta(x, y, \alpha) = \alpha x + (1 - \alpha)y$, and X has a Hausdorff topology on it so that η is continuous, then X is called a *topological convex set*. If for every $x \in X$ and U open with $x \in U$, there exists a convex neighborhood K of x such that $K \subset U$, then X is called *weakly locally convex*. Note that the neighborhood K need not be open. If an open convex neighborhood K can always be chosen, then X is called *strongly locally convex*. Two topological convex sets X and Y are *affinely homeomorphic* if there exists a homeomorphism h from X onto Y such that for every $x, y \in X$ and $\alpha \in [0, 1]$, $h(\alpha x + (1 - \alpha)y) = \alpha h(x) + (1 - \alpha)h(y)$. In [**3**], the authors prove that if X is a compact topological convex set that is strongly locally convex, then X is affinely homeomorphic to a compact convex subset of a locally convex topological vector space. They indicate that the same result with strongly locally convex replaced by weakly locally convex is not known. The purpose of this paper is to prove that the above result holds for weakly locally convex sets.

2. The embedding theorem. The method of proof is similar to [**3**], i.e. we shall construct a barycenter map and from the existence of such a map the theorem will follow. Suppose now that X is a compact topological convex set which is weakly locally convex. We shall denote the regular Borel probability measures on X by $\Omega(X)$. For each $x \in X$ we let $e(x)$ denote the point mass measure at x. A continuous map ψ from $\Omega(X)$ to X is called a *barycenter map* if

(i) ψ is affine, i.e. $\psi(\alpha\mu + (1 - \alpha)\gamma) = \alpha\psi(\mu) + (1 - \alpha)\psi(\gamma)$ for every μ, $\gamma \in \Omega(X)$ and $\alpha \in [0, 1]$ and
(ii) $\psi(e(x)) = x$ for every $x \in X$.

Received August 11, 1975 and in revised form, July 20, 1977.

If X possesses a barycenter map, then by Theorem 3.5 of [**5**] or by Proposition 8 of [**3**], it follows that X is affinely homeomorphic to a compact convex subset of a locally convex topological vector space. To produce the map ψ we first define a map $S: \Omega(X) \to \Omega(X)$ by

$$(2.1) \quad \int f dS(\mu) = \int f(\tfrac{1}{2}x + \tfrac{1}{2}y) d\mu(x) \times \mu(y).$$

for every $f \in C(X)$ and $\mu \in \Omega(X)$. Note that the right hand side of (2.1) defines a norm one positive linear functional on $C(X)$. This corresponds to a measure in $\Omega(X)$ which we have called $S(\mu)$. Note further that S is continuous because the map $(x, y) \to \frac{1}{2}x + \frac{1}{2}y$ is continuous. We shall show that for every $\mu \in \Omega(X)$, there exists $x \in X$ such that $\langle S^n(\mu) \rangle$ weak* converges to $e(x)$. We then define $\psi(\mu) = x$ and show that ψ is a barycenter map. To show that ψ is a barycenter map there are three major points that must be established. (1) We show that for $\mu \in \Omega(X)$, $\langle S^n(\mu) \rangle$ weak* converges to a measure $\gamma(\gamma$ is a fixed point of $S)$. (2) We show that the fixed points of S are precisely the point mass measures. (3) Finally we must show that ψ is continuous.

A function $f: X \to R$ is *convex* if for every $x, y \in X$ and $\alpha \in [0, 1]$, $f(\alpha x + (1 - \alpha)y) \leqq \alpha f(x) + (1 - \alpha)f(y)$. We let C denote the convex Borel measurable functions on X. To obtain (1), (2), and (3) it will be necessary to prove that the class C is fairly large. To do this we apply the assumption of local convexity. If $x \in X$ and O is an open set containing x, then there exists an open neighborhood V of x such that cl $(V) \subset O$. If K is a convex neighborhood of x such that $K \subset V$, then cl $(K) \subset O$. It is easily verified that cl (K) is also convex. Thus every point of X has a neighborhood base consisting of closed convex sets. Now let $x \in X$ and let K be a closed convex neighborhood of x. We define the Minkowski functional p for x and K as in the classical case by

$$p(y) = \inf \{1/\alpha: \alpha y + (1 - \alpha)x \in K, \alpha > 0\}$$

for every $y \in X$. Notice that since K is a neighborhood of x and $\lim_{\alpha \to 0} \alpha y + (1 - \alpha)x = x$ there exists $\alpha > 0$ such that $\alpha y + (1 - \alpha)x \in K$. We now prove the embedding theorem by a sequence of lemmas.

LEMMA 1. *Suppose $x \in X$, K is a closed convex neighborhood of x, and p is the Minkowski functional of K and x. Then*

(i) *p is bounded and convex;*

(ii) *p is lower semicontinuous; and*

(iii) *if $x \in K$, then $p(x) \leqq 1$ if and only if $x \in K$.*

Proof. The proof that p is convex is the same as in the classical case. For every $\alpha \in (0, 1]$, define $\phi_\alpha(y) = (1 - \alpha)x + \alpha y$. Each ϕ_α is continuous. Hence $\{\phi_\alpha(X): \alpha \in (0, 1]\}$ is a monotone collection of closed convex sets with intersection $\{x\}$. Since X is compact and $x \in K^0$, there exists $\alpha > 0$ such that $\phi_\alpha(K) \subset K^0$. Thus $0 \leqq p \leqq 1/\alpha$. The proof of (ii) and (iii) is straightforward.

LEMMA 2. *$C - C$ is a vector lattice and if E and F are disjoint closed sets, then there exists $f \in C - C$ such that*

$$f(X) \subset [0, 1], \quad f(E) = \{0\}, \quad \textit{and} \quad f(F) = \{1\}.$$

Proof. It is clear that $C - C$ is a vector space since C is a cone. Also, if $f, g \in C$, then $\max \{f, g\} \in C$. Thus if $f_1, g_1, f_2, g_2 \in C$, then $\max \{f_1 - g_1, f_2 - g_2\} = \max \{f_1 + g_2, f_2 + g_1\} - (g_1 + g_2) \in C - C$. Therefore $C - C$ is a vector lattice. Now suppose E and F are closed with $E \cap F = \phi$ and $x \in E$. Let K_x be a closed convex neighborhood of x such that $K \cap F = \emptyset$. If p_x is the Minkowski functional of x and K_x then $\sup p_x(K_x) = 1 < \inf p_x(F)$. The collection $\{K_x^0 : x \in E\}$ is an open cover of E and therefore has a finite subcover $\{K_{x_1}^0, \ldots, K_{x_m}^0\}$. If we let $q_i = \max \{p_{x_i}, 1\}$ and $g = \min \{q_1, \ldots, q_n\}$, then $g \in C - C$, $g(E) = \{1\}$, and $\inf g(F) > 1$. Let $h = g - 1$, and let $\alpha = \inf h(F) > 0$. Now define f by $f = 1/\alpha \min \{h, \alpha\}$. It is clear that $f \in C - C$, $f(E) = \{0\}$, and $f(F) = \{1\}$.

LEMMA 3. *$C(X)$ is contained in the uniform closure of $C - C$.*

Proof. Suppose $f \in C(X)$ and for convenience of notation suppose $\inf f(X) = 0$ and $\sup f(X) = 1$. Let N be a positive integer. For $0 \leqq n \leqq N$ define $E_n = f^{-1}([0, n/N])$ and $F_n = f^{-1}([n/N, 1])$. By Lemma 2 there exists $f_n \in C - C$ such that $f_n(E_{n-1}) = \{0\}$, $f_n(F_n) = \{1\}$, and $f_n(X) \subset [0, 1]$ for $1 \leqq n \leqq N$. Now let

$$g = \frac{1}{N} \sum_{n=1}^{N} f_n.$$

Then $g \in C - C$ and $||f - g|| \leqq 1/N$.

LEMMA 4. *If $\mu \in \Omega(X)$ and $f \in C$, then $\int f dS(\mu) \leqq \int f d\mu$.*

Proof. Let μ' be the restriction of μ to the Baire sets on X. Then $\mu' \times \mu'$ is a Baire measure on $X \times X$. This may be extended to a regular Borel measure $\overline{\mu \times \mu}$ on $X \times X$. If f is a bounded Borel measurable function on X, it is clear that

$$\int f(x) d\mu(x) = \int f(x) d\overline{\mu \times \mu}(x, y).$$

Let $\phi: X \times X \to X$ be defined by $\phi(x, y) = \frac{1}{2}x + \frac{1}{2}y$. Since ϕ is continuous $\overline{\mu \times \mu} \circ \phi^{-1} = S(\mu)$ by [**2**, Theorem 12-46, p. 180]. Hence for $f \in C$,

$$\begin{aligned} \int f dS(\mu) &= \int f(\tfrac{1}{2}x + \tfrac{1}{2}y) d\overline{\mu \times \mu}(x, y) \\ &\leqq \int \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) d\overline{\mu \times \mu}(x, y) \\ &= \int f d\mu. \end{aligned}$$

Recall now that if $\mu \in \Omega(X)$, then the support of μ, denoted supp μ, is defined by

$$\text{supp } \mu = \bigcap \{E \subset X\colon E \text{ is closed}, \mu(E) = 1\}.$$

The support of μ has the following properties:

(1) $\mu(\text{supp } \mu) = 1$; and
(2) if O is open and $O \cap \text{supp } \mu \neq \emptyset$, then $\mu(O) > 0$.

We shall use Property (2) in the following lemma.

LEMMA 5. *If $\mu \in \Omega(X)$, then $S(\mu) = \mu$ if and only if μ is a point mass measure.*

Proof. If $\mu = e(x)$ for some $x \in X$, then it is clear that $S(\mu) = \mu$. Suppose now that μ is not a point mass measure. Let $a, b \in \text{supp } \mu$ with $a \neq b$. Then by Lemma 1, there exist open disjoint sets U and V with $a \in U$, $b \in V$ and a Minkowski functional f such that $\sup f(U) \leqq 1 < \beta \leqq \inf f(V)$. Since $f \geqq 0$, $g = f^2$ is convex. If $u \in U$ and $v \in V$, then

$$\begin{aligned}
&\tfrac{1}{2}g(u) + \tfrac{1}{2}g(v) - g(\tfrac{1}{2}u + \tfrac{1}{2}v) \\
&\quad = \tfrac{1}{2}f(u)^2 + \tfrac{1}{2}f(v)^2 - f(\tfrac{1}{2}u + \tfrac{1}{2}v)^2 \\
&\quad \geqq \tfrac{1}{2}f(u)^2 + \tfrac{1}{2}f(v)^2 - (\tfrac{1}{2}f(u) + \tfrac{1}{2}f(v))^2 \\
&\quad = \tfrac{1}{4}(f(u) - f(v))^2 \geqq \tfrac{1}{4}(\beta - 1)^2
\end{aligned}$$

But then

$$\begin{aligned}
\int g dS(\mu) &= \int g(\tfrac{1}{2}u + \tfrac{1}{2}v) d\overline{\mu \times \mu}(u, v) \\
&\leqq \int_{(U \times V)^c} \tfrac{1}{2}g(u) + \tfrac{1}{2}g(v) d\overline{\mu \times \mu}(u, v) \\
&\qquad + \int_{U \times V} \tfrac{1}{2}g(u) + \tfrac{1}{2}g(v) - \tfrac{1}{4}(\beta - 1)^2 d\overline{\mu \times \mu}(u, v) \\
&= \int g d\mu - \tfrac{1}{4}(\beta - 1)^2 \mu(U)\mu(V).
\end{aligned}$$

Since U and V meet supp μ, $\mu(U) > 0$ and $\mu(V) > 0$. We have thus shown that $S(\mu) \neq \mu$.

LEMMA 6. *If $\mu \in \Omega(X)$, then there exists $x \in X$ such that $\langle S^n(\mu)\rangle$ weak* converges to $e(x)$.*

Proof. If $f \in C$, then $\langle \int f dS^n \mu \rangle$ is monotone decreasing and bounded and therefore convergent. Hence $\langle \int f dS^n \mu \rangle$ is convergent for $f \in C - C$. An easy approximation argument shows that $\langle \int f dS^n \mu \rangle$ is convergent for every function

f in the uniform closure of $C - C$ and thus for every continuous function. But then there exists $\gamma \in \Omega(X)$ such that $\langle S^n(\mu)\rangle$ weak* converges to γ. By the continuity of S we have

$$S(\gamma) = S \lim_{n\to\infty} S^n(\mu) = \lim_{n\to\infty} S^{n+1}(\mu) = \gamma.$$

By Lemma 5 there exists $x \in X$ such that $\gamma = e(x)$.

We now define $\psi: \Omega(X) \to X$ by $\psi(\mu) = x$ if $\langle S^n(\mu)\rangle$ converges to $e(x)$ in the weak* topology on $\Omega(X)$.

LEMMA 7. *If $\mu \in \Omega(X)$ and p is a Minkowski functional, then for every positive integer k,*

$$p(\psi(\mu)) \leqq \int pdS^k(\mu).$$

Proof. Define $g(\mu) = \int pd\mu$ for every $\mu \in \Omega(X)$. Since p is lower semicontinuous $p = \sup \{f \in C(X): f \leqq p\}$. By [**1**, Lemma 11.4, p. 180] $g(\mu) = \sup \{\int fd\mu: f \in C(X), f \leqq p\}$. Thus g is also lower semi-continuous. Since $S^n(\mu)$ weak* converges to $e(\psi(\mu))$,

$$p(\psi(\mu)) = g(e(\psi(\mu))) \leqq \lim_{n\to\infty} g(S^n(\mu)) \leqq \int pdS^k(\mu)$$

for every k.

LEMMA 8. *ψ is continuous.*

Proof. Suppose that O is open, $x \in O$, $\mu \in \Omega(X)$, and $\psi(\mu) = x$. Let K be a closed convex neighborhood of x such that $K \subset O$ and let U be the interior of K. Let p be the Minkowski functional for x and K. Then $\sup p(K) = 1 < \inf p(O^c)$. Choose α such that $1 < \alpha < \inf p(O^c)$. It is easily shown that there exists $\epsilon > 0$ such that if $\gamma \in \Omega(X)$ and $\gamma(K) > 1 - \epsilon$, then $\int pd\gamma < \alpha$. Since $\langle S^n\mu\rangle$ weak* converges to $e(x)$, there exists N such that for $n \geqq N$, $S^n(\mu)(U) > 1 - \epsilon$. We let

$$V = \{\gamma \in \Omega(X): S^N(\gamma)(U) > 1 - \epsilon\}.$$

It is easily verified that V is weak* open in $\Omega(X)$. If $\gamma \in V$, then $\int pdS^N(\gamma) < \alpha$. Hence by Lemma 7, $p(\psi(\gamma)) < \alpha$. Therefore $\psi(V) \subset O$ and $\mu \in V$. This proves that ψ is continuous.

THEOREM. *If X is a compact topological convex set that is weakly locally convex, then there is an affine homeomorphism mapping X onto a compact convex subset of a locally convex topological vector space.*

Proof. Let $x_1, \ldots, x_k \in X$, let L be the linear span of these points, and let Y be their convex hull. By Proposition 1 of [**3**] Y is compact and the topology on Y is stronger than the usual topology on L restricted to Y. Thus the two topologies coincide since both are compact Hausdorff. Now suppose that $\alpha_1, \ldots, \alpha_k \geqq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$. Let $\mu = \sum_{i=1}^{k} \alpha_i e(x_i)$. Then supp $S^n(\mu) \subset Y$ for every n. If f is a linear functional on L, then since f and $-f$ are both convex $\int f dS^n(\mu) = \int f d\mu$ for every n. Since $S^n(\mu) \to e(\psi(\mu))$ in the weak* topology on $\Omega(Y)$ and f is continuous on Y,

$$f\left(\sum_{i=1}^{K} \alpha_i x_i\right) = \lim_{n\to\infty} \int f dS^n(\mu) = f(\psi(\mu)).$$

Since the linear functionals on L separate points, $\psi(\mu) = \sum_{i=1}^{K} \alpha_i x_i$. We let F denote the probability measures on X with finite support. The above argument proves that ψ coincides with the usual barycenter map on F. But then ψ is an affine map when restricted to F, i.e. for every $\alpha \in [0, 1]$ and $\mu, \gamma \in F$, $\psi(\alpha\mu + (1 - \alpha)\gamma) = \alpha\psi(\mu) + (1 - \alpha)\psi(\gamma)$. But since F is dense in $\Omega(X)$ and ψ is continuous, ψ is affine on $\Omega(X)$. By Theorem 3.5 of [**5**] or Proposition 8 of [**3**], X is affinely homeomorphic to a compact convex subset of a locally convex topological vector space.

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