

UNIVERSAL QUADRATIC FORMS AND ELEMENTS OF SMALL NORM IN REAL QUADRATIC FIELDS

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Abstract

For any positive integer M we show that there are infinitely many real quadratic fields that do not admit M -ary universal quadratic forms (without any restriction on the parity of their cross coefficients).

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1. Introduction

The study of universal quadratic forms can be said to have started in 1770 with the four square theorem of Lagrange, which one can formulate as the statement that the positive definite form $x^2 + y^2 + z^2 + w^2$ is *universal*, that is, it represents every positive integer. This has been followed by a large number of results, most dealing only with *diagonal* forms $\sum a_i x_i^2$ or with *classical* forms $\sum_{i \leq j} a_{ij} x_i x_j$ which have all cross coefficients a_{ij} even for $i \neq j$. These common restrictions are somewhat unnatural, but usually greatly simplify the arguments and results. Compare, for example, the Conway–Schneeberger 15-theorem that a positive classical quadratic form is universal if and only if it represents $1, 2, \dots, 15$ with the Bhargava–Hanke [1] 290-theorem that a (possibly nonclassical) form is universal if and only if it represents $1, 2, \dots, 290$. Another general result is the 451-theorem of Rouse [8] that an integral quadratic form represents all odd natural numbers if and only if it represents $1, 3, \dots, 451$. (The proof is conditional on a conjecture that three specific ternary forms represent all odd positive integers.)

One can also consider totally positive definite forms over totally real number fields. Such a form is *universal* if it represents all totally positive integers. Again, much less is known in general than in the special case of classical forms. To give two examples, Deutsch [4] proved that the nonclassical form $x^2 + xy + y^2 + z^2 + zw + w^2$ is universal over $\mathbb{Q}(\sqrt{5})$ and Sasaki [9] classified all quaternary universal forms over $\mathbb{Q}(\sqrt{13})$. In the latter case, there are only two up to equivalence and they are both nonclassical.

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It may seem that the number of variables required by a universal quadratic form should grow with the discriminant of the (real quadratic) number field. This is not entirely true, as Kim [7] constructed an infinite family of fields of the form $\mathbb{Q}(\sqrt{n^2 - 1})$ admitting positive diagonal octonary universal forms.

On the other hand, Blomer and the author [2] have recently shown that for each M there exist infinitely many real quadratic fields which have no M -ary positive classical universal forms. The goal of the present short note is to extend this result to nonclassical forms, to strengthen some of the statements (in particular Proposition 2.1) and to greatly simplify the proofs. Our main result is the following theorem.

THEOREM 1.1. *For each positive integer M there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ which do not admit M -ary totally positive integral universal (possibly nonclassical) quadratic forms.*

The basic idea is the same as in [2]. By Proposition 2.1, it suffices to produce $M + 1$ suitable elements $\alpha_i = p_i + q_i \sqrt{D}$ of \mathcal{O}_K , which we do by considering finite approximations p_i/q_i to the continued fraction for \sqrt{D} . These elements have small norms of size approximately \sqrt{D} (see Proposition 3.3), which means that it is hard to represent them nontrivially by a quadratic form. Surprisingly, the lower bound saying that their norms are not too small also plays a key role in the proof. An important difference from the previous arguments is that we are considering a different class of continued fractions, which (together with the aforementioned improved estimate on the norms of α_i) allows us to bypass most of the delicate technical arguments of [2]. However, the present paper does not entirely supersede the previous one, as it produces a much thinner sequence of suitable values of D .

The estimates on the norms of α_i have a surprising corollary (Corollary 3.5) on the sizes of certain coefficients of the continued fraction for \sqrt{D} . It is perhaps already known, but we have not found it in the literature.

The proof of Proposition 2.1 is formulated in the language of lattices associated to quadratic forms. It is also common to consider \mathcal{O}_K -lattices that are not necessarily free as \mathcal{O}_K -modules. Our arguments work in this setting as well and we state the resulting generalisation of Theorem 1.1 as Corollary 4.3.

Although Proposition 2.1 holds for any totally real number field, we do not know how to extend the rest of the arguments to fields K of higher degree over \mathbb{Q} , leaving open the tantalising question of what happens in general. In fact, we are not aware of any results concerning universal quadratic forms over \mathcal{O}_K . Surprisingly, the situation is very different over rings of S -integers, as Collinet [3] recently proved that the sum of five squares is universal over $\mathcal{O}_K[\frac{1}{2}]$ for any number field K .

2. Universal forms

In this section, let K be a totally real number field of degree N over \mathbb{Q} . Let $\sigma_1 = \text{id}, \sigma_2, \dots, \sigma_N : K \hookrightarrow \mathbb{R}$ be the (distinct) real embeddings of K . The norm of $a \in K$ is then $N(a) = \sigma_1(a) \cdots \sigma_N(a)$. We write $a > b$ to mean $\sigma_i(a) > \sigma_i(b)$ for all $1 \leq i \leq N$; $a \geq b$ denotes $a > b$ or $a = b$.

PROPOSITION 2.1. *Assume that there exist totally positive elements $a_1, a_2, \dots, a_M \in \mathcal{O}_K$ such that, for all $1 \leq i \neq j \leq M$, if $4a_i a_j \geq c^2$ for $c \in \mathcal{O}_K$, then $c = 0$. Then there are no universal totally positive $(M - 1)$ -ary quadratic forms over \mathcal{O}_K .*

This result is similar to [2, Proposition 4], but we do not need items (1)–(3) from there. The assumption in the former item (4) was $a_i a_j > c^2$. The change here is only needed in order to obtain the result for all forms, not only the classical ones.

The general idea of the proof is the same as before, but we will still give the details (which are somewhat different) and state things more explicitly.

PROOF. Assume that $Q(x_i) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$ is a universal totally positive quadratic form over \mathcal{O}_K and let $A = (b_{ij})$ be its matrix with $b_{ii} = a_{ii}$ and $b_{ij} = b_{ji} = a_{ij}/2$ for $i \neq j$.

Let $B(x_i, y_i) = \sum_{1 \leq i, j \leq n} b_{ij} x_i y_j$ be the associated bilinear form and consider the corresponding \mathcal{O}_K -lattice L . By this we mean that $L \subset \mathbb{R}^n$ has an \mathcal{O}_K -basis ℓ_1, \dots, ℓ_n and $B(x_i, y_i) = \langle \sum x_i \ell_i, \sum y_i \ell_i \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{R}^n . Note that for $x_i, y_i \in \mathcal{O}_K$ we have $B(x_i, y_i) \in \frac{1}{2}\mathcal{O}_K$.

Since Q is universal, for each totally positive integer α , the lattice L contains a vector $v(\alpha)$ that represents α , that is, $\alpha = \langle v(\alpha), v(\alpha) \rangle$. In particular, for $1 \leq i \leq M$, there is $v_i \in L$ such that $a_i = \langle v_i, v_i \rangle$. From now on, we work only with the lattice L .

Let us show that v_i and v_j are orthogonal for $i \neq j$ by computing the inner product $\langle v_i, v_j \rangle =: c/2$ with $c \in \mathcal{O}_K$. By the Cauchy–Schwarz inequality for v_i and v_j ,

$$\frac{c^2}{4} = \langle v_i, v_j \rangle^2 \leq \langle v_i, v_i \rangle \langle v_j, v_j \rangle = a_i a_j.$$

Since Q and L are totally positive definite, we also see that $4\sigma_h(a_i)\sigma_h(a_j) \geq \sigma_h(c)^2$ for all $1 \leq h \leq N$, that is, $4a_i a_j \geq c^2$. But this implies that $c = 0$ by assumption, proving that v_i and v_j are orthogonal.

Thus, the lattice $L \subset \mathbb{R}^n$ contains M pairwise orthogonal elements and $n \geq M$. □

3. Continued fractions

In this section, we collect some useful results on continued fractions.

Let $\gamma = [u_0, u_1, \dots]$ be an infinite continued fraction of a real number $\gamma > 0$ and let $p_i/q_i = [u_0, \dots, u_i]$ be its i th approximation ($u_i, p_i, q_i \in \mathbb{N}$). Then it is well known (and easy to see) that $p_{i+1} = u_{i+1}p_i + p_{i-1}$ and $q_{i+1} = u_{i+1}q_i + q_{i-1}$ (with initial conditions $p_0 = u_0, p_1 = u_1 u_0 + 1, q_0 = 1, q_1 = u_1$) and

$$\left| \frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} \right| = \frac{1}{q_i q_{i+1}}.$$

Also note that

$$\frac{p_{2i}}{q_{2i}} < \frac{p_{2i+2}}{q_{2i+2}} < \gamma < \frac{p_{2i+1}}{q_{2i+1}} < \frac{p_{2i-1}}{q_{2i-1}}.$$

LEMMA 3.1. *We have*

$$\frac{1}{(u_{i+1} + 2)q_i^2} < \left| \frac{p_i}{q_i} - \gamma \right| < \frac{1}{u_{i+1}q_i^2}.$$

PROOF. For the upper bound,

$$\left| \frac{p_i}{q_i} - \gamma \right| < \left| \frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} \right| = \frac{1}{q_i q_{i+1}} < \frac{1}{u_{i+1} q_i^2}.$$

We show the lower bound by repeatedly using the recurrence for q_j as follows:

$$\begin{aligned} \left| \frac{p_i}{q_i} - \gamma \right| &> \left| \frac{p_i}{q_i} - \frac{p_{i+2}}{q_{i+2}} \right| = \left| \left(\frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} \right) - \left(\frac{p_{i+1}}{q_{i+1}} - \frac{p_{i+2}}{q_{i+2}} \right) \right| \\ &= \frac{1}{q_i q_{i+1}} - \frac{1}{q_{i+1} q_{i+2}} = \frac{u_{i+2}}{q_i q_{i+2}} = \frac{u_{i+2}}{q_i (u_{i+2} q_{i+1} + q_i)} \\ &\geq \frac{1}{q_i (q_{i+1} + q_i)} = \frac{1}{q_i (u_{i+1} q_i + q_i + q_{i-1})} > \frac{1}{(u_{i+1} + 2) q_i^2}. \quad \square \end{aligned}$$

Assume now that $\gamma = \sqrt{D}$ with squarefree $D > 0$. This means that $\gamma = \sqrt{D} = [k, u_1, u_2, \dots, u_{s-1}, 2k]$ is periodic with period s and that the sequence $(u_1, u_2, \dots, u_{s-1}) = (u_1, u_2, \dots, u_r, \dots, u_2, u_1)$ is symmetric, that is, $u_{s-i} = u_i$ and the central element u_r (for $r = \lceil (s-1)/2 \rceil$) appears once or twice. Note that $u_0 = k, u_{si} = 2k$ and $u_{si+j} = u_j$ for $i > 0$ and $j \geq 0$.

Friesen proved the following theorem, which says that it is often possible to find such a D . Note that the q_i in the statement of the theorem are independent of k and so the condition is well defined.

THEOREM 3.2 [5]. *Let $(u_1, u_2, \dots, u_{s-1})$ be a fixed symmetric sequence of positive integers such that*

$$q_{s-2} \quad \text{or} \quad \frac{q_{s-2}^2 - (-1)^s}{q_{s-1}}$$

is even. Then there exist infinitely many k such that $\sqrt{D} = [k, u_1, u_2, \dots, u_{s-1}, 2k]$ for a squarefree D .

Throughout the rest of the paper, let $K = \mathbb{Q}(\sqrt{D})$ with squarefree $D > 0$. Then $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$ when $D \equiv 2, 3 \pmod{4}$ and $\mathcal{O}_K = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})]$ when $D \equiv 1 \pmod{4}$. We denote the conjugate of $a \in K$ by a' . The norm is just $N(a) = aa'$ and $a > b$ means that $a > b$ and $a' > b'$.

Let

$$\alpha_i = p_i + q_i \sqrt{D}, \quad N(\alpha_i) = p_i^2 - Dq_i^2.$$

The elements α_i clearly satisfy the recurrence $\alpha_{i+1} = u_{i+1}\alpha_i + \alpha_{i-1}$. Note that α_i is totally positive if and only if i is odd.

To prove Theorem 1.1, we will use Proposition 2.1 and take for a_j some of the approximations α_i for a suitable value of \sqrt{D} . The following proposition gives us crucial information about the approximate sizes of the norms of the elements α_i , which we shall use several times in the following proofs. Note that when $D \equiv 1 \pmod{4}$ and p_i and q_i are both odd, we could consider $\alpha_i/2 \in \mathcal{O}_K$ instead of α_i , which would improve the bounds by a factor of four.

PROPOSITION 3.3. *We have*

$$\frac{2\sqrt{D}}{u_{i+1} + 2.5} < |N(\alpha_i)| < \frac{2\sqrt{D}}{u_{i+1} - 0.5}.$$

PROOF. Using Lemma 3.1, we first see that

$$2q_i\sqrt{D} - \frac{1}{u_{i+1}q_i} < p_i + q_i\sqrt{D} < 2q_i\sqrt{D} + \frac{1}{u_{i+1}q_i},$$

and then we obtain the upper bound as follows:

$$\begin{aligned} |N(\alpha_i)| &= (p_i + q_i\sqrt{D})|p_i - q_i\sqrt{D}| < \left(2q_i\sqrt{D} + \frac{1}{u_{i+1}q_i}\right)\frac{1}{u_{i+1}q_i} \\ &= \frac{2\sqrt{D}}{u_{i+1}} + \frac{1}{u_{i+1}^2q_i^2} < \frac{2\sqrt{D}}{u_{i+1} - 0.5}. \end{aligned}$$

The proof of the lower bound is similar:

$$|N(\alpha_i)| > \left(2q_i\sqrt{D} - \frac{1}{u_{i+1}q_i}\right)\frac{1}{(u_{i+1} + 2)q_i} = \frac{2\sqrt{D}}{u_{i+1} + 2} - \frac{1}{u_{i+1}(u_{i+1} + 2)q_i^2} > \frac{2\sqrt{D}}{u_{i+1} + 2.5}. \quad \square$$

Conversely, we shall also need to know that every element of sufficiently small norm is one of the approximations α_i .

LEMMA 3.4.

- (a) *Let $\mu \in \mathbb{Z}[\sqrt{D}] \setminus \mathbb{Z}$ be such that $0 < |N(\mu)| < \frac{1}{2}\sqrt{D}$. Then $\mu = n\alpha_i$ or $\mu = n\alpha'_i$ for some $i \geq 0$ and $n \in \mathbb{Z}$.*
- (b) *Let $D \equiv 1 \pmod{4}$ and $\mu \in \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})] \setminus \mathbb{Z}$. If $0 < |N(\mu)| < \frac{1}{8}\sqrt{D}$, then $\mu = n\alpha_i$ or $\mu = n\alpha'_i$ for some $i \geq 0$ and $n \in \frac{1}{2}\mathbb{Z}$.*

PROOF. The proof is essentially the same as the last part of the proof of Proposition 12 in [2], so we give only a sketch.

For (a), let $\mu = x + y\sqrt{D}$ with $x, y \in \mathbb{Z}$ and $y \neq 0$. By factoring out $n = \pm \gcd(x, y)$, we can assume that x and y are coprime and $x > 0$. Also, by replacing μ by its conjugate μ' if necessary, we can assume that also $y > 0$. We need to distinguish two cases depending on the sign of $N(\mu)$.

Case 1. $N(\mu) < 0$, that is, $x - y\sqrt{D} < 0$ and $y^2D - x^2 = |N(\mu)| < \sqrt{D}/2$. Hence, $y^2D - \frac{1}{2}\sqrt{D} - x^2 < 0$ and \sqrt{D} lies between the roots of the quadratic polynomial $2y^2T^2 - T - 2x^2$. Thus,

$$\sqrt{D} < \frac{1 + \sqrt{1 + 16x^2y^2}}{4y^2} < \frac{1}{4y^2} + \frac{\sqrt{1 + 8xy + 16x^2y^2}}{4y^2} = \frac{x}{y} + \frac{1}{2y^2}.$$

We also have $x/y < \sqrt{D}$ by assumption and so $|(x/y) - \sqrt{D}| < 1/(2y^2)$, which implies [6, Theorem 184] that $\mu = \alpha_i$ for some i .

Case 2. $N(\mu) > 0$. This case is similar, and so we omit it (for details see [2, proof of Proposition 12]).

To prove (b), just consider $2\mu \in \mathbb{Z}[\sqrt{D}]$ in (a). □

The two preceding results imply some basic information on the existence of large coefficients in the continued fraction for \sqrt{D} . We do not need this result in the paper, but it seems interesting in its own right. Let us remind the reader that by $u \asymp D^\delta$ we here mean that given $\delta > 0$, there are positive constants c_δ, d_δ (independent of u and D) such that $c_\delta D^\delta < u < d_\delta D^\delta$.

COROLLARY 3.5. *Let $\sqrt{D} = [k, u_1, u_2, \dots, u_{s-1}, 2k]$, where D is a squarefree positive integer, and let α_i be as above. Let $0 < \varepsilon < 1/2$ and assume that $u_i \asymp D^{1/2-\varepsilon}$ for some i . Then one of the following is true:*

- (1) α_{i-1} is divisible by a prime ramified in $\mathbb{Q}(\sqrt{D})$; or
- (2) for each $m \in \mathbb{N}$ with $m\varepsilon < 1/2$, there is $j(m) \in \mathbb{N}$ such that $u_{j(m)} \asymp D^{1/2-m\varepsilon}$.

If $D = P \equiv 1 \pmod{4}$ is a prime, then (1) never happens, and so (2) holds.

PROOF. By Proposition 3.3, we know that $N(\alpha_{i-1}) \asymp D^\varepsilon$. Assume that α_{i-1} is not divisible by any ramified prime. Since p_{i-1} and q_{i-1} are coprime, it is also not divisible by any inert prime, and so the only primes which divide it are split. The same holds also for α_{i-1}^m , and so $n \nmid \alpha_{i-1}^m$ for all $n \geq 2$. We have that $N(\alpha_{i-1}^m) \asymp D^{m\varepsilon} \ll D^{1/2}$ and $\alpha_{i-1}^m > 1$; hence, by Lemma 3.4, $\alpha_{i-1}^m = \alpha_{j(m)-1}$ for some $j(m)$. Then again by Proposition 3.3 we conclude that $u_{j(m)} \asymp D^{1/2-m\varepsilon}$.

Finally, if $D = P \equiv 1 \pmod{4}$ is a prime, then the only ramified prime is $(P) = \mathfrak{p}^2$ and $N(\mathfrak{p}) = P$. If $\mathfrak{p} \mid \alpha_{i-1}$, then $P = N(\mathfrak{p}) \leq N(\alpha_{i-1}) \asymp P^\varepsilon$, which is not possible. □

Note that the proof actually shows how the constants implicit in $u_{j(m)} \asymp D^{1/2-m\varepsilon}$ depend on m : there are positive constants $c_\varepsilon, d_\varepsilon$ such that

$$c_\varepsilon^m D^{1/2-m\varepsilon} < u_{j(m)} < d_\varepsilon^m D^{1/2-m\varepsilon}.$$

The assumption in Corollary 3.5 on the existence of $u_i \asymp D^{1/2-\varepsilon}$ is unknown in general, but if the class number of $\mathbb{Q}(\sqrt{D})$ is 1, then it follows from the Riemann hypothesis for $L(s, \chi_D)$ that there are many elements α with norm $\asymp D^\varepsilon$, and hence coefficients $u_i \asymp D^{1/2-\varepsilon}$ for any $0 < \varepsilon < \frac{1}{2}$. (See [2, proof of Proposition 5].)

4. The construction

We are finally ready to construct the continued fractions which give us many elements satisfying the assumptions of Proposition 2.1.

Given $M \in \mathbb{N}$, let u_1, \dots, u_{s-1} be a symmetric sequence of integers such that:

- (1) $r := \lceil \frac{1}{2}(s-1) \rceil \geq 2M+1$;
- (2) the assumptions of Theorem 3.2 are satisfied; and
- (3) u_i is rapidly increasing, that is, $u_1 \geq 2$ and, for $1 \leq i \leq r$, we have $u_{i+1} \geq u_i^3$.

For example, one can take $s \equiv 2 \pmod{3}$ and $u_i = 3^{3^{i-1}}$ for $1 \leq i \leq r$.

Let us show that in this setting, the assumption of Proposition 2.1 is satisfied.

PROPOSITION 4.1. *Assume that (1)–(3) above are satisfied and let i, j be odd such that $1 \leq i < j \leq r$. If $4\alpha_i\alpha_j \geq c^2$ for $c \in \mathcal{O}_K$, then $c = 0$.*

PROOF. We can assume that $c > 0$. We shall first prove that it suffices to consider $c = \frac{1}{2}\alpha_h$ for some h , then that $h < j$ and finally $h \geq j$, obtaining a contradiction.

(a) $c = \frac{1}{2}\alpha_h$. By Proposition 3.3,

$$|N(c)| = |cc'| \leq 4\sqrt{N(\alpha_i)N(\alpha_j)} < \frac{8\sqrt{D}}{\sqrt{(u_{i+1} - 0.5)(u_{j+1} - 0.5)}} < \frac{\sqrt{D}}{8}$$

(in the last inequality we used $j \geq i + 2 \geq 3$ and hence $u_{j+1} \geq u_4 \geq 2^{3^3}$), and so we can apply Proposition 3.4. If $c = n\alpha'_h$ ($n \in \frac{1}{2}\mathbb{N}$), then

$$1 < (\frac{1}{2}\alpha_1)^2 \leq (n\alpha'_i)^2 = c'^2 \leq 4\alpha'_i\alpha'_j < 1$$

(the first inequality follows from the definition of α_1 and the last one from Lemma 3.1), which is not possible. Hence, $c = n\alpha_h$. Then $4\alpha_i\alpha_j \geq c^2 = n^2\alpha_h^2 \geq (\frac{1}{2}\alpha_h)^2$, and so we can assume that $n = \frac{1}{2}$ and $c = \frac{1}{2}\alpha_h$.

(b) $h < j$. The sequence α_t is increasing, as we have $\alpha_{t+1} = u_{t+1}\alpha_t + \alpha_{t-1} > u_{t+1}\alpha_t$. Hence, if $h \geq j$, then $\alpha_h \geq \alpha_j$ and (since $j \geq i + 2$)

$$\alpha_h \geq \alpha_{i+2} > u_{i+2}\alpha_{i+1} > u_{i+2}u_{i+1}\alpha_i \geq 16\alpha_i.$$

This implies that $(\frac{1}{2}\alpha_h)^2 > 4\alpha_i\alpha_j$, which is a contradiction.

(c) $h \geq j$. By Proposition 3.3,

$$\frac{\sqrt{D}}{2(u_{h+1} + 2.5)} < |N(c)| \leq 4\sqrt{N(\alpha_i)N(\alpha_j)} < \frac{8\sqrt{D}}{\sqrt{(u_{i+1} - 0.5)(u_{j+1} - 0.5)}},$$

and so $16^2(u_{h+1} + 2.5)^2 > (u_{i+1} - 0.5)(u_{j+1} - 0.5) > u_{j+1} - 0.5$. But this is not possible if $h < j$, because the sequence u_t is rapidly increasing (and $j \geq 3$). □

Theorem 1.1 now follows directly from Propositions 2.1 and 4.1.

REMARK 4.2. The constructed elements α_i have other interesting properties. For example, when $D \equiv 2, 3 \pmod{4}$, then α_i is not a sum of totally positive elements and is irreducible (under a mild additional assumption on the size of k), that is, $\alpha_i = xy$ for $x, y \in \mathcal{O}_K$ implies that x or y is a unit.

As we mentioned in the introduction, one can formulate the main theorem (Theorem 1.1) more generally for \mathcal{O}_K -lattices that are not necessarily free. Let us first briefly review the definitions.

For a (totally positive) number field K , a *quadratic space* is a pair (V, Q) , where V is a finite-dimensional K -vector space and Q is a quadratic form on V . An \mathcal{O}_K -module (L, Q) is a (quadratic) \mathcal{O}_K -lattice if it is a lattice in (V, Q) of full rank. An \mathcal{O}_K -lattice (L, Q) is *totally positive* if Q is a totally positive quadratic form, it is *integral* if

$Q(v) \in \mathcal{O}_K$ for all $v \in L$ and it is *universal* if for each $0 < a \in \mathcal{O}_K$ there is $v \in L$ such that $Q(v) = a$. In the proof of Proposition 2.1, we have used the fact that to each quadratic form there corresponds an \mathcal{O}_K -lattice. Such an \mathcal{O}_K -lattice is always free as an \mathcal{O}_K -module, but there exist also nonfree \mathcal{O}_K -lattices in the case when \mathcal{O}_K is not a principal ideal domain.

All of the arguments, especially (the second part of) the proof of Proposition 2.1, still apply almost verbatim in this setting and we obtain the following result.

COROLLARY 4.3. *For each positive integer M , there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ which do not admit totally positive integral universal \mathcal{O}_K -lattices of rank M .*

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References

- [1] M. Bhargava and J. Hanke, ‘Universal quadratic forms and the 290-theorem’, Preprint.
- [2] V. Blomer and V. Kala, ‘Number fields without universal n -ary quadratic forms’, *Math. Proc. Cambridge Philos. Soc.* **159** (2015), 239–252.
- [3] G. Collinet, ‘Sums of squares in S -arithmetic rings’, Preprint.
- [4] J. I. Deutsch, ‘Universality of a non-classical integral quadratic form over $\mathbb{Q}[\sqrt{5}]$ ’, *Acta Arith.* **136** (2009), 229–242.
- [5] C. Friesen, ‘On continued fractions of given period’, *Proc. Amer. Math. Soc.* **103** (1988), 8–14.
- [6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edn (Clarendon Press–Oxford University Press, New York, 1979).
- [7] B. M. Kim, ‘Universal octonary diagonal forms over some real quadratic fields’, *Comment. Math. Helv.* **75** (2000), 410–414.
- [8] J. Rouse, ‘Quadratic forms representing all odd positive integers’, *Amer. J. Math.* **136** (2014), 1693–1745.
- [9] H. Sasaki, ‘Quaternary universal forms over $\mathbb{Q}[\sqrt{13}]$ ’, *Ramanujan J.* **18** (2009), 73–80.

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