Proceedings of the Edinburgh Mathematical Society (2010) **53**, 299–310 ©
DOI:10.1017/S0013091508000448 Printed in the United Kingdom

FROM PERMUTAHEDRON TO ASSOCIAHEDRON

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(Received 16 April 2008)

Abstract For each finite real reflection group W, we identify a copy of the type-W simplicial generalized associahedron inside the corresponding simplicial permutahedron. This defines a bijection between the facets of the generalized associahedron and the elements of the type-W non-crossing partition lattice that is more tractable than previous such bijections. We show that the simplicial fan determined by this associahedron coincides with the Cambrian fan for W.

Keywords: reflection; permutahedron; associahedron; non-crossing partition

2010 *Mathematics subject classification:* Primary 20F55
Secondary 05E15

1. Introduction

Let W be an irreducible finite real reflection group of rank n acting on $\mathbb{R}^n$. The *type-W simplicial permutahedron* is the simplicial complex obtained by intersecting the unit sphere $\boldsymbol{S}^{n-1}$ with the fan defined by the reflecting hyperplanes of W. The *type-W simplicial generalized associahedron* is obtained by intersecting the unit sphere $\boldsymbol{S}^{n-1}$ with the cluster fan associated to a chosen Coxeter element c of W [**3**]. Since its introduction, similarities have been noticed between the local structures of the generalized associahedron and the corresponding permutahedron. In the $W = A_n$ case, this relationship was investigated in [**8**]. In [**5**], a combinatorial isomorphism (which is linear for bipartite factorizations of c) is constructed between the cluster fan and the Cambrian fan, a certain coarsening of the fan defined by the reflecting hyperplanes of W. In [**4**] it is shown that the Cambrian fan is the normal fan of a simple polytope.

This paper constructs the c-Cambrian fan for a bipartite Coxeter element c, without using Coxeter-sortable elements. Our approach exhibits the c-Cambrian fan as the fan determined by the image $\mu(AX(c))$ of an isometric copy $AX(c)$ of the simplicial generalized associahedron under the linear isomorphism $\mu = 2(I - c)^{-1}$ from [**2**]. The vertex set of the complex $AX(c)$ consists of the positive roots and the first n negative roots relative to the total ordering on roots defined in [**2**]. We show that the codimension-1 simplices of

$\mu(AX(c))$ are pieces of the original reflecting hyperplanes and that each facet is a union of permutahedron facets. Thus, the fan defined by $\mu(AX(c))$ is a coarsening of the fan determined by the reflection hyperplanes and we show that this fan coincides with the c-Cambrian fan.

The set of facets of $\mu(AX(c))$ and the non-crossing partition lattice, NCP_c, are equinumerous (see, for example, [**1**]). In the current setting, this can be shown with the following easily described bijection.

For each $w \in \mathrm{NCP}_c$, define a region $F(w)$ in $\mathbb{R}^n$ as follows. Let $\{\delta_1, \dots, \delta_k\}$ be the simple system for the parabolic subgroup determined by w (with the reflection set consisting of those reflections in W whose fixed hyperplanes contain the fixed subspace of w) and let $\{\theta_1, \dots, \theta_{n-k}\}$ be the simple system for the parabolic subgroup determined by cw^{-1}. Now set

$$F(w) = \{x \in \mathbb{R}^n \mid x \cdot \delta_i \leqslant 0 \text{ and } x \cdot \theta_j \geqslant 0\}.$$

Our main theorem is the following.

Theorem 1.1. *The collection* $\{F(w) \mid w \in \mathrm{NCP}_c\}$ *is the set of facets of a complete simplicial fan. Moreover, this fan is linearly isomorphic to the corresponding cluster fan.*

Remark. The recent paper [**6**] defines cones for a general (not necessarily finite) W via Coxeter-sortable elements. We expect that these cones should coincide with the facets $F(w)$ for finite W and bipartite c, but this has not been shown.

2. Preliminaries

Fix a fundamental chamber C for the action of W on $\mathbb{R}^n$, denote the inward unit normals by $\alpha_1, \dots, \alpha_n$ and let $R_1, \dots, R_n$ be the corresponding reflections. Assume that $S_1 = \{\alpha_1, \dots, \alpha_s\}$ and $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ are orthonormal sets. Let $c = R_1 R_2 \cdots R_n$ be the corresponding bipartite Coxeter element. Letting $\boldsymbol{T}$ be the set of all reflections in W, the total reflection length function on W is defined by

$$\ell(w) = \min\{k > 0 \mid w = T_1 T_2 \cdots T_k,\ T_i \in \boldsymbol{T}\}.$$

We recall from [**2**] that $\ell(w)$ is the dimension of $M(w)$, the orthogonal complement of the fixed subspace of the orthogonal transformation w. The total reflection order on W is defined by

$$u \preceq w \quad \text{if and only if } \ell(u) + \ell(u^{-1}w) = \ell(w)$$

and the set of W-non-crossing partitions, NCP_c, is defined to be the subset of W consisting of those elements w satisfying $w \preceq c$. Associated to each $w \in \mathrm{NCP}_c$ is a parabolic subgroup W_w, which is the finite reflection group with reflection set consisting of those $T \in \boldsymbol{T}$ with $T \preceq w$. The W fundamental domain C lies in a unique chamber for the action of W_w on $\mathbb{R}^n$ and hence determines a simple system Π_w for W_w.

In [**2**], a total order, $\leqslant$, on the roots (vectors of the form $w(\alpha_i)$ for $w \in W$ and $1 \leqslant i \leqslant n$) is defined, following [**7**], by

$$\rho_i = R_1 R_2 \cdots R_{i-1}(\alpha_i),$$

with R_j and α_i defined cyclically modulo n. Furthermore, a simplicial complex $EX(c)$ is constructed with vertex set

$$\{\rho_{-n+s+1},\dots,\rho_0,\rho_1,\dots,\rho_{nh/2},\rho_{nh/2+1},\dots,\rho_{nh/2+s}\}$$

(where $\rho_{-k}=\rho_{nh-k}$) and a simplex on each subset $\{\tau_1,\tau_2,\dots,\tau_k\}$ of the vertices satisfying

$$\tau_1<\tau_2<\dots<\tau_k \quad\text{and}\quad \ell(R(\tau_1)\cdots R(\tau_k)\gamma)=n-k.$$

It is shown in [**2**] that $EX(c)$ coincides with the type-W generalized associahedron. We will continue to use the notation from [**2**]. In particular, $X(w)$ will denote the subcomplex of $EX(c)$ consisting of those simplices whose vertices are positive roots in the subspace $M(w)$ for $w\in\mathrm{NCP}_c$ and μ will denote the linear operator $2(I-c)^{-1}$. We recall that if τ is a root of unit length, then $\mu(\tau)$ is the unique vector in the fixed subspace of the length $n-1$ element $R(\tau)c$ satisfying $\mu(\tau)\cdot\tau=1$. Furthermore, $\{\mu(\rho_1),\dots,\mu(\rho_n)\}$ is the dual basis to $\{\alpha_1,\dots,\alpha_n\}$ and $c[\mu(\rho_i)]=\mu(\rho_{i+n})$.

3. The intermediate complex $AX(c)$

Since S_1 and S_2 are orthonormal sets, c factors as a product of two involutions, $c=c_+c_-$, where

$$c_+=R(\alpha_1)\cdots R(\alpha_s) \quad\text{and}\quad c_-=R(\alpha_{s+1})\cdots R(\alpha_n).$$

It follows that $c_+(S_1)=-S_1$ and that $c_+(S_2)=c_+c_-c_-(S_2)=-c(S_2)$.

Definition 3.1. We define the simplicial complex $AX(c)$ to be the result of applying the involution c_+ to $EX(c)$.

The vertices and simplices of $AX(c)$ have the following characterization.

Proposition 3.2. *The simplicial complex $AX(c)$ has vertex set*

$$\{\rho_1,\dots,\rho_{nh/2+n}\},$$

and a simplex on $\{\tau_1,\dots,\tau_k\}$ provided that

$$\rho_1\leqslant\tau_1<\tau_2<\dots<\tau_k\leqslant\rho_{nh/2+n} \quad\text{and}\quad \ell[R(\tau_1)\cdots R(\tau_k)c]=n-k.$$

Proof. It follows from [**2**, §§ 3 and 8] that the ordered set of roots

$$\{\rho_{-n+s+1},\dots,\rho_0,\rho_1,\dots,\rho_{nh/2},\rho_{nh/2+1},\dots,\rho_{nh/2+s},\dots,\rho_{nh/2+n}\}$$

is partitioned into the ordered sequence of subsets

$$-S_2,S_1,c(-S_2),c(S_1),\dots,c^{-1}(-S_1),S_2,-S_1,c(S_2).$$

Since c_+ and c_- are involutions, it follows that $c_+cc_+=c^{-1}$ and hence that

$$c_+(c^k(S_1))=c_+(c^k)c_+c_+(S_1)=-c^{-k}(S_1), \tag{3.1}$$

$$c_+(c^k(S_2))=c_+(c^k)c_+c_+(S_2)=-c^{1-k}(S_2). \tag{3.2}$$

Thus, the action of c_+ on the ordered sequence of subsets is

$$\begin{array}{ccccccc} -S_2 & S_1 & -c(S_2) & c(S_1) & -c^2(S_2) & c^2(S_1) & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ c(S_2) & -S_1 & S_2 & -c^{-1}(S_1) & c^{-1}(S_2) & -c^{-2}(S_1) & \cdots \end{array}$$

As the vertex set of $EX(c)$ is

$$-S_2 \cup S_1 \cup c(-S_2) \cup c(S_1) \cup \cdots \cup c^{-1}(-S_1) \cup S_2 \cup -S_1,$$

the vertex set of $AX(c)$ is

$$S_1 \cup c(-S_2) \cup c(S_1) \cup \cdots \cup c^{-1}(-S_1) \cup S_2 \cup -S_1 \cup c(S_2),$$

which is equal to $\{\rho_1, \rho_2, \dots, \rho_{n+nh/2}\}$.

Next suppose that τ and σ are vertices of $AX(c)$ with $\tau < \sigma$. We will show that an edge in $AX(c)$ joins τ and σ if and only if

$$c = R(\sigma)R(\tau)x \quad \text{for some } x \in W \text{ with } \ell(x) = n-2.$$

Indeed, by definition, an edge in $AX(c)$ joins τ and σ if and only if an edge in $EX(c)$ joins $c_+(\tau)$ and $c_+(\sigma)$ and this holds if and only if either

(i) $c_+(\sigma) < c_+(\tau)$ and $c = R(c_+(\tau))R(c_+(\sigma))y$ for some $y \in W$ with $\ell(y) = n-2$, or

(ii) $c_+(\tau) < c_+(\sigma)$ and $c = R(c_+(\sigma))R(c_+(\tau))y$ for some $y \in W$ with $\ell(y) = n-2$.

Since the c_+ action inverts the order of the subsets $\pm c^k(S_i)$, the relation $c_+(\tau) < c_+(\sigma)$ can only occur when τ and σ belong to the same subset $\pm c^k(S_j)$. Because these subsets are orthogonal, it follows that τ and σ are joined by an edge in $AX(c)$ if and only if

$$c = R(c_+(\tau))R(c_+(\sigma))y \quad \text{for some } y \text{ with } \ell(y) = n-2.$$

However, using the fact that the set of reflections in W is closed under conjugation, we deduce that $c = R(c_+(\tau))R(c_+(\sigma))y$ is equivalent to

$$c^{-1} = c_+ c c_+ = R(\tau)R(\sigma)z = tR(\tau)R(\sigma),$$

which in turn is equivalent to $c = R(\sigma)R(\tau)x$, where y, z and t are length $n-2$ elements in W, z is conjugate to y, t is conjugate to z and $x = t^{-1}$. In particular, $c = R(c_+(\tau))R(c_+(\sigma))y$ for some y with $\ell(y) = n-2$ if and only if $c = R(\sigma)R(\tau)x$ for some x with $\ell(x) = n-2$. This establishes the characterization of edges in $AX(c)$. As both $EX(c)$ and $AX(c)$ are determined by their 1-skeletons, the proposition follows. □

4. Vertex type revisited

In this section we construct a bijection between facets of $AX(c)$ and elements of NCP_c by partitioning the vertices of each facet F of $AX(c)$ into forward and backward vertices in a manner similar to the way vertices of facets are partitioned into right and left vertices in [**1**]. The two notions of vertex type in a facet are different. We choose the one below because we can give a uniform characterization of both forward vertices and backward vertices of facets. We recall that, for $w \in \mathrm{NCP}_c$, $X(w)$ is the subcomplex of $EX(c)$ (and hence also of $AX(c)$) consisting of those simplices whose vertices are positive roots in the subspace $M(w)$. The total order on the vertices of $EX(c)$ allows us to put a lexicographic order on simplices. In particular, the lexicographically first facets (which we will often refer to simply as the first facets) of the subcomplexes $X(w)$, for $w \in \mathrm{NCP}_c$, will be used to define the bijection.

In this section F will be a facet of $AX(c)$ with ordered vertices $\tau_1 < \tau_2 < \cdots < \tau_n$ so that $c = R(\tau_n)R(\tau_{n-1}) \cdots R(\tau_1)$.

Definition 4.1. For $1 \leqslant i \leqslant n$ we define the non-crossing partitions

$$u_i = u_i(F) = R(\tau_n)R(\tau_{n-1}) \cdots R(\tau_i),$$
$$v_i = v_i(F) = R(\tau_i) \cdots R(\tau_2)R(\tau_1),$$

and say that τ_i is a *forward* vertex in F if τ_i is a vertex of the first facet of $X(v_i)$. Otherwise, we say that τ_i is a *backward* vertex in F.

Lemma 4.2.

(i) *If $\tau_i \in \{\rho_1, \dots, \rho_n\}$, then τ_i must be a forward vertex of F.*

(ii) *If $\tau_i \in \{\rho_{nh/2+1}, \dots, \rho_{nh/2+n}\}$, then τ_i must be a backward vertex of F.*

Proof. (i) Suppose that $\tau_i = \rho_j$ is one of the first n roots. We claim that

$$M(v_i) \cap \{\rho_1, \rho_2, \dots, \rho_j = \tau_i\} = \{\tau_1, \dots, \tau_i\}.$$

Indeed, the inclusion $\{\tau_1, \dots, \tau_i\} \subseteq M(v_i) \cap \{\rho_1, \rho_2, \dots, \rho_j = \tau_i\}$ follows from the definition of v_i and the ordering of the τs. If the reverse inclusion did not hold, we would have had $R(\rho_k) \preceq v_i$ for some ρ_k satisfying

$$\rho_k < \rho_j = \tau_i \quad \text{and} \quad \rho_k \notin \{\tau_1, \dots, \tau_i\}.$$

However, since the first n roots are linearly independent, $\{\tau_1, \dots, \tau_i, \rho_k\}$ is a set of $i+1$ linearly independent vectors in $M(v_i)$, contradicting $\ell(v_i) = i$. Thus, the above equality of sets holds and the first facet of v_i is forced to have vertex set $\{\tau_1, \dots, \tau_i\}$. In particular, τ_i is a forward vertex of F.

(ii) The first facet of $X(w)$ is necessarily a set of positive roots for any $w \in \mathrm{NCP}_c$. Thus, any vertex of a facet F that is also a negative root must be a backward vertex of F. □

https://doi.org/10.1017/S0013091508000448 Published online by Cambridge University Press

The characterization of forward vertices of a facet uses [**1**, Lemma 3.3], which we now recall. Here $\delta_1, \dots, \delta_k$ is the ordered simple system for W_w.

Lemma 4.3 (Athanasiadis *et al.* [1, Lemma 3.3]). *Let τ be a positive root in $M(w)$ and fix $1 \leqslant i \leqslant k$. Then $\tau \in \{\varepsilon_1, \dots, \varepsilon_i\}$ if and only if $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau$ is a negative root. In particular, $\tau \in \{\varepsilon_1, \dots, \varepsilon_k\}$ if and only if $w^{-1}(\tau)$ is a negative root.*

Lemma 4.4. *The root τ_i is a forward vertex of F if and only if $v_i^{-1}(\tau_i)$ is a negative root.*

Proof. ($\Rightarrow$) By definition, if τ_i is a forward vertex of F, then τ_i is a vertex of the first facet of $X(v_i)$ and, by Lemma 4.3, $v_i^{-1}(\tau_i)$ is a negative root.

($\Leftarrow$) Conversely, assume that $v_i^{-1}(\tau_i)$ is a negative root. We deal separately with the two cases in which τ_i is positive and negative. If τ_i is a positive root, then τ_i is a vertex of the first facet of $X(v_i)$ by Lemma 4.3. On the other hand, if τ_i is negative, then

$$\tau_i \in \{\rho_{nh/2+1}, \dots, \rho_{nh/2+n}\} = (-S_1) \cup c(S_2).$$

Hence, $-\tau_i$ belongs to $S_1 \cup c(-S_2)$ and is one of the first n roots. Since the first n roots form a linearly independent set, the vectors in

$$\{\rho_1, \dots, \rho_n\} \cap M(v_i)$$

must all lie in the first facet of $X(v_i)$. In particular, $-\tau_i$ lies in the first facet of $X(v_i)$. Thus, $v_i^{-1}(-\tau_i)$ is a negative root by Lemma 4.3. However, this gives a contradiction, since $v_i^{-1}(\tau_i)$ is assumed to be negative. □

The following is an immediate consequence.

Corollary 4.5. *The root τ_i is a backward vertex of F if and only if $v_i^{-1}(\tau_i)$ is a positive root.*

We now turn to the characterization of backward vertices of facets in $AX(c)$. We begin with an elementary observation.

Lemma 4.6. *If θ_i is the root defined by $\theta_i = c^{-1}(\tau_i)$, then*

$$v_i^{-1}(\tau_i) = -c^{-1}u_i c(\theta_i).$$

Proof. For convenience, let r denote the reflection $R(\tau_i)$ and note that $c = u_i r v_i$. Thus, $v_i^{-1} = c^{-1}u_i r$ and hence

$$v_i^{-1}(\tau_i) = c^{-1}u_i r(\tau_i) = c^{-1}u_i(-\tau_i) = -c^{-1}u_i c(\theta_i).$$

□

The characterization of backward vertices also uses a result of [**1**], which we now recall.

Lemma 4.7 (Athanasiadis *et al.* [1, Corollary 3.15]). *Let τ be a positive root in $M(w)$. Then $w(\tau)$ is a negative root if and only if τ is a vertex of the last facet of $X(w)$.*

Lemma 4.8. *The root τ_i is a backward vertex of F if and only if $\tau_i = c(\theta_i)$ for some vertex θ_i in the last facet of $X(c^{-1}u_ic)$.*

Proof. ($\Leftarrow$) Suppose that $\tau_i = c(\theta_i)$ for some vertex θ_i in the last facet of $X(c^{-1}u_ic)$. Then Lemma 4.6 gives $v_i^{-1}(\tau_i) = -c^{-1}u_ic(\theta_i)$. However, the fact that θ_i is in the last facet of $X(c^{-1}u_ic)$ means that $c^{-1}u_ic(\theta_i)$ is negative by Lemma 4.7. Thus, $v_i^{-1}(\tau_i)$ is positive and τ_i is a backward vertex of F by Corollary 4.5.

($\Rightarrow$) Conversely, suppose that τ_i is a backward vertex of F. Then, by Lemma 4.2 (i), τ_i is not one of the first n roots. However, since $c(\rho_i) = \rho_{i+n}$, this means that $c^{-1}\tau_i$ is a positive root. Let θ_i be this positive root. By Lemma 4.7, it remains to show that $c^{-1}u_ic(\theta_i)$ is a negative root. However, by Lemma 4.6, $c^{-1}u_ic(\theta_i) = -v_i^{-1}(\tau_i)$ and this root is negative, by Corollary 4.5, since we are assuming that τ_i is backward. Thus, θ_i is a vertex of the last facet of $X(c^{-1}u_ic)$. □

As in [**1**, Lemma 5.3], forward and backward vertices of a facet F of $AX(c)$ are orthogonal if they appear in the wrong order in the factorization of c determined by F. The induction proof of [**1**, Lemma 5.3] could be adapted here but it is possible to give a more conceptual proof.

Lemma 4.9. *If τ_i is a backward vertex of F and τ_j is a forward vertex of F with $\tau_i < \tau_j$, then $\tau_i \cdot \tau_j = 0$.*

Proof. Let $\{\varepsilon_1, \dots, \varepsilon_j\}$ be the ordered vertex set of the first facet of $X(v_j)$, where v_j is the non-crossing partition

$$v_j = R(\tau_j) \cdots R(\tau_1) = R(\varepsilon_j) \cdots R(\varepsilon_1).$$

Since τ_j is forward, τ_j must, by definition, be one of $\varepsilon_1, \dots, \varepsilon_j$. Moreover, since the set $\{\tau_1, \tau_2, \dots, \tau_j\}$ is linearly independent, we must have $\tau_j = \varepsilon_j$. If $\tau_i \notin \{\varepsilon_1, \dots, \varepsilon_{j-1}\}$, then [**1**, Lemma 3.4] gives $\tau_j \cdot \tau_i = 0$. Thus, it remains to show that τ_i is not one of $\varepsilon_1, \dots, \varepsilon_{j-1}$.

In order to show this, let $\{\varepsilon'_1, \dots, \varepsilon'_i\}$ be the ordered vertex set of the first facet of $X(v_i)$, where v_i is the non-crossing partition

$$v_i = R(\tau_i) \cdots R(\tau_1) = R(\varepsilon'_i) \cdots R(\varepsilon'_1).$$

Since τ_i is backward, $\tau_i \notin \{\varepsilon'_1, \dots, \varepsilon'_i\}$ by definition. However, since $\{\varepsilon'_1, \dots, \varepsilon'_i\}$ is a basis for $M(v_i)$ and $\tau_i > \varepsilon'_i$, it follows that the root τ_i lies in the linear span of the set

$$\{\rho \in M(v_i) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \tau_i\}.$$

Since $v_i \preceq v_j$, we deduce that τ_i lies in the linear span of

$$\{\rho \in M(v_j) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \tau_i\}.$$

However, the linear span of $\{\rho \in M(v_j) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \varepsilon_k\}$ does not contain ε_k for any $1 \leqslant k \leqslant j-1$, by [**2**, Corollary 6.12]. Thus, $\tau_i \neq \varepsilon_k$ for $1 \leqslant k \leqslant j-1$. □

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Theorem 4.10. *The function ϕ from facets of $AX(c)$ to* NCP_c *taking a facet F with forward vertices $\tau_{i_1} < \tau_{i_2} < \cdots < \tau_{i_k}$ to the product $R(\tau_{i_k})R(\tau_{i_{k-1}})\cdots R(\tau_{i_1})$ is a bijection.*

Proof. Suppose that F is a facet with $\phi(F) = v$. By Lemma 4.9, appropriate pairs of factors of

$$c = R(\tau_n)R(\tau_{n-1})\cdots R(\tau_1)$$

can be commuted until the product $R(\tau_{i_k})R(\tau_{i_{k-1}})\cdots R(\tau_{i_1})$ appears on the right. By Lemma 4.4, the forward vertices of F are precisely the vertices of the first facet of $X(v)$. Since $c = (cv^{-1})v$ and $c^{-1}(cv^{-1})c = v^{-1}c$, Lemma 4.8 implies that the backward vertices of F are the images under c of the vertices of the last facet of $X(v^{-1}c)$. Thus, the vertex set of F is completely determined by v and hence ϕ is injective. On the other hand, we know from [**1**, Theorem 6.4] that the number of facets of $EX(c)$ is the same as the number of elements of NCP_c. Since $AX(c)$ is the image of $EX(c)$ under the isometry c_+, it follows that ϕ is a bijection. □

The following result is immediate from Theorem 4.10 and its proof.

Corollary 4.11. *For each $v \in \mathrm{NCP}_c$ there is a facet of $AX(c)$ whose vertex set consists of the vertices of the first facet of $X(v)$ and the images under c of the vertices of the last facet of $X(v^{-1}c)$. Moreover, every facet of $AX(c)$ arises in this way.*

5. Applying the μ operator

Definition 5.1. We define the simplicial complex $\mu(AX(c))$ to be the result of applying the operator $\mu = 2(I - c)^{-1}$ to $AX(c)$.

The vertices and simplices of $\mu(AX(c))$ have a simple characterization which follows immediately from Proposition 3.2.

Proposition 5.2. *The simplicial complex $\mu(AX(c))$ has vertex set*

$$\{\mu(\rho_1), \ldots, \mu(\rho_{nh/2+n})\},$$

and a simplex on $\{\mu(\tau_1), \ldots, \mu(\tau_k)\}$, provided that

$$\rho_1 \leqslant \tau_1 < \tau_2 < \cdots < \tau_k \leqslant \rho_{nh/2+n} \quad \text{and} \quad \ell[R(\tau_1)\cdots R(\tau_k)c] = n - k.$$

Now we are in a position to show that the cones on the facets of $\mu(AX(c))$ are precisely the cones $F(w)$ defined in § 1. Recall that

$$F(w) = \{x \in \mathbb{R}^n \mid x \cdot \delta_i \leqslant 0 \text{ and } x \cdot \theta_j \geqslant 0\},$$

where $\{\delta_1, \ldots, \delta_k\}$ is the simple system for the parabolic subgroup determined by w and $\{\theta_1, \ldots, \theta_{n-k}\}$ is the simple system for the parabolic subgroup determined by $w' = cw^{-1}$. We note that $F(w)$ is a simplicial cone of dimension n since

$$c = w'w = R(\theta_1)\cdots R(\theta_{n-k})R(\delta_1)\cdots R(\delta_k)$$

means that $\{\delta_1,\dots,\delta_k,\theta_1,\dots,\theta_{n-k}\}$ is a linearly independent set. We first determine the rays of each $F(w)$.

Proposition 5.3. *Suppose that $w \in \mathrm{NCP}_c$ and $F(w)$ is the simplicial cone defined above. Then the rays of $F(w)$ are generated by*

$$\{\mu(\varepsilon_1),\dots,\mu(\varepsilon_{n-k}),\mu[c(\eta_{n-k+1})],\dots,\mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1},\dots,\eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$.

Proof. Suppose that $\{\tau_1,\dots,\tau_n\}$ is an arbitrary set of positive roots satisfying $c = R(\tau_1)\cdots R(\tau_n)$. We are interested in the case when

$$\tau_i = \begin{cases} \theta_i & \text{for } 1 \leqslant i \leqslant n-k, \\ \delta_{i-n+k} & \text{for } n-k+1 \leqslant i \leqslant n, \end{cases}$$

so that the τ_i are positive but may not be in increasing order even though the subsets $\{\delta_1,\dots,\delta_k\}$ and $\{\theta_1,\dots,\theta_{n-k}\}$ are in increasing order. We define

$$\varepsilon_i = R(\tau_1)\cdots R(\tau_{i-1})\tau_i \quad \text{and} \quad \eta_i = R(\tau_n)\cdots R(\tau_{i+1})\tau_i.$$

As in § 4, we can define the non-crossing partitions

$$a_i = R(\tau_1)\cdots R(\tau_i) \quad \text{and} \quad b_i = R(\tau_i)\cdots R(\tau_n).$$

Thus, $\varepsilon_i = -a_i(\tau_i)$ and $\eta_i = -b_i^{-1}(\tau_i)$. Moreover, since $c = a_i R(\tau_i) b_i$, we have $c(\eta_i) = -a_i R(\tau_i)[\tau_i] = a_i[\tau_i] = -\varepsilon_i$. We deduce from

$$R(\varepsilon_i) = R(\tau_1)\cdots R(\tau_{i-1})R(\tau_i)R(\tau_{i-1})\cdots R(\tau_1)$$

that

$$R(\varepsilon_i)c = R(\tau_1)\cdots R(\tau_{i-1})R(\tau_{i+1})\cdots R(\tau_n)$$

and hence, by [**1**, Lemma 2.2], that $\mu(\varepsilon_i)$ is orthogonal to τ_j for $j \neq i$. Also, by [**1**, Lemmas 2.3 and 2.4],

$$\begin{aligned} \tau_i \cdot \mu(\varepsilon_i) &= \tau_i \cdot \mu[-a_i(\tau_i)] \\ &= -\tau_i \cdot a_i(\mu[\tau_i]) \\ &= -\tau_i \cdot (\mu[\tau_i] - 2\tau_i) \\ &= -1 + 2 \\ &= 1. \end{aligned}$$

Thus, $\mu(\varepsilon_i)$ lies on each of the hyperplanes $\tau_j^{\perp}$ for $j \neq i$ and on the positive side of $\tau_i^{\perp}$. Since $c(\eta_i) = -\varepsilon_i$, it follows that $\mu(c[\eta_i])$ lies on each of the hyperplanes $\tau_j^{\perp}$ for $j \neq i$ but on the negative side of $\tau_i^{\perp}$.

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Now, suppose that

$$\tau_i = \begin{cases} \theta_i & \text{for } 1 \leqslant i \leqslant n-k, \\ \delta_{i-n+k} & \text{for } n-k+1 \leqslant i \leqslant n, \end{cases}$$

corresponding to the factorization

$$c = (cw^{-1})c = R(\theta_1)\cdots R(\theta_{n-k})R(\delta_1)\cdots R(\delta_k),$$

where $\{\delta_1,\dots,\delta_k\}$ is the simple system for the parabolic subgroup W_w and $\{\theta_1,\dots,\theta_{n-k}\}$ is the simple system for the parabolic $W_{cw^{-1}}$. The ray of $F(w)$ that is opposite the $\theta_i^{\perp}$ wall and on its positive side is generated by $\mu(\varepsilon_i)$, while the ray of $F(w)$ that is opposite the $\delta_i^{\perp}$ wall and on its negative side is generated by $\mu(c(\eta_{n-k+i}))$. We deduce that the rays of $F(w)$ are generated by

$$\{\mu(\varepsilon_1),\dots,\mu(\varepsilon_{n-k}),\mu[c(\eta_{n-k+1})],\dots,\mu[c(\eta_n)]\}.$$

To conclude, we note that the roots $\varepsilon_1,\dots,\varepsilon_{n-k}$ are the vertices of the lexicographically first facet of $X(cw^{-1})$ and the roots $\eta_{n-k+1},\dots,\eta_n$ are the vertices of the lexicographically last facet of $X(w)$, by [**1**, propositions 3.6 and 3.14]. □

Corollary 5.4. *For each $w \in NCP_c$, the rays of $F(w)$ are generated by a subset of the set of vertices of $\mu(AX(c))$.*

Proof. By Proposition 5.3, the rays of $F(w)$ are generated by

$$\{\mu(\varepsilon_1),\dots,\mu(\varepsilon_{n-k}),\mu[c(\eta_{n-k+1})],\dots,\mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1},\dots,\eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$. Since $c\rho_i = \rho_{i+n}$, the rays of $F(w)$ are generated by a subset of the set $\{\mu(\rho_1),\dots,\mu(\rho_{nh/2+n})\}$. □

Proof of Theorem 1.1. We wish to show that the set of simplicial cones $\{F(w)\}$, where w ranges over the elements of NCP_c, is precisely the set of cones on simplices of $\mu(AX(c))$. If $w \in \mathrm{NCP}_c$, then, by Proposition 5.3, the rays of $F(w)$ are generated by

$$V = \{\mu(\varepsilon_1),\dots,\mu(\varepsilon_{n-k}),\mu[c(\eta_{n-k+1})],\dots,\mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1},\dots,\eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$. On the other hand, by Corollary 4.11 with $v = cw^{-1}$, there is a facet of $AX(c)$ whose vertex set is the union of the vertices of the first facet of $X(cw^{-1})$ and the images under c of the vertices of the last facet of $X(w)$. Since $\mu(AX(c))$ is the image of $AX(c)$ under the action of μ, the complex $\mu(AX(c))$ has a facet with vertex set V. Since every facet of $\mu(AX(c))$ arises in this way by the bijectivity of ϕ and the invertibility of the linear transformation μ, the set of $F(w)$s coincides with the set of cones on simplices of $\mu(AX(c))$. □

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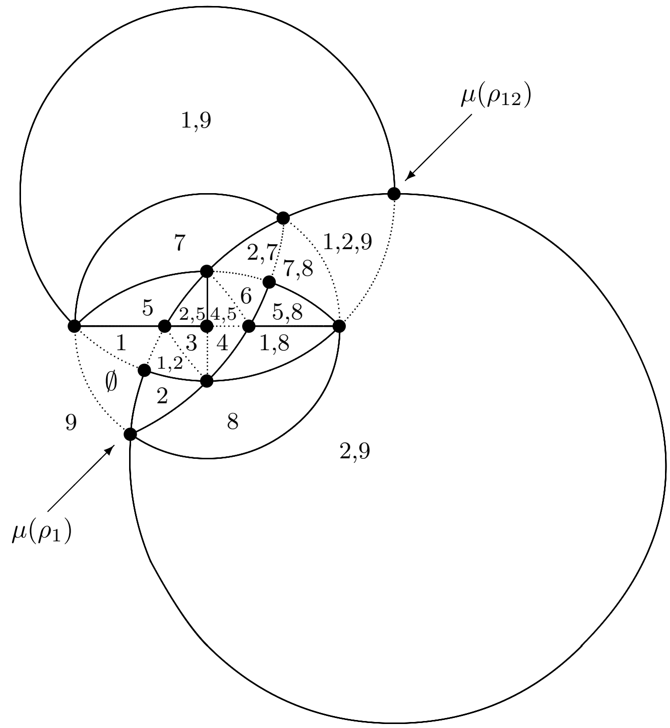

Figure 1. The cyclohedron inside the C_3 permutahedron.

Theorem 5.5. *The fan determined by the cones $F(w)$ for $w \in NCP_c$ coincides with the c-Cambrian fan.*

Proof. Reading and Speyer [**5**] show a linear isomorphism L from the c-cluster fan of a bipartite Coxeter element c to the c-Cambrian fan. We prove that, up to a scalar multiple, this map L coincides with $\mu \circ c_+$. Indeed, the map L is defined on the basis $\{\alpha_1, \dots, \alpha_n\}$ by

$$\alpha_i \mapsto \begin{cases} -\omega_i & \text{for } i = 1, \dots, s, \\ \omega_i & \text{for } i = s+1, \dots, n. \end{cases}$$

Here $\{\omega_1, \dots, \omega_n\}$ is the dual basis to the basis of coroots $\{\alpha_i^\vee\}$, where

$$\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$

Since we have chosen our simple roots to have unit length, the coroot $\alpha^\vee$ is simply $2\alpha_i$ and the 'weight' ω_i is simply $\frac{1}{2}\mu(\rho_i)$ by [**2**, § 3]. However, by (3.1) and (3.2) we have

$$c_+(\alpha_i) = \begin{cases} -\alpha_i & \text{for } i = 1, \dots, s, \\ -c(\alpha_i) & \text{for } i = s+1, \dots, n. \end{cases}$$

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Recalling from [**2**] that

$$\rho_i = \begin{cases} \alpha_i & \text{for } i = 1, \dots, s, \\ -c(\alpha_i) & \text{for } i = s+1, \dots, n. \end{cases}$$

we see that L coincides with $\frac{1}{2}(\mu \circ c_+)$. □

Example 5.6. Let W be the group C_3 (or B_3) of symmetries of the cube in $\boldsymbol{R}^3$. The type-C_n generalized associahedron is known as the cyclohedron. We can choose a simple system

$$\alpha_1 = (1,0,0), \quad \alpha_2 = \tfrac{1}{2}\sqrt{2}(0,1,-1), \quad \alpha_3 = \tfrac{1}{2}\sqrt{2}(-1,0,1),$$

so that the dual basis is

$$\mu(\rho_1) = (1,1,1), \quad \mu(\rho_2) = \sqrt{2}(0,1,0), \quad \mu(\rho_3) = \sqrt{2}(0,1,1).$$

Here the Coxeter element is the orthogonal transformation defined by $c(x,y,z) = (-z,x,y)$, so that $h = 6$, $\frac{1}{2}nh = 9$ and $\frac{1}{2}nh + n = 12$. The complex $\mu(AX(c))$ is shown in Figure 1, where the 2-sphere has been stereographically projected onto the plane from the point $(-1,0,1)$. Only the vertices $\mu(\rho_1)$ and $\mu(\rho_{12})$ are labelled in the figure, but the other vertices occur consecutively on the dotted polygonal path between the labelled pair. The reflecting hyperplanes intersect the sphere in circles, and segments of these circles form the edges of facets of $\mu(AX(c))$. The position of a particular hyperplane can be deduced from the fact that $\rho_i^{\perp}$ passes through $\mu(\rho_{i+1})$ and $\mu(\rho_{i+2})$, since

$$c = R(\rho_{i+2})R(\rho_{i+1})R(\rho_i) \quad \text{for } 1 \leqslant i \leqslant 9.$$

The figure also incorporates the map $\phi' = \phi \circ \mu^{-1}$ defined by the bijection ϕ from § 4. Each $\mu(AX(c))$ region F' is labelled by a set of integers, the corresponding positive roots forming the simple system for $c[\phi'(F')]^{-1}$. Thus, the set of integers labelling a region F' corresponds to a subset of the walls of F' with a wall contributing to the subset if and only if F' lies on the negative side of the wall.

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