STRUCTURE OF SEMIGROUPS

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The treatment of semigroups given in a previous paper **(3)** is based upon representations of a semigroup by means of transformations of a set (cf. also **12**). In this paper we try to remove the assumption of the existence of a zero element proposed in **(3)**. In accordance with our general programme explained at the beginning of **(3)** we utilize certain minimum conditions in order to gain more information on the structure of semigroups.

Our main results are structure theorems on primitive semigroups which have irreducible right ideals generated by idempotents (§§15–17). As we have shown in **(5)**, these theorems permit the explicit construction of primitive semigroups. The form of these theorems corresponds to a similar statement on primitive rings with minimal right ideals which arises if the density theorem of Chevalley–Jacobson in Jacobson's form (**7**, p. 75) is reformulated in an equivalent purely algebraical manner. In the case of semigroups, we cannot expect a density theorem but only a sequence of transitivity conditions (for a finite degree of transitivity) whose limit would be the density condition equivalent to countable transitivity).

In contrast to the main results mentioned above, the lemmas and theorems of §§2–14 are preliminary in character. Nevertheless they are indispensable for our more general purpose to build up a systematic theory of semigroups. Thus these lemmas and theorems fall into three classes: Either they reformulate our fundamental concepts, or they elucidate these concepts in some simple cases, or they yield applications in subsequent sections (mainly §§15–17). Especially, we point out Theorem 9.4, which shows that the analogue of Schur's Lemma for semigroups holds not only for totally irreducible S-systems. The results, for instance, of §§15–17 depend upon this observation.

1. Terminology and notation. Let S be a semigroup with multiplication as its binary operation. It is not assumed that S contains a zero element. Let M be a set on which the elements $a \in S$ act as right multipliers inducing mappings

$$\rho_a : x \longrightarrow x\rho_a = xa \qquad (x \in M)$$

of M into M. Such a set M is called an *S-system* if

$$(xa)b = x(ab) \qquad \text{for all } x \in M \text{ and } a, b \in S.$$

It follows that the correspondence $\Delta : a \longrightarrow \rho_a$ is a homomorphism of S onto a subsemigroup S_M of the semigroup T_M of all transformations (single-valued

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mappings) of the set M into itself. S_M is called the *representation* of S generated by the S-system M. The representation S_M is said to be *faithful* if Δ is an isomorphism into T_M. Two S-systems M_i $(i = 1, 2)$ are *homomorphic*, $M_1 \simeq M_2$ [*isomorphic*, $M_1 \simeq M_2$], if there is a single-valued [and invertible] mapping ϕ of M_1 onto M_2 such that

$$\phi(xa) = (\phi x)a \qquad \text{for all } x \in M_1 \text{ and } a \in S.$$

If $M_1 \simeq M_2$, then $S_{M_1} \simeq S_{M_2}$ and S_{M_1}, S_{M_2} are called *equivalent*. A homomorphism ϕ of M_1 onto M_2 is *trivial* if ϕ is an isomorphism or if M_2 has only one element. An *S-subsystem* of M is a non-void subset L of M such that $LS \subset L$. If $\phi : M_1 \rightarrow M_2$ is a homomorphism into, then $\phi(M_1)$ is an S-subsystem of M_2. Any non-void subset of the set

$$FM = \{x | x \in M, xa = x \text{ for all } a \in S\}$$

of all elements of M invariant with respect to S is an S-subsystem of M. An S-subsystem $L \subset M$ is *trivial* if $L = M$ or if $|L| = 1$ ($|L|$ is the cardinal number of L).

A *congruence* in M is an equivalence relation λ (regarded as a subset of $M \times M$) such that

$$(x_1, x_2) \in \lambda \Rightarrow (x_1 a, x_2 a) \in \lambda \qquad (a \in S).$$

$[x]_\lambda$ (shorter $[x]$ if no misunderstanding is possible) denotes the congruence class containing the element $x \in M$, and M/λ is the set of all congruence classes of M with respect to λ. Under the composition

$$[x]_\lambda a = [xa]_\lambda, \qquad x \in M, a \in S,$$

M/λ becomes an S-system. Let L be an S-subsystem of M. The *difference system* $M/L = M/\lambda$ is defined by means of the congruence

$$(x_1, x_2) \in \lambda \Leftrightarrow \begin{cases} x_1 = x_2 \\ \quad \text{or} \\ x_1, x_2 \in L. \end{cases}$$

Evidently, $F(M/L) \neq \emptyset$ as $L \in F(M/L)$. If $FM \subset L \neq \emptyset$, then $F(M/L) = \{L\}$, i.e., $|F(M/L)| = 1$.

An S-system M (a representation S_M) is called *irreducible* if

(1.1) $$MS \not\subset FM,$$

(1.2) M has no non-trivial S-subsystems.

Condition (1.1) yields $FM \neq M$; by (1.2)

(1.3) $$|FM| \leqslant 1.$$

Further

(1.4) $$MS = M,$$

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for if $|MS| = 1$, $MS = \{x\}$, then $xa = x$ (for all $a \in S$) and $MS \subset FM$ contrary to (1.1).

An S-system M (a representation S_M) is *totally irreducible* if

$$MS \not\subset FM, \tag{1.5}$$

(1.6) M has no non-trivial homomorphisms.

Every totally irreducible S-system M is irreducible. For if M contains a non-trivial S-subsystem L, then the canonical mapping of M onto M/L yields a non-trivial homomorphism, contrary to (1.6).

If the representation S_M contains a zero element $\sigma = \rho_s$, the elements of S that are mapped onto σ form an ideal (= two-sided ideal) $\Delta^{-1}\{\sigma\}$ of S. This ideal is the *kernel* of the representation S_M. We note that $FM = M\sigma$. If S itself contains a zero element 0, then M and S_M are said to be 0-*faithful* if

$$\Delta^{-1}\{\rho_0\} = \{0\}.$$

2. The radical rad S. The representation S_M defines a congruence $\delta_M \subset S \times S$ in S through

$$(a_1, a_2) \in \delta_M \Leftrightarrow \rho_{a_1} = \rho_{a_2}.$$

Let $\kappa \subset \lambda$ be two congruences in S and let $[a]_\kappa$ be the congruence class with respect to κ containing $a \in S$. In the semigroup S/κ of all congruence classes with respect to κ, the congruence λ/κ is defined through

$$([a]_\kappa, [b]_\kappa) \in \lambda/\kappa \Leftrightarrow (a, b) \in \lambda.$$

It satisfies $(S/\kappa)/(\lambda/\kappa) \simeq S/\lambda$. The following lemma is obvious.

2.1. Lemma. *Let λ be any congruence in S.*

(a) *If*

$$\lambda \subset \delta_M(S), \tag{2.2}$$

then the S-system M becomes an S/λ-system under the rule

$$x[a]_\lambda = xa \ (x \in M, a \in S)$$

and we have

$$\delta_M(S/\lambda) = \delta_M(S)/\lambda. \tag{2.3}$$

(b) *Conversely, an S/λ-system M becomes an S-system satisfying* (2.2) *and* (2.3) *if we define $xa = x[a]_\lambda$.*

(c) *Any congruence in the S-system M remains a congruence in M regarded as an S/λ-system and vice versa.*

(d) *An element of M is invariant with respect to S if and only if it is invariant with respect to S/λ.*

The congruence

$$\text{rad}\, S = \bigcap_{M \in I} \delta_M$$

where I is the set of all irreducible S-systems is the *radical* of S. By convention, rad $S = \mathbf{1}$ (where $\mathbf{1}$ is the universal relation) if $I = \emptyset$. If rad $S = \mathbf{0}$ (where $\mathbf{0}$ is the identical relation), S is said to be *radical-free*. If rad $S = \mathbf{1}$, then S is called a *radical* semigroup.

2.4 THEOREM. rad $(S/\text{rad } S) = \mathbf{0}$.

Proof. Applying Lemma 2.1 with rad $S \subset \delta_M$ $(M \in I)$, we see that every irreducible S-system M remains irreducible as an $(S/\text{rad } S)$-system and vice versa. Therefore

$$\begin{aligned} \text{rad } (S/\text{rad } S) &= \bigcap_{M \in I} \delta_M(S/\text{rad } S) = \bigcap_{M \in I} (\delta_M(S)/\text{rad } S) \\ &= (\bigcap_{M \in I} \delta_M(S))/\text{rad } S = (\text{rad } S)/\text{rad } S = \mathbf{0}. \end{aligned}$$

3. The 0-radical rad^0S. With each S-system M we associate the set

$$M^0 = M^0(S) = \{a | a \in S \text{ and every equation } xab = x \text{ where } x \in M \text{ and } b \in S^1 \text{ implies } x \in FM\}.$$

Here S^1 is the semigroup obtained from S by adjoining an identity-element 1. M^0 is void or an ideal in S.

Let K and L be subsets of M, $K \neq \emptyset$. Define

$$K^{-1}L = \{a | a \in S, Ka \subset L\}.$$

3.1. LEMMA. *Let M be an irreducible S-system. Then $M^0 = M^{-1}FM$.*

Proof. (1) Suppose $a \in M^{-1}FM$ and $xab = x (x \in M, b \in S^1)$. Then $xa \in FM$ and $x = xab \in (FM)b \subset FM$.

(2) Let $a \in M^0, x \in M$. Assume $xa \notin FM$. This would imply that $xaS^1 = M$ and $xab = x$ with suitable $b \in S^1$. Hence $x \in FM$ and $xa \in FM$.

3.2. COROLLARY. *Let M be an irreducible S-system, $M^0 \neq \emptyset$. Then M^0 is the kernel of the representation S_M.*

With every element $a \in S$ we associate the congruence $\kappa(a)$ defined in S by means of $S/\kappa(a) = S/S^1aS^1$. Here S/S^1aS^1 is the difference semigroup of S in the sense of Rees with respect to the principal ideal S^1aS^1.

3.3. LEMMA. *Let $a \in S$, $M \in I$. Then*

$$\kappa(a) \subset \delta_M \Leftrightarrow a \in M^0.$$

Proof. (1) Let $\kappa(a) \subset \delta_M$ and $xab = x$ $(x \in M, b \in S^1)$. From $c \in S$, $(ab, abc) \in \kappa(a)$, we infer that $yab = yabc$ for every $y \in M$. In particular we obtain $x = xab = xabc = xc$ for every $c \in S$ and hence $x \in FM$.

(2) Let $a \in M^0$; hence $S^1aS^1 \subset M^0$. By Corollary 3.2, M^0 is a congruence class with respect to δ_M; whence $\kappa(a) \subset \delta_M$.

An element $a \in S$ such that

$$as = a \text{ for every } s \in S$$

is a *left zero* of S. The set $O(S)$ of all left zeros of S is either void or an ideal in S. The difference semigroup $S/O(S)$ then contains zero. The *nilradical* nil rad $(S/O(S))$ (i.e., the sum of all nil right ideals or equivalently the sum of all nil ideals of $S/O(S)$) defines an ideal $N(S)$ in S through

$$N(S)/O(S) = \text{nil rad}\,(S/O(S)), \qquad N(S) \supset O(S) \neq \emptyset.$$

If $O(S) = \emptyset$, put $N(S) = \emptyset$. For any S-system M, we have $MO(S) \subset FM$, hence $O(S) \subset M^{-1}FM = M^0$ for an irreducible M.

3.4. Lemma. *Let $O(S) \neq \emptyset$. Then every irreducible S-system M is an irreducible $S/O(S)$-system and vice versa. Moreover,*

$$M^0\,(S/O(S)) = M^0(S)/O(S). \tag{3.5}$$

Proof. $O(S) \subset M^0$ implies $M^0 \neq \emptyset$. Hence M^0 is a congruence class with respect to δ_M. Since $O(S) \subset M^0$, Lemma 2.1 can be applied. The relation (3.5) follows from Lemma 2.1 (d) and Lemma 3.1.

3.6. Theorem. *Let*

$$\text{rad}^0\, S = \bigcap\nolimits_{M \in I} M^0 \tag{3.7}$$

(by convention $\text{rad}^0\, S = S$ if $I = \emptyset$). *Then* $N(S) \subset \text{rad}^0\, S$.

Remark. Obviously $\text{rad}^0\, S$ may be void. If S contains zero, then $\text{rad}^0\, S$ is the 0-radical in the sense of **(3)**. In general $\text{rad}^0\, S$ is called the *generalized* 0-*radical.*

Proof of Theorem 3.6. If $N(S) \neq \emptyset$ and $M \in I$, then as in **(3)**

$$\begin{aligned} N(S)/O(S) &= \text{nil rad}\,(S/O(S)) \\ &\subset \bigcap\nolimits_{M \in I} M^0(S/O(S)) = \bigcap\nolimits_{M \in I} M^0(S)/O(S) \\ &= (\text{rad}^0\, S)/O(S). \end{aligned} \tag{3.8}$$

The following theorem is a special consequence of a surprising characterization of our congruence rad S **(6; 11)**. It is stronger than Theorem 3.6. We state it without proof.

3.9. Theorem. $N(S) = \text{rad}^0\, S$.

4. Primitive semigroups. A semigroup S is called (*right*) *primitive* (0-*primitive, totally primitive*) if it has a faithful irreducible (0-faithful irreducible, faithful totally irreducible) S-system. Since the regular representation of a group G is transitive, every group (distinct from identity) is a primitive semigroup. Obviously every primitive permutation group is a totally primitive semigroup. The semigroup T_M is primitive but not left primitive. If a completely simple semigroup S is faithfully represented by a regular matrix semigroup (over a group or group with zero) with the defining matrix P, then S is primitive if no two different columns of P have a common right multiple **(12)**.

A congruence π in a semigroup S is *primitive* (*totally primitive*) if S/π is a primitive (totally primitive) semigroup. This definition implies

4.1. LEMMA. *The congruence π in S is primitive* (*totally primitive*) *if and only if $\pi = \delta_M$ where M is an irreducible* (*totally irreducible*) *S-system.*

4.2. LEMMA. *Primitive semigroups are radical free.*

4.3. THEOREM. *A radical-free semigroup S satisfies*

$$S = \bigcap_{M \in I} S/\delta_M.$$

Thus it is subdirectly decomposable into primitive semigroups S/δ_M.

Proof. By Lemma 2.1, every irreducible S-system M is an irreducible S/δ_M-system. Hence S/δ_M is primitive if $M \in I$. Since $\bigcap_{M \in I} \delta_M = 0$, the Theorem follows immediately from a theorem of Birkhoff (**1**, p. 92, Theorem 9).

It is natural to introduce the following notion. A semigroup S with zero 0 is *weakly free of zero-divisors* if for all a and $b \in S$:

$$aS^1b = \{0\} \Rightarrow a = 0 \text{ or } b = 0. \tag{4.4}$$

This condition is equivalent to

$$aSb = \{0\} \Rightarrow a = 0 \text{ or } b = 0. \tag{4.5}$$

Obviously (4.5) implies (4.4). Conversely, if (4.4) is true and $aSb = \{0\}$, $b \neq 0$, then $asS^1b = \{0\}$ (for all $s \in S$); hence $as = 0$, i.e., $aS = \{0\}$ and in particular $ab = 0$. Hence $aS^1b = \{0\}$ and (4.4) implies $a = 0$.

A semigroup S is *weakly left cancelling* if for all a, b_1, and $b_2 \in S$ the following statement is true:

(4.6) If $asb_1 = asb_2$ for all $s \in S^1$ and if $b_1 \neq b_2$, then S contains left zeros and $a \in O(S)$.

4.7. LEMMA. *Every weakly left cancelling semigroup with zero is weakly free of zero-divisors.*

Proof. $aS^1b = \{0\}$ implies $asb_1 b = asb_2 b = 0$ for all b_1, b_2, $s \in S^1$; hence $a = 0$ or $b_1 b = b_2 b$. Choose $b_1 = 1$, $b_2 = 0$. Then the second case yields $b = 0b = 0$.

The congruence π in a semigroup S is a (*right*) *prime congruence* if S/π is weakly left cancelling (in commutative semigroups with identity this concept of a prime congruence is equivalent to a definition due to K. Drbohlav). As usual, an ideal P in S is a *prime ideal* if $AB \subset P$, $B \not\subset P$ implies $A \subset P$ whenever A and B are ideals in S.

4.8. LEMMA. *The ideal $P \subset S$ is a prime ideal of S if and only if S/P is weakly free of zero-divisors.*

Proof. (1) Let S/P be weakly free of zero-divisors and $AB \subset P$, $B \not\subset P$. For all $a \in A$ and $b \in B$ we have $aS^1b \subset AB \subset P$. Since b may be chosen such that $b \notin P$, we deduce that $a \in P$.

(2) Let P be a prime ideal and $aS^1b \subset P$. Then $(S^1aS^1)(S^1bS^1) \subset P$ and $a \in S^1aS^1 \subset P$ or $b \in S^1bS^1 \subset P$.

4.9. THEOREM. *Let P be an ideal of the semigroup S and let π be the corresponding congruence in S such that $S/\pi = S/P$. If π is a prime congruence, then P is a prime ideal.*

Proof. Cf. Lemmas 4.7 and 4.8.

4.10. THEOREM. *Every primitive congruence in a semigroup S is a prime congruence.*

Proof. Let π be a primitive congruence in S. Then S/π has a faithful irreducible S-system M. If $asb_1 = asb_2$ for all $s \in S^1$, then $xasb_1 = xasb_2$ for all $x \in M$. If $a \notin O(S)$, then M contains at least one element x such that $xaS^1 = M$. For if $xaS^1 \subset FM = \{x_0\}$ holds for every $x \in M$, then $xas = xa$ for all $x \in M$, i.e., $as = a$ for all $s \in S$, whence $a \in O(S)$, a contradiction. From $xaS^1 = M$, it follows that $yb_1 = yb_2$ for all $y \in M$; hence $b_1 = b_2$.

5. Cyclic S-systems. An S-system Z is *cyclic* (*strictly cyclic*) if Z contains an element z such that $Z = \{z\} \cup zS$ ($Z = zS$). The element z is a *generator* (*strict generator*) for Z. If Z is strictly cyclic, then every generator for Z is strict. The set $\hat{Z}$ of all the non-generators of the cyclic S-system Z is void or an S-subsystem of Z distinct from Z. If Z contains at least two different elements, then

$$FZ \subset \hat{Z} \neq Z. \tag{5.1}$$

Indeed $x \in FZ$ implies that $xS = \{x\} \neq Z$ and $x \in \hat{Z}$.

5.2. THEOREM. (a) *An irreducible S-system M is strictly cyclic with $FM = \hat{M}$.*

(b) *Let Z be a strictly cyclic S-system containing at least two elements. If $\hat{Z} = \emptyset$, then Z is irreducible. If $\hat{Z} \neq \emptyset$, then the difference system $Z/\hat{Z}$ is irreducible.*

Proof. (a) $xS = M$ holds for every $x \in M - FM$. For otherwise $xS \subset FM$, i.e.,

$$L = \{y|y \in M, yS \subset FM\} \not\subset FM;$$

hence $L = M$ (note that L is an S-subsystem of M). The property $LS \subset FM$ of L implies $MS \subset FM$, a contradiction. At the same time, it follows that $\hat{M} \subset FM$. By (5.1), $FM \subset \hat{M}$; thus $FM = \hat{M}$.

(b) $Z = zS$ yields $ZS = Z$ and, by (5.1),

$$ZS \not\subset \hat{Z}, \qquad ZS \not\subset FZ. \tag{5.3}$$

Let W be an S-subsystem of Z; $W \not\subset \hat{Z}$. Then W contains an element w such that $wS = Z$; therefore $Z \subset WS \subset W$, and hence $Z = W$. From (5.3), we deduce that Z is irreducible if $\hat{Z} = \emptyset$. Let $\hat{Z} \neq \emptyset$. Then $F(Z/\hat{Z}) = \{\hat{Z}\}$, by (5.1). On account of (5.3), we have

$$(Z/\hat{Z})S \not\subset F(Z/\hat{Z}).$$

Since every S-subsystem of $Z/\hat{Z}$ is expressible as a quotient $W/\hat{Z}$ where W is an S-subsystem of Z such that $\hat{Z} \subset W$, we obtain (1.2) for $M = Z/\hat{Z}$, i.e. M is irreducible.

6. Some characterizations of rad S**.** Interpreting the semigroup S as an S-system with respect to right multiplication, we have $O(S) = FS$. A *right congruence* μ of the semigroup S then is identical with a congruence in the S-system S. The S-system S/μ is defined as in §1. We call a right congruence μ in S *modular* if there is an element $e \in S$ such that $[e]a = [a] \in S/\mu$ for all $a \in S$. The following Lemma is due to Tully **(12)**.

6.1. Lemma. *The S-system Z is strictly cyclic if and only if $Z \simeq S/\mu$, where μ denotes a modular right congruence in the semigroup S.*

For the sake of completeness we give a short proof. Let $Z = zS$ be strictly cyclic. $\psi : a \to za$ is a homomorphism of the S-system S onto Z and

$$(a, b) \in \mu \Leftrightarrow \psi a = \psi b$$

yields a right congruence in the semigroup S such that $S/\mu \simeq Z$. Choosing $e \in S$ such that $ze = z$, we have $zea = za$, i.e., $[e]a = [ea] = [a]$ for all $a \in S$. The converse is obvious.

Let M be an irreducible S-system. Since M is strictly cyclic, there exists a modular right congruence μ in S such that $M \simeq S/\mu$. This remark immediately yields

6.2. Theorem. *An S-system M is totally irreducible if and only if $M \simeq S/\mu$, where μ is a maximal modular right congruence in S.*

6.3. Theorem. *For every maximal right congruence μ in S the following two conditions are equivalent:*

(a) *S/μ is a totally irreducible S-system.*

(b) *At most one of the right congruence classes of S with respect to μ is a right ideal R of S and, if this is the case, then $S^2 \not\subset R$.*

Proof. A right congruence class of S with respect to μ belongs to $F(S/\mu)$ if and only if it is a right ideal R of S.

Remark. Comparing Theorems 6.2 and 6.3, the following question arises, which corresponds to a question on rings due to Kertész and solved by Leavitt **(9**, p. 84**)**: Let μ be a maximal right congruence in the semigroup S satisfying condition (b). If S has a left identity or if S is commutative, then μ is modular. (This is obvious if a left identity exists. If S is commutative, the assertion is easy to verify.) Is this true in general? As the following example shows, the answer is negative (in analogy with the situation in rings). Let K be the free semigroup generated by two non-commuting symbols a and b. Set $S = K/\kappa$ where κ is the congruence in S generated by the two pairs (a, a^2) and (a, ab). Then S consists of the elements $[a]$, $[b]^n[a]$, $[b]^m$ $(m, n = 1, 2, \ldots)$. Let R be the maximal right ideal of S containing all elements of S but $[a]$. Obviously, the right congruence μ in S defined by $S/\mu = S/R$ is maximal. Since $[a]^2 = [a] \in S^2$, we have $S^2 \not\subset R$. We can easily verify directly that there is no element $[e] \in S$ such that $([e]\,[x], [x]) \in \mu$ for all $[x] \in S$.

From **(3**, Theorems 16 and 17**)**, we readily obtain a necessary and sufficient condition for a maximal right congruence to be modular.

6.4. Lemma. *Let α be a right congruence in the semigroup S. Let A be a right congruence class of S with respect to α. Let $\mathfrak{R}$ denote the complete lattice of all right congruences in S. Then*

$$\mu_A = \sup_{\mathfrak{R}}\{\mu | \mu \in \mathfrak{R},\ A \textit{ is a right congruence class with respect to } \mu\}$$

is the unique maximal right congruence relative to A (i.e., relative to the property of having A as a class). μ_A is modular if and only if S contains an element e such that

$$ea \in A \Leftrightarrow a \in A \qquad \textit{for all } a \in S.$$

Proof. Let β be the equivalence relation

$$\beta = (A \times A) \cup ((S - A) \times (S - A)).$$

Set

$$\beta_* = \sup_{\mathfrak{R}}\{\mu | \mu \in \mathfrak{R},\ \mu \subset \beta\}$$

and

$$\beta C = \{(a, b) | (as, bs) \in \beta \text{ for all } s \in S^1\}. \tag{6.5}$$

Then $\mu_A = \beta_* = \beta C$, as can be seen by arguments similar to those given in the proof of **(3**, Theorem 15**)**. Since

$$\alpha \subset \beta_* = \beta C \subset \beta,$$

A is a class with respect to μ_A. By (6.5), μ_A is modular if and only if S contains an element e such that $(eas, as) \in \beta$ for all $a \in S$ and $s \in S^1$.

6.6. Theorem. *The maximal right congruence μ in S is modular if and only if at least one right congruence class A of S with respect to μ satisfies the condition*

$$ea \in A \Leftrightarrow a \in A \qquad \textit{for all } a \in S \tag{6.7}$$

with a suitable $e \in S$.

Proof. Since μ is maximal, we have $\mu = \mu_A$. Hence by Lemma 6.4, the assertion (6.7) is obvious.

REMARK. Contrary to the situation in rings (cf. the theorem in (**9**, p. 84)), condition (6.7) is not equivalent to

$$e(S - A) \subset S - A \qquad \text{for suitable } e \in S. \tag{6.8}$$

Indeed, let S be the semigroup $S = \{a, b\}$, where $a^2 = ab = a$, $b^2 = ba = b$. The identical relation $\mathbf{0}$ is maximal in S and (6.8) is satisfied, e.g., $A = \{a\}$, $e = b$. But $\mathbf{0}$ is not modular.

6.9. LEMMA. *Let μ be a modular right congruence in S. Suppose S contains an ideal Ω with the following property: If ω denotes the congruence in S defined by $S/\omega = S/\Omega$, then the right congruence $\mu_0 = \sup_{\mathfrak{R}}\{\omega, \mu\}$ in S is distinct from $\mathbf{1}$. Then μ is contained in a maximal (necessarily modular) right congruence in S.*

Proof. The modular right congruence μ_0 determines a modular right congruence $\mu_0/\omega \neq 1$ in S/ω. Since the class Ω is the zero-element of S/ω, (**3**, Theorem 14) implies that μ_0/ω is contained in a maximal right congruence μ^*/ω in S/ω. Hence μ^* is a maximal right congruence in S and $\mu \subset \mu_0 \subset \mu^*$.

Choose $\Omega = O(S) \neq \emptyset$. Then $\mu_0 \neq \mathbf{1}$. This yields

6.10. THEOREM. *Let $O(S) \neq \emptyset$. Then every modular right congruence $\mu \neq \mathbf{1}$ in S is contained in a maximal one.*

6.11. THEOREM. *Let μ be a modular right congruence in S. If $M = S/\mu$ is an irreducible S-system and $M^0 \neq \emptyset$, then μ is contained in a maximal modular right congruence in S.*

Proof. Let $\Omega = M^0$. Then $\mu_0 \neq 1$, for the ideal

$$M^0 = M^{-1}FM = S^{-1}R \qquad (FM = \{R\})$$

is a congruence class with respect to δ_M. This implies $\omega \subset \delta_M$. Hence by Lemma 2.1, M is an irreducible S/ω-system. Because μ is modular, there exists $e \in S$ such that $[e]_\mu a = [a]_\mu$ ($\in M$) for all $a \in S$. Going back to our definition, we obtain $[a]_\mu [b]_\omega = [a]_\mu b$ for $[b]_\omega \in S/\omega$ ($b \in S$). The relation

$$([a]_\omega, [b]_\omega) \in \mu'_0/\omega \Leftrightarrow [e]_\mu [a]_\omega = [e]_\mu [b]_\omega$$
$$\Leftrightarrow [a]_\mu = [b]_\mu \Leftrightarrow (a, b) \in \mu$$

defines a right congruence μ'_0/ω in S/ω, μ'_0 being the corresponding right congruence in S. By our construction, $\mu'_0 = \omega \circ \mu \circ \omega$ ($\circ$ being the usual product of relations), hence $\mu'_0 = \mu_0 = \sup_{\mathfrak{R}}\{\omega, \mu\}$. Furthermore,

$$M \simeq (S/\omega)/(\mu_0/\omega)$$

relative to S/ω; hence $\mu_0 \neq \mathbf{1}$.

Let M be an S-system and $x \in M$. Then

$$(a, b) \in \mu_x \Leftrightarrow xa = xb$$

defines a right congruence μ_x in S.

6.12. LEMMA. *Let M be any S-system. Then*

$$\delta_M = \bigcap_{x \in M} \mu_x.$$

If $x \in FM$, or more generally $|xS| = 1$, then $\mu_x = \mathbf{1}$.

Proof.

$$(a, b) \in \delta_M \Leftrightarrow (\forall x \in M) xa = xb \Leftrightarrow (\forall x \in M)(a, b) \in \mu_x.$$

6.13. LEMMA. *If ϵ is a modular right congruence in S, then*

$$\delta_{S/\epsilon} \subset \epsilon.$$

Proof. Suppose $e \in S$ satisfies $[e]_\epsilon a = [a]_\epsilon$ for all $a \in S$. Then

$$\mu_{[e]_\epsilon} = \epsilon;$$

hence, by Lemma 6.12, $\delta_{S/\epsilon} \subset \epsilon$.

6.14. THEOREM. *Let M be an irreducible S-system. Let*

$$E_M = \{\epsilon | \epsilon \in \mathfrak{R},\ \epsilon \text{ modular},\ S/\epsilon \simeq M\}.$$

Then

$$\delta_M = \bigcap_{\epsilon \in E_M} \epsilon.$$

Proof. If $S/\epsilon \simeq M$, then $\delta_{S/\epsilon} = \delta_M$; hence, by Lemma 6.13, $\delta_M \subset \epsilon$ for $\epsilon \in E_M$. On the other hand, every μ_x of

$$\delta_M = \bigcap_{x \in M - FM} \mu_x$$

belongs to E_M; hence

$$\bigcap_{\epsilon \in E_M} \epsilon \subset \delta_M,$$

and therefore equality holds. In particular, we can verify that

$$E_M = \{\mu_x | x \in M - FM\}.$$

6.15. COROLLARY. *Let E be the set of all those modular right congruences ϵ in S for which S/ϵ is irreducible. Then*

$$\text{rad } S = \bigcap_{\epsilon \in E} \epsilon.$$

Similar to rad S we may consider the congruence

$$\overline{\text{rad}}\ S \underset{\text{def}}{=} \bigcap_{M \in \bar{I}} \delta_M$$

where $\bar{I}$ is the set of all the totally irreducible S-systems. We note that

$$\overline{\text{rad}}\ (S/\overline{\text{rad}}\ S) = \mathbf{0}.$$

6.16. COROLLARY. *Let $\bar{E}$ be the set of all the maximal modular right congruences in S. Then*

$$\overline{\text{rad}}\ S = \bigcap_{\epsilon \in \bar{E}} \epsilon\ (\supset \text{rad } S).$$

6.17. THEOREM. *Let M be an irreducible S-system. Put*

$$K_M = \{\epsilon | \epsilon \in \mathfrak{R}, S/\epsilon \simeq M\}.$$

Then

$$\kappa_M \underset{\text{def}}{=} \bigcap_{\epsilon \in K_M} \epsilon$$

is a congruence in S contained in δ_M.

Proof. Let $(a, b) \in \kappa_M$ and $(xa, xb) \notin \kappa_M$ for a suitable $x \in S$. Then $(xa, xb) \notin \epsilon$ for a suitable $\epsilon \in K_M$. Hence $[x]_\epsilon S = S/\epsilon$ and

$$S/\mu_{[x]_\epsilon} \simeq S/\epsilon \simeq M.$$

This yields

$$\mu_{[x]_\epsilon} \in K_M \text{ and } (a, b) \in \mu_{[x]_\epsilon},$$

i.e., $[x]_\epsilon a = [x]_\epsilon b$ and $[xa]_\epsilon = [xb]_\epsilon$, respectively, a contradiction.

6.18. COROLLARY. *Let K be the set of all those right congruences ϵ in S for which S/ϵ is irreducible. Then $\bigcap_{\epsilon \in K} \epsilon$ is a congruence in S contained in* rad S.

6.19. COROLLARY. *Let $\bar{K}$ denote the set of all those right congruences ϵ in S for which S/ϵ is totally irreducible. Then*

$$\bigcap_{\epsilon \in \bar{K}} \epsilon$$

is a congruence in S contained in $\overline{\text{rad}}\, S$.

An S-system M is 2-*minimal* if

(6.20) $$|M| \geqslant 2,$$

(6.21) M contains no non-trivial S-subsystems.

Every irreducible S-system is 2-minimal. If the right congruence μ in S is maximal, then S/μ is 2-minimal.

6.22. LEMMA. *If μ is a right congruence in S such that S/μ is reducible and 2-minimal, then μ is a congruence in S.*

Proof. Let $(a, b) \in \mu$. If S contains an element x such that $(xa, xb) \notin \mu$, then $[x]_\mu S = S/\mu$ and $(S/\mu)S \not\subseteq F(S/\mu)$. Hence S/μ is irreducible, a contradiction.

6.23. THEOREM. *The intersection of all maximal right congruences in S is a congruence in S contained in* $\overline{\text{rad}}\, S$.

REMARK. In an oral communication, A. Kertész pointed out to me that an analogous statement holds for the intersection of all the maximal right ideals of a ring; cf. A. Kertész, *Vorlesungen über Artinsche Ringe* (in preparation).

Proof. If μ is a maximal right congruence in S, then S/μ is either totally irreducible or 2-minimal reducible. Thus our theorem follows from Corollary 6.19 and Lemma 6.22.

Kertész **(8)** stated a connection between the Frattini-subgroup of a group and the Jacobson-radical of a general ring. An analogous characterization of the 0-radical of a semigroup with zero was given in **(5)**. We now proceed to characterize the congruence $\overline{\text{rad}}\, S$ in a similar manner.

Let $\Phi(S)$ be the set of all the pairs $(a, b) \in S \times S$ such that (sa, sb) for every $s \in S$ may be omitted from every *right generating relation* of $\mathbf{1}$ in S (containing it); thus if α is any relation in S, then

$$\{\{(sa, sb)\} \cup \alpha\}_{\mathrm{r}} = \mathbf{1} \Rightarrow \{\alpha\}_{\mathrm{r}} = \mathbf{1}. \tag{6.24}$$

Here $\{\alpha\}_{\mathrm{r}}$ denotes the right congruence in S generated by α, i.e. the intersection of all $\gamma \in \mathfrak{R}$ such that $\alpha \subset \gamma$.

6.25. THEOREM. $\overline{\text{rad}}\, S = \Phi(S)$.

Proof. 1. Let $(a, b) \in S \times S$ and $(a, b) \notin \Phi(S)$. Then there exist $x \in S$ and $\beta \subset S \times S$ such that

$$\{\{(xa, xb)\} \cup \beta\}_{\mathrm{r}} = \mathbf{1} \tag{6.26}$$

and $\{\beta\}_{\mathrm{r}} \neq \mathbf{1}$. Choose $\mu \in \mathfrak{R}$ maximal such that

$$\beta \subset \mu, \qquad (xa, xb) \notin \mu.$$

By (6.26), μ is maximal in $\mathfrak{R}$. Hence S/μ is totally irreducible. Suppose that

$$(a, b) \in \cap_{M \in \bar{I}} \delta_M.$$

Then $[y]_\mu a = [y]_\mu b$ for all $y \in S$; in particular $(xa, xb) \in \mu$, a contradiction. Hence $(a, b) \notin \overline{\text{rad}}\, S$.

2. Conversely, if $(a, b) \in S \times S$ and $(a, b) \notin \overline{\text{rad}}\, S$, then by definition there exists $M \in \bar{I}$ such that $yua = yub$ does not hold for all $y \in M$ and $u \in S$. Hence, $zva \neq zvb$ for some $z \in M$ and $v \in S$; i.e. $(va, vb) \notin \mu_z$. On the other hand, μ_z is a maximal right congruence in S, whence $\{\{(va, vb)\} \cup \mu_z\}_{\mathrm{r}} = \mathbf{1}$, $\mu_z \neq \mathbf{1}$. Thus $(a, b) \notin \Phi(S)$.

REMARK. Consider the set $\Phi^*(S)$ of all the pairs $(a, b) \in S \times S$ that may be omitted from every right generating relation of $\mathbf{1}$ in S (containing (a, b)). Then, by a general principle, $\Phi^*(S)$ will be the intersection of all maximal right congruences in S (therefore $\Phi^*(S) \subset \Phi(S)$). Indeed, let A be any set. Let $\mathscr{A}$ be a class of subsets of A with the intersection property, i.e., $\mathscr{B} \subset \mathscr{A}$ implies

$$\cap_{B \in \mathscr{B}} B \in \mathscr{A}.$$

If $\mathscr{B} = \emptyset$, define

$$\cap_{B \in \mathscr{B}} B = A.$$

In addition, assume that $\cup_{B \in \mathscr{B}} B \in \mathscr{A}$ for every $\mathscr{B} \subset \mathscr{A}$ that is simply ordered with respect to set inclusion. It is natural to regard the subsets of A belonging to $\mathscr{A}$ as the *$\mathscr{A}$-substructures* of A. Let X be any subset of A. Put

$$\mathscr{B} = \{B | B \in \mathscr{A}, X \subset B\}, \qquad \{X\}_{\mathscr{A}} = \cap_{B \in \mathscr{B}} B.$$

Obviously $\{X\}_{\mathscr{A}} \in \mathscr{A}$. The subset X is called a *generating system* for A if $\{X\}_{\mathscr{A}} = A$. Let $\Phi_{\mathscr{A}}(A) \subset A$ be the set of all those elements $x \in A$ that may be omitted from every generating system for A (containing x). Zorn's Lemma implies that $\Phi_{\mathscr{A}}(A)$ is the intersection of all maximal elements of $\mathscr{A}$. (An element $B \in \mathscr{A}$ is called *maximal* if $B \neq A$ and if, for every $C \in \mathscr{A}$, $B \subset C$ implies $A = C$ or $B = C$.) Hence $\Phi_{\mathscr{A}}(A)$ is an $\mathscr{A}$-substructure, the *Frattini-$\mathscr{A}$-substructure* of A. Note that, in contrast to (**10**, pp. 73–86), this definition of $\Phi_{\mathscr{A}}(A)$ is independent of the special concepts of "A-substructures relative to a set of axioms A" and "substructures of the same kind." In our case take $A = S \times S$ and $\mathscr{A} = \mathfrak{K}$. Then $\Phi^*(S) = \Phi_{\mathfrak{K}}(S \times S)$.

7. The socle of an S-system. In this section we investigate the concept of a socle for S-systems corresponding to the socle of a module (**7**, p. 63).

7.1. LEMMA. *Every 2-minimal S-system M is either irreducible or it satisfies $MS = FM$; in the latter case we have either*

$$|FM| = 1, \qquad |M| = 2,$$

or

$$|FM| = 2, \qquad M = FM.$$

Proof. If M is reducible, then $MS \subset FM$. Because M is 2-minimal, we have $|MS| = 1$ or $MS = M = FM$. The first case yields $|FM| = 1$, and $MS = FM$. For $FM = M$ would imply $MS = M$, $|MS| > 1$, a contradiction. The set $\{x, y\}$ (where $x \in M - FM$ and $y \in FM$) is an S-subsystem of M, consequently $\{x, y\} = M$.

In the case $MS = M = FM$, every non-void subset of M is an S-subsystem; hence $|M| = 2$.

7.2. LEMMA. *Let $\phi : M_1 \rightarrow M_2$ be a homomorphism of the irreducible (of the 2-minimal) S-system M_1 into the S-system M_2. Either $\phi(M_1)$ is an irreducible (a 2-minimal) S-subsystem of M_2 or $|\phi(M_1)| = 1$, $\phi(M_1) \subset FM_2$.*

Proof. Let M_1 be irreducible. The pre-image $\phi^{-1}L$ of any non-trivial S-subsystem $L \subset \phi(M_1)$ is a non-trivial S-subsystem of M_1. Thus $\phi(M_1)$ contains only trivial S-subsystems, whence $|F\phi(M_1)| \leqslant 1$. For otherwise

$$F\phi(M_1) = \phi(M_1). \tag{$*$}$$

Every subset of $F\phi(M_1)$ is an S-subsystem and $(*)$ would imply

$$\phi(M_1) = \{x_1, x_2\}, \quad x_1 \neq x_2, \quad \phi^{-1}\{x_1\} \cap \phi^{-1}\{x_2\} = \emptyset;$$

hence, by the irreducibility of M_1, either $\phi^{-1}\{x_1\} = M_1$ or $\phi^{-1}\{x_2\} = M_1$, a contradiction. We therefore have

$$\phi(M_1)S = \phi(M_1 S) = \phi(M_1) \not\subset F\phi(M_1)$$

if $\phi(M_1) \neq 1$; while for $|\phi(M_1)| = 1$, the S-system $\phi(M_1)$ consists of only one fixed element of M_2.

Let J be the set of all the irreducible S-subsystems of an S-system M. The sum

$$(7.3) \qquad \mathfrak{S} = \bigcup_{L \in J} L \cup FM$$

is called the *socle* of M. If H_α is the set of all the irreducible S-subsystems of M isomorphic to a given irreducible S-subsystem K of M, then

$$\mathfrak{H}_\alpha = \bigcup_{L \in H_\alpha} L \cup FM$$

is called the *homogeneous component* of the socle determined by K.

If $H_\alpha \neq \emptyset$ and $H_\beta \neq \emptyset$, then

$$\mathfrak{H}_\alpha \cap \mathfrak{H}_\beta = FM \qquad \text{for } \alpha \neq \beta.$$

Otherwise, there would exist $L_\alpha \in H_\alpha$ and $L_\beta \in H_\beta$ such that $L_\alpha \cap L_\beta \not\subset FM$, whence $L_\alpha = L_\beta$ and $H_\alpha = H_\beta$, $\alpha = \beta$.

Given two S-subsystems X and Y of M, we write $X \sim Y$ if there is a finite number of S-subsystems $M_i \subset M (i = 1, 2, \ldots, n)$ with $M_1 = X$, $M_n = Y$, such that

$$M_i \simeq M_{i+1} \text{ or } M_{i+1} \simeq M_i$$

for every $i = 1, 2, \ldots, n - 1$. Obviously this relation is an equivalence that induces a decomposition of the set $J = \bigcup J_\sigma$ into mutually disjoint classes J_σ. The S-subsystems

$$\mathfrak{I}_\sigma = \bigcup_{L \in J_\sigma} L \cup FM$$

are called the *semi-homogeneous components* of the socle $\mathfrak{S}$. If $J_\sigma \neq \emptyset$ and $J_\tau \neq \emptyset$, then

$$\mathfrak{I}_\sigma \cap \mathfrak{I}_\tau = FM \qquad \text{for } \sigma \neq \tau.$$

M is said to be *completely reducible* (*semi-homogeneous*, *homogeneous*) if $M = \mathfrak{S} (M = \mathfrak{I}_\sigma, M = \mathfrak{H}_\alpha)$.

Lemma 7.2 implies the following

7.4. THEOREM. (a) *Let M_i be an S-system with the socle $\mathfrak{S}_i \neq \emptyset$ $(i = 1, 2)$. Every homomorphism ϕ of M_1 into M_2 induces a homomorphism of $\mathfrak{S}_1$ into $\mathfrak{S}_2$.*

(b) *Every homomorphic image of a completely reducible S-system is completely reducible.*

7.5. THEOREM. *Every S-subsystem M' of a completely reducible S-system M is completely reducible.*

Proof. Choosing $\mathfrak{S} = M$ in (7.3), we obtain

$$M' = M' \cap M = \bigcup_{L \in J} (M' \cap L) \cup (M' \cap FM)$$

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where $M' \cap FM = FM'$ and

$$M' \cap L = \emptyset \quad \text{or} \quad M' \cap L \subset FM' \quad \text{or} \quad M' \cap L = L.$$

Hence

$$M' = \bigcup_{M' \supset L \in J} L \cup FM'.$$

8. The socle of a semigroup. A right ideal R of a semigroup S is called *irreducible* (*2-minimal*) if it is irreducible (2-minimal) as an S-system. Considering S as an S-system, its socle $\mathfrak{S}$, its homogeneous components $\mathfrak{H}_\alpha$, and its semi-homogeneous components $\mathfrak{J}_\sigma$ are called the (*right*) *socle* of the semigroup S, its (*right*) *homogeneous*, and its (*right*) *semi-homogeneous* components respectively. Every element $u \in S$ induces an endomorphism $a \to ua$ $(a \in S)$ of the S-system S. Therefore by Theorem 7.4, $\mathfrak{S}$ is either void or an ideal of the semigroup S such that $O(S) \subset \mathfrak{S}$. By Lemma 7.2, every non-void $\mathfrak{J}_\sigma$ is an ideal of S. Through $\mathfrak{S} = \bigcup \mathfrak{J}_\sigma$, the socle $\mathfrak{S}$ is decomposed into the ideals $\mathfrak{J}_\sigma$ such that

$$\mathfrak{J}_\sigma \cap \mathfrak{J}_\tau = O(S) \qquad \text{if } \sigma \neq \tau.$$

This implies that

$$\mathfrak{J}_\sigma \cdot \mathfrak{J}_\tau = O(S) \qquad \text{if } \sigma \neq \tau.$$

Thus $\mathfrak{S}$ has at most one semi-homogeneous component, $\mathfrak{S} = \mathfrak{J}_\sigma$ (or $\mathfrak{S} = \emptyset$) if $O(S)$ is void.

8.1. Lemma. (a) *An irreducible S-system M is irreducible as a T-system for every ideal T of S for which $MT \not\subset FM$ holds.*

(b) *Every 2-minimal reducible S-system M is 2-minimal relative to each subsemigroup of S.*

Proof. (a) Let M be an irreducible S-system; $x \in M - FM$. Then $xT = M$; for $xT = FM$ would imply that the S-subsystem $\{x | x \in M, xT \subset FM\}$ contains at least two distinct elements; hence it would be equal to M while $MT \not\subset FM$.

(b) Cf. Lemma 7.1.

8.2. Theorem. *Suppose the semigroup S satisfies the following condition*:

Either $O(S) = \emptyset$, or $S/O(S)$ is a semigroup without nilpotent ideals distinct from zero.

Under this condition we have either $\mathfrak{S} = \emptyset$ or $\mathfrak{S}$ is completely reducible.

Proof. Let R be an irreducible right ideal of S. If $R\mathfrak{S} \not\subset FR$, then R is an irreducible $\mathfrak{S}$-system by Lemma 8.1 (a). If $R\mathfrak{S} \subset FR$, we obtain

$$R^2 \subset FR = R \cap O(S).$$

Hence $R \subset O(S)$ and $RS \subset FR$, a contradiction.

8.3. THEOREM. *Let S be a completely reducible semigroup.*

(a) *An S-system M satisfying $MS = M$ is completely reducible.*

(b) *Every irreducible S-system is the homomorphic image of a right ideal of S.*

Proof. Let $S = \bigcup R \cup O(S)$ where R ranges over the set of irreducible right ideals of S.

(a) Observing $MO(S) \subset FM$, we obtain

$$M = MS = \bigcup_R MR \cup FM = \bigcup_R \bigcup_{x \in M} xR \cup FM.$$

The mapping $r \to xr$ $(r \in R)$ defines a homomorphism of R onto xR $(x \in M)$. Hence, by Lemma 7.2, xR is either irreducible or contained in FM.

(b) If M is an irreducible S-system, then $MS \not\subset FM$. Thus there exists an irreducible right ideal $R \subset S$ such that $MR \not\subset FM$. Hence $xR \not\subset FM$ for some $x \in M$, i.e., $xR = M$ and $R \simeq M$ by $r \to xr$ $(r \in R)$.

9. An analogue of Schur's Lemma. Let Hom (M_1, M_2) denote the set of all the homomorphisms $\phi : M_1 \to M_2$ of the S-system M_1 into the S-system M_2. We write ϕx for the image of $x \in M_1$. Then Hom $(M, M) = \Gamma_M$ is the *centralizer* of M if $M_1 = M_2 = M$ **(3)**.

9.1. THEOREM. *The centralizer of a cyclic (of a strictly cyclic) S-system is a homomorphic image of a suitable subsemigroup of S^1 (of S).*

Proof. Let $M = xS^1$ (Let $M = xS$ $(x \in M)$). If $\gamma \in \Gamma_M$, then S^1 (then S) contains an element c such that $\gamma x = xc$. Thus $\gamma(xa) = (\gamma x)a = xca$, a being any element of S^1 (of S). Since γ is a mapping we have

(9.2) $\qquad xa = xb \Rightarrow xca = xcb \qquad$ for all a, b in S^1 (in S).

Conversely if C is the set of all the elements of S^1 (of S) which satisfy (9.2), then C is non-void (Γ_M contains the identical mapping); moreover C is a subsemigroup of S. If $d \in C$, then $xa = xb$ implies $xda = xdb$. This yields $xcda = xcdb$ for every $c \in C$; hence $cd \in C$. Each element $c \in C$ induces an endomorphism $\gamma_c \in \Gamma_M$ through $\gamma_c(xa) = xca$. The correspondence $c \to \gamma_c$ yields a homomorphism of C onto Γ_M.

REMARK. The mapping $c \to \gamma_c$ induces a homomorphism $gCg \simeq \Gamma_M$ for every element g of the pre-image of the identity of Γ_M. In our two cases these pre-images are the semigroups

$$\{g | g \in S^1, xg = x\} \text{ and } \{g | g \in S, xga = xa \text{ for all } a \in S\}.$$

9.3. LEMMA. *Let e be an idempotent of S and let M be an S-system. Then* Hom $(eS, M) = Me$; *in particular* $\Gamma_{eS} = eSe$.

Proof. If $\phi \in$ Hom (eS, M) and $a \in S$, then

$$\phi(ea) = \phi(eea) = (\phi e)ea.$$

This implies that $\phi e = (\phi e)e \in Me$. Conversely, each $y \in Me$ determines a homomorphism $\phi_y : eS \to M$ through $\phi_y(ea) = ya = yea$; thus $\phi_y e = y$. The correspondence $y \to \phi_y$ is one–one. If $M = eS$, $x = e$, it is an isomorphism. Thus y and ϕ_y may be identified.

9.4. THEOREM. *Let e be an idempotent of S.*

(a) *If the right ideal $M = eS$ is irreducible, then*

$$\text{(9.5)} \qquad \begin{array}{l} \textit{either } eSe \textit{ is a group, or it is a group with zero and } e \notin O(S), \\ \textit{also, } eSe = \{e\} \textit{ or } O(eSe) = eSe \cap O(S). \end{array}$$

(b) *Assume that either $O(S)$ is void or that $S/O(S)$ has no nilpotent ideals but zero. Then* (9.5), *conversely, implies the irreducibility of eS.*

Proof. (*a*) Let T be a right ideal of eSe. Then $TS = eTS \subset eS$. If $TS \subset F(eS)$, then $T = Te \subset F(eS)$ and $|T| = 1$. If $TS \not\subset F(eS)$, we conclude that $TS = eS$ and $T \subset eSe = TeSe \subset T$, i.e., $T = eSe$. Hence the only right ideals of eSe are eSe and (if $F(eS) \neq \emptyset$, i.e., $|F(eS)| = 1$) $F(eS)$. In particular, we obtain $O(eSe) = eSe$ or $O(eSe) = F(eS) \subset O(S)$; in the former case we have $eSe = \{e\}$. Let a be any element of eSe. From $aeSe \subset F(eS)$, it follows that a is the zero-element of eSe. For on the one hand, we have $a \in F(eS)$, i.e., $as = a$ for all $s \in S$. On the other hand, let $b \in eSe$. Then $bas = ba$ for all $s \in S$ where $ba \in eSe \subset eS$; hence $ba \in F(eS) = \{a\}$ and $ba = a$. Now let $aeSe \not\subset F(eS)$. Then ($aS = eS$ and) $aeSe = eSe$, i.e., for every $y \in eSe$ the equation $ax = y$ has a solution $x \in eSe$. Thus eSe is a group or a group with zero.

(b) Conversely, assume that eSe satisfies (9.5). If eSe is a group with zero, then $O(eSe)$ contains only this zero-element. If eSe is a group, then $O(eSe) = \emptyset$ or $O(eSe) = eSe = \{e\}$. Observing that $e \notin O(S)$, we deduce that $e \notin F(eS)$ and $(eS)S \not\subset F(eS)$. Let ea be any element of $eS - F(eS)$. Then $eaS \not\subset O(S)$. For otherwise $ea \in Q = O(S)S^{-1}$. From $QS \subset O(S)$ it follows that $Q^2 \subset O(S) \subset Q$. Hence $Q/O(S)$ is a nilpotent ideal of $S/O(S)$, i.e., $Q = O(S)$. This would imply that $ea \in O(S)$, while $ea \notin F(eS)$.

Now $eaS \not\subset O(S)$ implies that $(eaS)^2 \not\subset O(S)$; therefore $eaSe \not\subset O(S)$; hence by (9.5), either $eSe = \{e\}$ or $eaSe \not\subset O(eSe)$. In the latter case, there exists $b \in S$ such that $eabe \notin O(eSe)$. Since eSe is a group or a group with zero, the equation $eabc = e$ has a solution $c \in eSe$. Hence the equation $eau = ed$ has a solution $u \in S$ for every $ed \in eS$. This is true even if $eSe = \{e\}$. Thus eS is irreducible.

9.6. COROLLARY. *Let S be a semigroup satisfying at least one of the following two conditions:*

(a) *S contains no one-sided zero-elements.*

(b) *S contains zero but no nilpotent ideals except* $\{0\}$.

Let e be an idempotent of S. Then eS is irreducible if and only if Se is left irreducible.

An idempotent $e \in S$ is called (right) primitive if

$$eu = ue = u \Rightarrow u = e$$

for every idempotent $u \notin O(S)$.

9.7. LEMMA. *Let e be an idempotent of S. If eS is irreducible, then e is primitive.*

Proof. Assume that $eu = ue = u$ where $u^2 = u \notin O(S)$. Then $uS = eS$. Hence $e = uv$ for some v and thus $u = ue = u^2v = uv = e$.

9.8. LEMMA. *Let eS and fS be two irreducible right ideals of S generated by the idempotents e and f. Then*

(a) $eS \simeq fS \Leftrightarrow eSf \not\subset O(S)$.
(b) $eS \simeq fS \Rightarrow eSe \simeq fSf$.
(c) *Every homomorphism of fS into eS is trivial.*

Proof. Lemma 9.3 yields Hom $(fS, eS) = eSf$.

(a) If $eS \simeq fS$, there is $c \in eSf$ such that $cfS = eS$. Since $eS \not\subset O(S)$, we have $c = cf \notin O(S)$.

Conversely, $eSf \not\subset O(S)$ implies that $eSf \not\subset F(eS)$, $eSfS = eS$, and hence $eSffSe = eSe \not\subset O(S)$. We can, therefore, find elements $a \in eSf$ and $b \in fSe$ such that $ab = e$. Obviously, ba is idempotent and $a(ba)b = e \notin O(S)$ implies that $ba \notin O(S)$. Moreover, $fba = baf = ba$. Since f is primitive, we obtain $ba = f$. Hence $ab = e$ implies that b induces an isomorphism of eS onto fS.

(b) The mapping $s \to bsa$ $(s \in eSe)$ is an isomorphism of the semigroup eSe onto fSf.

(c) If $eSf \subset O(S)$, then every homomorphism of fS into eS is the trivial mapping onto the fixed element of eS. We may therefore assume that $eSf \not\subset O(S)$.

Let a be an element of eSf not contained in $O(S)$. Then $aS = eS$; hence $afSe = eSe$ and repeating the argument in the proof of (a), we conclude that a induces an isomorphism of fS onto eS.

10. Vector sets. Let Δ be a group or a group with 0, the identity of Δ being 1. Put

$$^{-}\Delta = \begin{cases} \Delta - \{0\} & \text{if } 0 \in \Delta, \\ \Delta & \text{otherwise.} \end{cases}$$

A left Δ-system M is called a *(left) vector set* over Δ if M is *unital* (i.e., $1x = x$ for all $x \in M$) and the following four conditions are fulfilled:

$(10.1)_l$ $\quad |FM| \leqslant 1$ or $FM = M$.

$(10.2)_l$ $\quad FM = M \Rightarrow |\Delta| = 1$.

$(10.3)_l$ $\quad \emptyset \neq FM \neq M \Rightarrow (\Delta(M - FM)) \cap FM \neq \emptyset$.

$(10.4)_l$ For all $\gamma, \delta \in \Delta$, $x \in M$, $\gamma x = \delta x$ implies $\gamma = \delta$ or $x \in FM$.

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10.5. LEMMA. *Let M be a vector set over Δ satisfying $\emptyset \neq FM \neq M$. Then*

$$|FM| = 1, \qquad 0 \in \Delta, \qquad FM = 0M.$$

Proof. Choose $x \in M - FM$ and $\delta \in \Delta$ according to $(10.3)_1$ such that $\delta x \in FM$. Then $\gamma\delta x = \delta x$ and, by $(10.4)_1$, $\gamma\delta = \delta$. Hence $\delta = 0 \in \Delta$ or $\gamma = 1$ for all $\gamma \in \Delta$. But $\Delta = \{1\}$ and $FM \neq M$ are inconsistent.

Relative to the group ${}^{-}\Delta$ the set M decomposes according to

$$M = \bigcup_{x \in N} {}^{-}\Delta x \tag{10.6}$$

into domains of transitivity ${}^{-}\Delta x$ where x ranges over a set of representants N of these domains. The *dimension* of M is defined by

$$\dim M = \begin{cases} |N| - 1 & \text{if } |M| > |FM| = 1, \\ |N| & \text{otherwise.} \end{cases}$$

For each $x \in M - FM$, the set Δx is an irreducible left Δ-system. On the one hand if $\Delta(\Delta x) \subset F(\Delta x) \subset FM$, then $|FM| = 1, |\Delta x| = 1, |\Delta| = 1$, and $FM = M$ while $M - FM \neq \emptyset$. On the other hand from $\delta x \notin F(\Delta x)$ we deduce $\delta \neq 0$ and $\Delta\delta x = \Delta\delta^{-1}\delta x = \Delta x$, i.e., $\delta x \notin \widehat{\Delta x}$.

Since either

$$M = \bigcup_{x \in M - FM} \Delta x \quad \text{or} \quad M = FM = \bigcup_{x \in FM} \Delta x$$

and since $\delta x \to \delta y (\delta \in \Delta)$ for $x, y \in M - FM$ is an isomorphism of Δx onto Δy, we see that M is homogeneous.

Let M be a vector set over Δ. Every homomorphism of M into the (left) vector set $\Delta = \Delta \cdot 1$ over Δ is called a *linear form* on M. The set M^* of all linear forms on M is a (right) Δ-system relative to the composition

$$x(f\delta) = (xf)\delta \qquad (x \in M, f \in M^*, \delta \in \Delta).$$

10.7. LEMMA. *If $|\Delta| \neq 1$, then*

$$|FM^*| \neq 0 \Leftrightarrow |FM^*| = 1, \qquad 0 \in \Delta, \qquad FM^* = M^*0.$$

Proof. Let $f \in FM^*$, $x \in M$, and $\delta \in \Delta$. Then $(xf)\delta = x(f\delta) = xf$. Since $|\Delta| \neq 1$, we have $xf = 0 \in \Delta$, $|FM^*| = 1$.

Note that

$$|M^*| = |\Delta|^{\dim M}. \tag{10.8}$$

Indeed, if x ranges over N, then there exists one and only one linear form $f \in M^*$ such that xf takes given values in Δ under the restriction that $(FM)f = \{0\}$ when $|M| > |FM| = 1$.

In particular, it follows from (10.8) that $M^* \neq \emptyset$. We prove that M^* is a right vector set over Δ.

Obviously $(10.1)_r$ (the analogue of $(10.1)_l$ with "left" and "right" interchanged) is true in M^*.

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$(10.2)_r$: Let $FM^* = M^*$ and $|\Delta| > 1$. By Lemma 10.7, $|M^*| = 1$. Since $\dim M \geqslant 1$, (10.8) implies that $|M^*| \geqslant |\Delta| > 1$, a contradiction.

$(10.3)_r$: From Lemma 10.7, it follows that $|\Delta| = 1$ implies that

$$|FM^*| = |M^*| = 1.$$

$(10.4)_r$: If $f\gamma = f\delta$, $\gamma \neq \delta$, and $f \in M^* - FM^* \neq \emptyset$, then $|\Delta| \neq 1$. For all $x \in M$, $(xf(\gamma = (xf)\delta$, and hence $xf = 0 \in \Delta$, and so $f \in FM^*$.

We call M^* the (*algebraic*) *conjugate* (*vector*) *set* of M. If the cardinals of Δ and M are finite,

$$\dim M^* = \begin{cases} \dfrac{|\Delta|^{\dim M} - 1}{|\Delta| - 1} & \text{if } |M| > |FM| = 1, \\ |\Delta|^{\dim M - 1} & \text{otherwise.} \end{cases}$$

Indeed, if $|M| > |FM| = 1$, we have $0 \in \Delta$. Then M^* is isomorphically represented by the system of all ordered sets $(0, \delta_1, \ldots, \delta_{\dim M})$ where $\delta_1, \ldots, \delta_{\dim M}$ are arbitrary elements of Δ. A set of representatives of the domains of transitivity relative to ${}^-\Delta$ is given by

$$(0, 1, \delta_2, \ldots, \delta_{\dim M}), \quad (0, 0, 1, \gamma_3, \ldots, \gamma_{\dim M}), \quad \ldots,$$
$$(0, 0, 0, \ldots, 0, 1), \quad (0, 0, \ldots, 0),$$

where $\gamma_i, \delta_j, \ldots$ run over Δ. We can argue similarly in the second case.

10.9. THEOREM. *Let M be an irreducible S-system with the centralizer Γ. $F_S M$ ($F_\Gamma M$) denotes the set of all the fixed elements of M with respect to S (to Γ).*

(a) *If $|F_S M| = 1$, then $0 \in \Gamma$ and $F_\Gamma M = 0M$.*

(b) *If $|\Gamma| \neq 1$, then $F_\Gamma M = F_S M$.*

(c) *Let Γ be a group or a group with zero. Then M is a vector set over Γ.*

Proof. (a) Let $F_S M = \{y\}$ and $0x = y$ for all $x \in M$. Then

$$0(xa) = y = ya = (0x)a$$

for all $a \in S$. Thus $0 \in \Gamma$. For $\gamma \in \Gamma$ and $x \in M$ we have

$$(0\gamma)x = 0(\gamma x) = y = 0x;$$

hence $0\gamma = 0$. On the other hand, we have $(\gamma y)a = \gamma(ya) = \gamma y$ if $a \in S$; therefore $\gamma y \in F_S M$ and $\gamma y = y$. Thus

$$(\gamma 0)x = \gamma(0x) = \gamma y = y = 0x$$

for all $x \in M$, i.e., $\gamma 0 = 0$. $|M| \neq 1$ implies that $0 \neq 1$. Therefore 0 is the zero-element of Γ.

(b) Let $x \in F_\Gamma M$, $\gamma \in \Gamma$, and $a \in S$. Then $\gamma xa = xa$. If $x \notin F_S M$, then $xS = M$ and $\gamma = 1$ for all $\gamma \in \Gamma$, contrary to $|\Gamma| \neq 1$. Hence $F_\Gamma M \subset F_S M$ and in virtue of $|F_S M| \leqslant 1$ equality holds. On the other hand, $|F_S M| = 1$ in connection with (a) implies that $0 \in \Gamma$ and $F_\Gamma M = 0M \neq \emptyset$. Therefore, $F_\Gamma M = \emptyset$ and $F_S M \neq \emptyset$ are inconsistent.

(c) From (b) it follows that $(10.1)_{\text{I}}$ and $(10.2)_{\text{I}}$ hold with $\Delta = \Gamma$. $(10.3)_{\text{I}}$: Let $\emptyset \neq F_\Gamma M \neq$ M. Obviously $|\Gamma| \neq 1$. Then (b) implies $F_S M \neq \emptyset$ and (a) yields $0 \in \Gamma$ and $F_\Gamma M = 0M$. Since $|F_\Gamma M| = 1$, we have also

$$F_\Gamma M = 0(M - F_\Gamma M).$$

$(10.4)_{\text{I}}$: Let $\gamma x = \delta x$ where $\gamma, \delta \in \Gamma$, $\gamma \neq \delta$, and $x \in M - F_\Gamma M$. Then $|\Gamma| \neq 1$, $F_\Gamma M = F_S M$ and $x \notin F_S M$. Hence $xS = M$. Therefore, from $\gamma xa = \delta xa$ $(a \in S)$ we obtain $\gamma = \delta$, a contradiction.

Suppose that Γ_M, the centralizer of an irreducible S-system M, is a group or a group with zero. Then the representation S_M of S, generated by M, can be regarded as a semigroup of certain monomial matrices over Γ_M; cf. **(3)**. More generally the following theorem holds.

10.10. THEOREM. *Suppose M is both an S-system and a vector set over the group or group with zero Δ such that*

$$(\delta x)a = \delta(xa) \text{ for all } \delta \in \Delta, x \in M, a \in S, \tag{10.11}$$

and

$$F_\Delta M \subset F_S M \text{ if } |M| > |F_\Delta M| = 1. \tag{10.12}$$

Then S_M, the representation of S generated by M, can be interpreted as a monomial representation of S over Δ.

Proof. In virtue of (10.6), each $a \in S$ determines a mapping ν_a of N into N and a mapping γ_a of N into Δ such that

$$xa = (x\gamma_a)(x\nu_a) \qquad (x \in N).$$

Here we set $x\gamma_a = 0$ and $x\nu_a = y$ if $xa \in F_\Delta M = \{y\}$ and $|M| > |F_\Delta M| = 1$. From

$$xab = (x\gamma_{ab})(x\nu_{ab}) = ((x\gamma_a)(x\nu_a\,\gamma_b))(x\nu_a\,\nu_b),$$

we have

$$x\gamma_{ab} = (x\gamma_a)(x\nu_a\,\gamma_b) \tag{10.13}$$

and

$$x\nu_{ab} = x\nu_a\,\nu_b. \tag{10.14}$$

We need only consider the case $|M| > |F_\Delta M| = 1$. If $xab \neq y$, then $x\nu_{ab} \neq y$, $x\nu_a\,\nu_b \neq y$, and (10.13), (10.14) are true. Next let $xab = y$. Then $x\gamma_{ab} = 0$, $x\nu_{ab} = y$. If $xa = y$, then $x\gamma_a = 0$, $x\nu_a = y$, and (10.13) is valid.

(10.14):

$$x\nu_a\,b = (x\nu_a\,\gamma_b)(x\nu_a\,\nu_b) = yb = y$$

implies that $x\nu_a\,\nu_b = y$. Now let $xa \neq y$; then $x\gamma_a \neq 0$ and $x\nu_a \neq y$. From $xab = y = (x\gamma_a)((x\nu_a)b)$, we deduce $(x\nu_a)b = y$, $x\nu_a\,\gamma_b = 0$, and $x\nu_a\,\nu_b = y$. Thus (10.13) and (10.14) are again true.

The relations (10.13) and (10.14) ensure that the mapping

$$\rho_a \longrightarrow (x\gamma_a\, \delta_{x\nu_a, y})$$

(where $\delta_{x,y}$ denotes the Kronecker symbol) is an isomorphism of S_M onto a semigroup of monomial matrices; cf. **(3)**.

11. The ideal SeS**.** The intersection $\mathfrak{N}$ of all the (two-sided) ideals of a semigroup S is either void or the kernel of S. If $O(S) \neq \emptyset$, then $\mathfrak{N} = O(S)$ is the intersection of all the left ideals of S. An element or a subset of S is said to be $\mathfrak{N}$-potent if some power of it is contained in $\mathfrak{N}$. Every $\mathfrak{N}$-potent right (left) ideal of S is contained in an $\mathfrak{N}$-potent ideal of S (**2**, p. 841, Lemma 5.2).

Let R be an irreducible right ideal of S. Either R is $O(S)$-potent and hence $R^2 \subset O(S)$ or R is not $O(S)$-potent and $R^2 = R$. In the latter case, there exists an $x \in R$ such that $xR = R$.

The sum $\mathfrak{P}$ of all the $O(S)$-potent ideals of S is either void or an ideal of S containing $O(S)$.

11.1. Lemma. *Let R be any non-$O(S)$-potent irreducible right ideal of S; $\mathfrak{J} = SR$.*

(a) *R is contained in the minimal non-$O(S)$-potent ideal $\mathfrak{J}$ of S.*

(b) *If $O(S) \neq \emptyset$, then $\mathfrak{J} \cap \mathfrak{P}$ is $O(S)$-potent.*

(c) *Either $\mathfrak{J}$ itself (if $O(S) = \emptyset$) or $\bar{\mathfrak{J}} = \mathfrak{J}/(\mathfrak{J} \cap \mathfrak{P})$ (if $O(S) \neq \emptyset$) is a simple semigroup with irreducible right ideals.*

Proof. (a) Since $R^2 = R$, we have $R \subset \mathfrak{J}$. The ideal $\mathfrak{J}$ is minimal non-$O(S)$-potent. For if B is an ideal of S contained in $\mathfrak{J}$, then $RB \subset R$, hence $RB = R$ or $RB \subset FR = R \cap O(S)$. In the latter case, $B^2 \subset \mathfrak{J}B \subset O(S)$ and B is $O(S)$-potent. In the former case, we have $R \subset B$ and $\mathfrak{J} \subset SB \subset B$. Therefore $\mathfrak{J} = B$.

(b) $\mathfrak{J} \cap \mathfrak{P}$ is an ideal of S contained in $\mathfrak{J}$. Suppose that $\mathfrak{J} \cap \mathfrak{P} = \mathfrak{J}$. Then $R \subset \mathfrak{J} \subset \mathfrak{P}$. Since $R \not\subset O(S)$, there exists an $O(S)$-potent ideal P of S such that $R \cap P \not\subset O(S)$. Since $R \cap P \subset R$ and the irreducibility of R implies $R \cap P = R$, we find $R \subset P$, i.e., R is $O(S)$-potent, a contradiction. Hence $\mathfrak{J} \cap \mathfrak{P} \neq \mathfrak{J}$ and, by (a), $\mathfrak{J} \cap \mathfrak{P}$ is $O(S)$-potent.

(c) Since $\bar{B}$ is an ideal of $\bar{\mathfrak{J}}$, there exists an ideal B of $\mathfrak{J}$ such that $\mathfrak{J} \cap \mathfrak{P} \subset B \subset \mathfrak{J}$ and $\bar{B} = B/(\mathfrak{J} \cap \mathfrak{P})$. Consider the ideal $\mathfrak{J}B\mathfrak{J}$ of S. Obviously, $\mathfrak{J}B\mathfrak{J} \subset B \subset \mathfrak{J}$. By (a), either $\mathfrak{J}B\mathfrak{J} = \mathfrak{J}$ or $\mathfrak{J}B\mathfrak{J}$ is $O(S)$-potent. In the former case, $\mathfrak{J} \subset B$ and thus $B = \mathfrak{J}$ and $\bar{B} = \bar{\mathfrak{J}}$. In the latter case, we have $\mathfrak{J}B\mathfrak{J} \subset \mathfrak{P}$; hence $(\mathfrak{J}B)^2 \subset \mathfrak{J} \cap \mathfrak{P}$ and $(B\mathfrak{J})^2 \subset \mathfrak{J} \cap \mathfrak{P}$. Therefore, by (b), $\mathfrak{J}B \cup B\mathfrak{J} \subset \mathfrak{P}$. Since the ideal SBS of S lies in $S\mathfrak{J}S \subset \mathfrak{J}$ we have $SBS = \mathfrak{J}$ or $SBS \subset \mathfrak{P}$. In either event, $(SB)^2 = SBS \cdot B \subset \mathfrak{J} \cap \mathfrak{P}$ and, by (b), $SB \subset \mathfrak{J} \cap \mathfrak{P} \subset B$. Similarly, $BS \subset B$ so that B is an ideal of S. Hence $B = \mathfrak{J}$ or $B = \mathfrak{J} \cap \mathfrak{P}$.

Since S is a domain of right operators,

$$\bar{R} = (R \cup (\mathfrak{J} \cap \mathfrak{P}))/(\mathfrak{J} \cap \mathfrak{P}) \simeq R/(R \cap \mathfrak{J} \cap \mathfrak{P}) = R/FR \simeq R \text{ if } O(S) \neq \emptyset.$$

Therefore $\bar{R} \simeq R$ with respect to both $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ because

$$R(\mathfrak{J} \cap \mathfrak{P}) \subset (R\mathfrak{J}) \cap (R\mathfrak{P}) \subset R \cap \mathfrak{P} = FR.$$

This implies that $|R(\mathfrak{J} \cap \mathfrak{P})| = 1$. Let $\bar{r} \in F_{\bar{\mathfrak{J}}}\bar{R}$ $(r \in R)$; then $ra = r$ for all $a \in \mathfrak{J}$, and $|r\mathfrak{J}| = 1$. Hence $r \in r\mathfrak{J} \subset FR$, and $F_{\bar{\mathfrak{J}}}\bar{R} = \overline{FR}$. The assumption $\bar{R}\bar{\mathfrak{J}} \subset F_{\bar{\mathfrak{J}}}\bar{R}$ yields $R\mathfrak{J} \subset FR$ and $R = R^3 \subset RSR \subset FR$, contrary to the irreducibility of R. Clearly, the irreducibility of $\bar{R}$ is proved if it is shown that $\bar{R}$ contains only trivial right ideals of $\bar{\mathfrak{J}}$ or, equivalently, that R contains only trivial $\mathfrak{J}$-subsystems. Let U be an $\mathfrak{J}$-subsystem of R. From $U\mathfrak{J} \subset U \subset R$, it follows that either $U\mathfrak{J} = R$ or $U\mathfrak{J} \subset FR$. The first alternative implies that $U = R$. The second alternative implies that

$$U \subset V \underset{\text{def}}{=} \{x | x \in R, x\mathfrak{J} \subset FR\}.$$

The right ideal V of S is contained in R. Hence either $V = FR$ and $U = FR$, or $V = R$ and $R = R^3 \subset RSR \subset FR$, a contradiction.

Note that the irreducibility of R, regarded as a right ideal of $\mathfrak{J}$, can also be obtained by Lemma 8.1 (a).

11.2. THEOREM. *Let R be any non-$O(S)$-potent irreducible right ideal of S. Suppose that $\mathfrak{J} = SR$ contains at least one minimal non-$O(S)$-potent left ideal of S. Then $\bar{\mathfrak{J}}$ is completely simple.*

Proof. A right ideal R of S is said to be [0-]*minimal* if either S contains zero, $R \neq \{0\}$ and R contains no right ideals of S except $\{0\}$ and R, or S has no zero-element and R contains no right ideals of S except R. This notion will also be used later in the paper.

Let L be any minimal non-$O(S)$-potent left ideal of S contained in $\mathfrak{J}$. The corresponding left ideal of $\bar{\mathfrak{J}}$ is $\bar{L}$. Set $\bar{L} = L$ and $\bar{\mathfrak{J}} = \mathfrak{J}$ if $O(S) = \emptyset$. Then $\bar{L}$ is [0-]minimal. For if $\bar{K}$ is a left ideal of $\bar{\mathfrak{J}}$ contained in $\bar{L}$ where K is the corresponding left ideal of $\mathfrak{J}$ such that

$$\mathfrak{J} \cap \mathfrak{P} \subset K \subset L \cup (\mathfrak{J} \cap \mathfrak{P}) \subset \mathfrak{J},$$

then

$$\mathfrak{J}K \subset L \cup (\mathfrak{J} \cap \mathfrak{P}) \quad \text{and} \quad \mathfrak{J}K = (\mathfrak{J}K \cap L) \cup (\mathfrak{J}K \cap \mathfrak{P}).$$

Since $\mathfrak{J}K \cap L$ is either void or a left ideal of S contained in L, we obtain either $\mathfrak{J}K \cap L \subset \mathfrak{J} \cap \mathfrak{P}$ or $\mathfrak{J}K \cap L = L$. In the former case,

$$K^2 \subset \mathfrak{J}K \subset \mathfrak{J} \cap \mathfrak{P},$$

and by Lemma 11.1(b) $K \subset \mathfrak{J} \cap \mathfrak{P} \subset K$, i.e., $K = \mathfrak{J} \cap \mathfrak{P}$. The latter case yields

$$L \cup (\mathfrak{J} \cap \mathfrak{P}) = (\mathfrak{J}K \cap L) \cup (\mathfrak{J} \cap \mathfrak{P}) = \mathfrak{J}K \cup (\mathfrak{J} \cap \mathfrak{P}) \subset K;$$

hence $K = L \cup (\mathfrak{J} \cap \mathfrak{P})$. Thus $\bar{K} = \bar{L}$ or $|\bar{K}| = 1$, $\bar{K} = \{\mathfrak{J} \cap \mathfrak{P}\}$. Moreover $|\bar{L}| > 1$ if $\bar{\mathfrak{J}}$ contains zero. Indeed, consider first the case $O(S) \neq \emptyset$. If $|\bar{L}| = 1$, then $\bar{L} = \{\mathfrak{J} \cap \mathfrak{P}\}$, since $\mathfrak{J} \cap \mathfrak{P}$ is the zero-element of $\bar{\mathfrak{J}}$. Hence $L \subset \mathfrak{J} \cap \mathfrak{P}$; thus by Lemma 11.1 (b), L is $O(S)$-potent, contrary to the hypothesis. Now let $O(S) = \emptyset$. Then $\bar{\mathfrak{J}} = \mathfrak{J}$ contains no zero-element. For otherwise, $\mathfrak{J}$ would contain an ideal distinct from $\mathfrak{J}$. But in the proof of Lemma 11.1 (c) we have seen that $\bar{\mathfrak{J}} = \mathfrak{J}$ contains no ideals except $\bar{\mathfrak{J}}$ and possibly $\{\mathfrak{J} \cap \mathfrak{P}\}$ (if $O(S) \neq \emptyset$). Since $\bar{\mathfrak{J}}$ contains [0-]minimal right ideals and [0-]minimal left ideals, $\bar{\mathfrak{J}}$ is completely simple.

11.3. Theorem. *Let R be any non-$O(S)$-potent irreducible right ideal of S. Suppose the minimal non-$O(S)$-potent ideal $\mathfrak{J} = SR$ of S has at least one minimal non-$O(S)$-potent left ideal of S. Then R contains an idempotent $e \notin O(S)$ such that $R = eS$.*

Proof. By Theorem 11.2, $\bar{\mathfrak{J}}$ is completely simple. Thus $\bar{R}$ contains an idempotent $\bar{e}$ (e being an element of R) such that $\bar{R} = \bar{e}\bar{\mathfrak{J}}$. If $e \in \mathfrak{J} \cap \mathfrak{P}$, then $\bar{R}$ would be the zero-ideal of $\bar{\mathfrak{J}}$; this contradicts the irreducibility of $\bar{R}$ stated in the proof of Lemma 11.1 (c).

11.4. Lemma. *Let $R = eS$ be an irreducible right ideal of S where e is an idempotent of S.*

(a) *The ideal $\mathfrak{J} = SR$ is equal to*

$$R' = \bigcup_{R \simeq Q} Q \cup O(S)$$

where Q ranges over the set of all the irreducible right ideals of S homomorphic to R.

(b) *If Q_1 is any irreducible right ideal of S contained in I, then $R \simeq Q_1$.*

(c) *$\mathfrak{J} \cap \mathfrak{P}$ is the sum $\mathfrak{R}$ of all the $O(S)$-potent irreducible right ideals of S homomorphic to R and of $O(S)$.*

Proof. (a) If s is any element of S, the correspondence $r \to sr$ where r ranges over R is a homomorphism of R onto sR. By Lemma 7.2, sR is either irreducible or contained in $O(S)$. Hence $sR \subset R'$ and $\mathfrak{J} \subset R'$. Conversely, let Q be an irreducible right ideal of S homomorphic to R. Then $Q\mathfrak{J}$ is a right ideal of S such that $Q\mathfrak{J} \subset Q \cap \mathfrak{J}$. Hence either $Q\mathfrak{J} = \mathfrak{J}$ or $Q\mathfrak{J} \subset FQ$. In the latter case, Lemma 9.3 would imply that

$$\operatorname{Hom}(eS, Q) = Qe \subset O(S)$$

and $eS = R \simeq Q$ would yield $Q \subset O(S)$, a contradiction. Hence $Q = Q\mathfrak{J} \subset \mathfrak{J}$. On the other hand, the ideal $\mathfrak{J}$ contains $O(S)$. Therefore $R' \subset \mathfrak{J}$ and equality holds.

(b) Since $Q_1 \subset \mathfrak{J}$, there exists a Q such that $R \simeq Q$ and $\emptyset \neq Q_1 \cap Q \not\subset O(S)$. The irreducibility of Q and Q_1 implies $Q \cap Q_1 = Q = Q_1$ and $R \simeq Q_1$.

(c) Let Q be an $O(S)$-potent irreducible right ideal and $R \simeq Q$. Then $Q \subset \mathfrak{J} \cap \mathfrak{P}$ yields $\mathfrak{R} \subset \mathfrak{J} \cap \mathfrak{P}$. Conversely, the relations

$$\mathfrak{J} \cap \mathfrak{P} = \bigcup_{R \simeq Q} (\mathfrak{P} \cap Q) \cup O(S)$$

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and

$$\mathfrak{P} \cap Q \begin{cases} \subset O(S) \text{ or} \\ = Q \subset \mathfrak{J} \cap \mathfrak{P} \end{cases}$$

imply $\mathfrak{J} \cap \mathfrak{P} \subset \mathfrak{R}$; thus equality holds.

12. Further properties of the socle. In the following discussion we have to keep in mind that $\mathfrak{P} = O(S)$ equivalently means either $O(S) = \emptyset$, or $S/O(S)$ contains no nilpotent ideals except zero. Note also that $O(S)$ is either void or the intersection of all the left ideals of S. We further adopt the convention that $T/\emptyset = T$ for any subsemigroup T of S.

12.1. THEOREM. *Let $\mathfrak{S}$ be the socle of any semigroup S; $\mathfrak{S} \not\subset \mathfrak{P}$.*

(a) *We have the decomposition*

$$\tilde{\mathfrak{S}} = \mathfrak{S}/(\mathfrak{S} \cap \mathfrak{P}) = \cup_\nu \tilde{\mathfrak{J}}_\nu$$

where $\tilde{\mathfrak{J}}_\nu$ is a certain ideal of $\tilde{\mathfrak{S}}$ and also a simple semigroup with irreducible right ideals.

(b) *If $O(S) \neq \emptyset$, then distinct $\tilde{\mathfrak{J}}_\nu$ annihilate one another.*

(c) *If $O(S) = \emptyset$, then $\mathfrak{S} = \mathfrak{J}_\nu$ is a simple semigroup with irreducible right ideals.*

Proof. Let $\mathfrak{S} \not\subset \mathfrak{P}$. Then $\mathfrak{S} = \cup R \cup (\mathfrak{S} \cap \mathfrak{P})$, where R runs over the set of all the non-$O(S)$-potent irreducible right ideals of S. By Lemma 11.1 (a), R is contained in a minimal non-$O(S)$-potent ideal $\mathfrak{J}_\nu$ of S. Since $R \subset \mathfrak{S} \cap \mathfrak{J}_\nu$ where $\mathfrak{J}_\nu$ is minimal non-$O(S)$-potent and since R is non-$O(S)$-potent, we have $\mathfrak{S} \cap \mathfrak{J}_\nu = \mathfrak{J}_\nu$, i.e., $\mathfrak{J}_\nu \subset \mathfrak{S}$. From the relation

$$\tilde{\mathfrak{J}}_\nu = (\mathfrak{J}_\nu \cup (\mathfrak{S} \cap \mathfrak{P}))/(\mathfrak{S} \cap \mathfrak{P}) \simeq \mathfrak{J}_\nu/(\mathfrak{J}_\nu \cap \mathfrak{P}) = \bar{\mathfrak{J}}_\nu$$

and Lemma 11.1 (c), we see that $\tilde{\mathfrak{J}}_\nu$ is a simple semigroup with irreducible right ideals. Since $\tilde{\mathfrak{J}}_\lambda \tilde{\mathfrak{J}}_\nu \subset \tilde{\mathfrak{J}}_\lambda \cap \tilde{\mathfrak{J}}_\nu$ and since $\tilde{\mathfrak{J}}_\lambda$ and $\tilde{\mathfrak{J}}_\nu$ are simple, we have $|\tilde{\mathfrak{J}}_\lambda \cap \tilde{\mathfrak{J}}_\nu| = 1$ or

$$\tilde{\mathfrak{J}}_\lambda \cap \tilde{\mathfrak{J}}_\nu = \tilde{\mathfrak{J}}_\lambda = \tilde{\mathfrak{J}}_\nu, \qquad \lambda = \nu.$$

When $O(S) \neq \emptyset$, this implies that $\tilde{\mathfrak{J}}_\lambda \tilde{\mathfrak{J}}_\nu$ must be zero for $\lambda \neq \nu$. When $O(S) = \emptyset$ (where $\mathfrak{P} = \emptyset$ and $\tilde{\mathfrak{J}}_\nu = \mathfrak{J}_\nu$ for all ν), $|\mathfrak{J}_\lambda \cap \mathfrak{J}_\nu| = 1$ is impossible; for otherwise S contains zero, contrary to $O(S) = \emptyset$. Hence in this case $\lambda = \nu$ and $\mathfrak{S} = \mathfrak{J}_\nu$.

The ideals $\tilde{\mathfrak{J}}_\nu$ are called the *simple constituents of* $\tilde{\mathfrak{S}}$.

12.2. COROLLARY. *Let S be a completely reducible semigroup and suppose that $S \neq O(S) = \mathfrak{P}$. Then $\bar{S} = S/O(S)$ decomposes into ideals that are simple semigroups with irreducible right ideals. Any two distinct simple constituents annihilate each other. If $O(S) = \emptyset$, then S is itself a simple semigroup with irreducible right ideals.*

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The following lemma is well known.

12.3. Lemma. *Let S be a semigroup with* [0-]*minimal right ideals. Then the sum $T = \cup_R R$ of all the* [0-]*minimal right ideals R of S is an ideal of S.*

12.4. Theorem. *Every simple semigroup S with* [0-]*minimal right ideals is semi-homogeneous.*

Proof. Let $T = \cup R$ be the sum of all the [0-]minimal right ideals R of S. If S contains zero, then $|S| > 1$ and $|T| > 1$; thus $T = S$. If S contains no zero-element, then either (i) $|S| = 1$ and $S = T$, or (ii) $|S| > 1$, $|T| > 1$, and again $S = T$. If $S = O(S)$, S is homogeneous by definition. Therefore, let $S \neq O(S)$. Then $|S| > 1$ and $|O(S)| \leqslant 1$. If R is [0-]minimal and $|R| = 1$, then $R \subset O(S)$, and hence $|O(S)| = 1$, i.e., S contains zero 0. But then $R = \{0\}$ contrary to the hypothesis that it is not [0-]minimal. Thus $|R| > 1$. By definition, each R contains no right ideals distinct from R and possibly $\{0\}$ (if $0 \in S$). Suppose $RS \subset FR$ ($\subset O(S)$). Then $RS = \{0\}$, i.e., $R \subset A$ where

$$A = \{a | a \in S, aS = \{0\}\}$$

is an ideal of S which is necessarily equal to S. Hence $S^2 = \{0\}$, contrary to the definition of simplicity. Therefore, $RS \not\subset FR$. Thus R is irreducible.

R is not nilpotent. For otherwise, R would belong to the ideal

$$B = \{b | b \in S, Rb = \{0\}\}$$

of S because of $R^2 = \{0\}$. This would lead to $RS = \{0\}$, a contradiction. Since $R \subset SR$, we have

$$S = SR = \cup_{s \in S} sR$$

where $R \precsim sR$ and either $sR = \{0\}$ or it is irreducible. Therefore S is semi-homogeneous.

Remark. By the hypothesis of Theorem 12.4, any two irreducible right ideals R_1 and R_2 of S satisfy $R_1 \precsim R_2$ and $R_2 \precsim R_1$, but not necessarily $R_1 \simeq R_2$.

13. Primitive semigroups with minimal ideals. Let S be any [0-] primitive semigroup (i.e., either a primitive or 0-primitive semigroup) with or without a zero. Then $|S| > 1$ and $|\mathrm{rad}^0 S| \leqslant 1$. Indeed if S is 0-primitive, then $\mathrm{rad}^0 S = \{0\}$. If S is primitive and $\mathrm{rad}^0 S \neq \emptyset$, then, by (3.7) and Corollary 3.2, $\mathrm{rad}^0 S$ is a congruence class with respect to $\mathrm{rad}\, S = \mathbf{0}$. Since

$$O(S) \subset \mathfrak{P} \subset N(S) \subset \mathrm{rad}^0 S,$$

the relation

$$O(S) = \mathfrak{P} = N(S) = \mathrm{rad}^0 S = \begin{cases} \{0\} & \text{if } 0 \in S, \\ \emptyset & \text{if } 0 \notin S \end{cases} \tag{13.1}$$

is true for every [0-]primitive semigroup.

13.2. LEMMA. *Suppose that S is a* [0-]*primitive semigroup with* [0-]*minimal right ideals.*

(a) *Every* [0-]*minimal right ideal R of S is irreducible and* [0-]*faithful* (*i.e., faithful and* 0-*faithful respectively*).

(b) *Every* [0-]*faithful irreducible S-system M is homomorphic to each irreducible right ideal of S.*

Proof. Let R be any [0-]minimal right ideal. Assume that $|R| = 1$. Then $R \subset O(S)$, i.e., $|O(S)| = 1$. Since $|S| > 1$, it follows that $0 \in S$ and $R = O(S) = \{0\}$, contrary to $R \neq \{0\}$. Thus, $|R| > 1$.

Let M be any [0-]faithful irreducible S-system; then $MR \not\subset FM$. For otherwise, we would have $|FM| = 1$ and, since M is [0-]faithful, $|R| = 1$, a contradiction. M being irreducible, it follows that $MR = M$. Hence there is an element $x \in M$ such that $xR = M$. The mapping $\phi : r \to xr \quad (r \in R)$ yields a homomorphism of R onto M. Hence R is a [0-]faithful S-system. Indeed, suppose S is 0-primitive. From $Ra = \{0\}$, $a \in S$, it follows that

$$FM = \phi(\{0\}) = \phi(Ra) = \phi(R)a = Ma,$$

whence $a = 0$. On the other hand, if S is primitive and $ra = rb$ for all $r \in R$ and fixed $a, b \in S$, then

$$\phi(r)a = \phi(ra) = \phi(r)b,$$

i.e., $ya = yb$ for all $y \in M$. Therefore $a = b$. Moreover, $RS \not\subset FR$, for otherwise $FR = \{0\}$ (observe that $|R| > 1$) and $RS = \{0\}$; hence R would not be 0-faithful. Thus R is irreducible.

13.3. THEOREM. *Let S be any* [0-]*primitive semigroup with* [0-]*minimal right ideals. Then the socle $\mathfrak{S}$ of S is a simple semigroup with irreducible right ideals.*

Proof. By Lemma 13.2, $|\mathfrak{S}| > 1$ and hence $\mathfrak{S} \not\subset \mathfrak{P}$. By Theorem 12.1, we have $\mathfrak{S} = \bigcup \mathfrak{J}_\nu$ where the ideals $\mathfrak{J}_\nu$ are simple semigroups having irreducible right ideals. If $0 \notin S$, then $\mathfrak{S} = \mathfrak{J}_\nu$.

If $0 \in S$, then $\mathfrak{J}_\lambda \mathfrak{J}_\nu = \{0\}$ $(\lambda \neq \nu)$. Every [0-]primitive semigroup with zero is obviously 0-primitive. In every 0-primitive semigroup, $\{0\}$ proves to be a prime ideal. Hence $\mathfrak{J}_\lambda = \{0\}$ or $\mathfrak{J}_\nu = \{0\}$, contrary to the hypothesis. Thus $\mathfrak{S} = \mathfrak{J}_\nu$ also if $0 \in S$.

13.4. THEOREM. *For every semigroup S with zero the following three conditions are equivalent:*

(a) *S is* 0-*primitive and has* 0-*minimal right ideals.*

(b) *S is weakly free of zero-divisors and has* 0-*minimal right ideals.*

(c) *S is weakly free of zero-divisors and contains an ideal that is a simple semigroup having* 0-*minimal right ideals.*

Proof. (b) $\Rightarrow$ (a). Let R be any 0-minimal right ideal. Assume $RS = \{0\}$. Then $rSb = \{0\}$ for every $r \in R$ and $b \in S$. Choose $b \neq 0$. Then by (4.5), $r = 0$ for all $r \in R$, i.e., $R = \{0\}$, a contradiction. Hence, $RS \not\subset FR$, i.e., R is irreducible and $R = aS$ $(a \neq 0)$. If $Rb = \{0\}$, then $aSb = \{0\}$. Hence by (4.5), $b = 0$ and R is 0-faithful.

(a) $\Rightarrow$ (c). This follows from Lemma 13.3.

(c) $\Rightarrow$ (b). Let U be any ideal of S which is a simple semigroup with 0-minimal right ideals. As the proof of Theorem 12.4 shows, U is the sum $U = \bigcup Q$ of the irreducible right ideals Q of U. We first prove that Q is a right ideal of S. Consider QU; it is a right ideal of S contained in Q. Thus $QU = Q$ or $QU = \{0\}$. In the latter case, $qSu = \{0\}$ for all $q \in Q$ and $u \in U$, hence $Q = \{0\}$ or $U = \{0\}$ contrary to the hypothesis. Therefore, the right ideal Q of S is irreducible with respect to $U \subset S$ and thus also with respect to S.

13.5. THEOREM. *For any semigroup S, the following three conditions are equivalent:*

(a) *S is primitive and contains* [0-]*minimal right ideals.*

(b) *S is weakly left cancelling, contains* [0-]*minimal right ideals, and satisfies the condition $|S| > 1$.*

(c) *S is weakly left cancelling and contains an ideal which is a simple semigroup with* [0-]*minimal right ideals; further the condition $|S| > 1$ holds.*

Proof. We first note that if S is weakly left cancelling, $|S| > 1$ and $asx = asy$ for $a \in O(S)$, $s \in S^1$, and $x \neq y$, imply $a = 0 \in S$ and $O(S) = \{0\}$.

(b) $\Rightarrow$ (a). Let R be any 0-minimal right ideal. To prove that R is an irreducible S-system, we first assume that $|R| = 1$. Then $R \subset O(S)$, $0 \in S$, and $R = O(S) = \{0\}$, contrary to $R \neq \{0\}$. Therefore $|R| > 1$. From $|O(S)| \leqslant 1$, we have $R \neq FR$; hence $|FR| \leqslant 1$. If $RS \subset FR$, then $|FR| = 1$, $0 \in S$, and $RS = \{0\}$, which by Lemma 4.7 and (4.5) would yield $R = \{0\}$, contrary to $|R| > 1$. Hence, R is irreducible and $R = xS = xS^1$ $(x \neq 0)$. Moreover, R is faithful. Indeed if $xsa = xsb$ for all $s \in S^1$ and fixed $a, b \in S$, then since S is weakly left cancelling and since $x \neq 0$, it follows that $a = b$.

(a) $\Rightarrow$ (c). This follows from Theorem 13.3.

(c) $\Rightarrow$ (b). This can be verified by using an argument analogous to that used to prove the assertion (c) $\Rightarrow$ (b) in Theorem 13.4.
(Note that, because $|O(S)| \leqslant 1$, every irreducible right ideal of S is also a [0-]minimal right ideal. For (a) $\Leftrightarrow$ (b) cf. also **(12)**.)

13.6. THEOREM. (a) *Every semigroup S with zero and without zero-divisors is* 0*-primitive.*

(b) *If S is a commutative semigroup with zero, then S is* 0*-primitive if and only if it contains no zero-divisors.*

Proof. (a) Since S is free of zero-divisors, let $M = \{m, 0\}$ where m is any symbol distinct from 0. Define

$$ma = \begin{cases} m & \text{if } a \in S - \{0\}, \\ 0 \in M & \text{if } a = 0 \in S, \end{cases}$$

and assume that $0S = \{0\}$ for $0 \in M$. Then M is a 0-faithful irreducible S-system, i.e., S is 0-primitive.

(b) If S is 0-primitive, then $\{0\}$ is prime, i.e., S contains no divisors of zero (except 0).

13.7. THEOREM. *Let S be any commutative semigroup with zero. For S to be a 0-primitive semigroup and to have 0-minimal ideals it is necessary and sufficient that S be of the form $S = H \cup \{0\}$ where H is any commutative homogroup.*

Proof. For every commutative 0-primitive semigroup S, the set $H = S - \{0\}$ is multiplicatively closed. If S contains 0-minimal ideals, then, by Theorem 13.3, the socle $\mathfrak{S}$ of S is a simple commutative semigroup. It has therefore the form $\mathfrak{S} = G \cup \{0\}$ where G is both a group and an ideal of H. Thus H is a homogroup. Conversely, if H is an arbitrary commutative homogroup, then by Theorem 13.4, $S = H \cup \{0\}$ (where $H0 = 0H = \{0\}$) is a commutative 0-primitive semigroup with 0-minimal ideals.

13.8. THEOREM. *The commutative semigroup S is primitive and has* [0-]*minimal ideals if and only if it is either an abelian group which contains at least two elements or an abelian group with zero added.*

Proof. The socle $\mathfrak{S}$ of any commutative primitive semigroup S with [0-]minimal ideals must be either a group containing at least two elements or a group with zero. Let a be any element of S and e be the identity of $\mathfrak{S}$. By (4.6), the conditions $ea = e(ea)$ and $e \neq 0$ imply that $a = ea \in \mathfrak{S}a \subset \mathfrak{S}$ and $S \subset \mathfrak{S}$. Thus equality holds.

14. Property A. We say that a semigroup S has the *property A* if the following three conditions are fulfilled:

$$\mathfrak{P} = O(S). \tag{14.1}$$

$$\mathfrak{S} \supset O(S), \qquad \mathfrak{S} \neq O(S). \tag{14.2}$$

(14.3) Every $O(S)$-*minimal ideal* of S (i.e., minimal with respect to the property of being an ideal of S that is different from $O(S)$) contains an $O(S)$-minimal left ideal of S.

14.4. THEOREM. *Let S be a semigroup with the property A.*

(a) *Every semi-homogeneous component $\mathfrak{J}_\sigma$ of the socle $\mathfrak{S}$ of S is homogeneous and $\mathfrak{J}_\sigma = \mathfrak{H}_\alpha$ for some α.*

(b) *$\mathfrak{H}_\alpha/O(S)$ is completely simple.*

(c) *$\mathfrak{S}/O(S)$ is the sum*

$$\mathfrak{S}/O(S) = \cup\ (\mathfrak{H}_\alpha/O(S))$$

of the ideals $\mathfrak{H}_\alpha/O(S)$.

(d) *If $O(S) \neq \emptyset$, then any two different $\mathfrak{H}_\alpha/O(S)$ annihilate each other.*

(e) *If $O(S) = \emptyset$, then $\mathfrak{S} = \mathfrak{H}_\alpha$ is completely simple.*

Proof. By (14.2), S possesses an irreducible right ideal R. By (14.1), R is non-$O(S)$-potent. Hence by Theorem 11.3 and (14.3), the $O(S)$-minimal ideal $\mathfrak{J} = SR$ contains an idempotent $e \notin O(S)$ such that $R = eS$. By Lemma 9.8 (c), every irreducible right ideal of S homomorphic to R is clearly isomorphic to R. Hence (a) is true. Lemma 11.4 implies $\mathfrak{J} = \mathfrak{H}_\alpha$ for a suitable α. From Theorem 11.2, we also obtain (b). Now (c) is evident, and (d) and (e) follow from Theorem 12.1.

14.5. THEOREM. *Let S be a semigroup with the property A. In addition suppose that S either contains a zero-element or has neither right nor left zeros.*

(a) *The socle $\mathfrak{S} = \bigcup \mathfrak{H}_\alpha$ of S is contained in the left socle $\mathfrak{S}_l$ of S.*

(b) *Every homogeneous component $\mathfrak{H}_\alpha$ of $\mathfrak{S}$ is both an ideal of S and a completely simple semigroup equal to some homogeneous left component $\mathfrak{G}_\beta$ of $\mathfrak{S}_l$.*

(c) *If S contains neither right nor left zeros, then $\mathfrak{S} = \mathfrak{S}_l = \mathfrak{G}_\beta = \mathfrak{H}_\alpha$ is completely simple.*

Proof. Let R be an irreducible right ideal. As we have seen in the proof of Theorem 14.4, R has the form $R = eS$ where e is an idempotent. By Corollary 9.6, Se is irreducible, i.e., $Se \subset \mathfrak{S}_l$. Since $\mathfrak{P}$ is the sum of all the nilpotent ideals of S (hence "self-dual" relative to the interchange of "right" and "left") and since $|\mathfrak{P}| \leqslant 1$, it follows that $\mathfrak{S}_l \not\subset \mathfrak{P}$, i.e., the assumption of Theorem 12.1 is fulfilled, whence $\mathfrak{S}_l = \bigcup \mathfrak{K}_\nu$ where the $\mathfrak{K}_\nu$ are ideals of $\mathfrak{S}_l$ which are simple semigroups with irreducible left ideals. Different $\mathfrak{K}_\nu$ annihilate each other. Furthermore, $\mathfrak{S} = \bigcup \mathfrak{H}_\alpha$ where each $\mathfrak{H}_\alpha$ is of the form $\mathfrak{H}_\alpha = SR = SeS$. Since $e \in Se \subset \mathfrak{S}_l$, there exists $\mathfrak{K}_\nu$ such that $e \in \mathfrak{K}_\nu$ and thus $\mathfrak{H}_\alpha \subset \mathfrak{K}_\nu$. Since $\mathfrak{K}_\nu$ is simple, equality holds. Let $\mathfrak{l}$ be any irreducible left ideal of S homomorphic to Se. Since $\mathfrak{K}_\nu = LS$, $L = Se$, Lemma 11.4 implies that $\mathfrak{l} \subset \mathfrak{K}_\nu$. Since $\mathfrak{K}_\nu = \mathfrak{H}_\alpha$ is completely simple, there exists an irreducible left ideal $\mathfrak{l}' = \mathfrak{K}_\nu e'$ generated by an idempotent e' $(\neq 0) \in \mathfrak{K}_\nu$ such that $\mathfrak{l} \cap \mathfrak{l}' \neq 0$. Hence $\mathfrak{l} \cap \mathfrak{l}' = \mathfrak{l}'$, i.e., $e' \in \mathfrak{l}' \subset L$. Since $\mathfrak{l}$ is irreducible and $Se' \subset \mathfrak{l}$, we see that $\mathfrak{l} = Se'$. Thus every irreducible left ideal of S homomorphic to Se is generated by an idempotent. By Lemma 9.8, every irreducible left ideal homomorphic to Se is also isomorphic to Se. By Lemma 11.4, $\mathfrak{K}_\nu = LS$ $(L = Se)$ is equal to the sum $\mathfrak{G}_\beta$ of all those irreducible left ideals of S that are isomorphic to L. Since different $\mathfrak{K}_\nu$ annihilate each other, we conclude that

$$\mathfrak{S}_l = \mathfrak{K}_\nu = \mathfrak{H}_\alpha = \mathfrak{G}_\beta$$

when $0 \notin S$.

14.6. COROLLARY. *Let S be a completely reducible semigroup with the property A. Further suppose that S contains either a zero-element or has neither right nor left zeros.*

(a) *S is completely left reducible.*

(b) *The semi-homogeneous left components of S are equal to the semi-homogeneous components as well as to the homogeneous left components and also to the homogeneous components of S and therefore they are completely simple.*

(c) *If S contains neither right nor left zeros, then it is homogeneous as well as left homogeneous, and completely simple.*

14.7. COROLLARY. *Let S be any semigroup satisfying the conditions of Theorem* 14.5. *Further let S satisfy the dual condition of* (14.3):

(14.8) *Every* [0-]*minimal ideal of S contains a* [0-]*minimal right ideal of S.*

(a) *The socle $\mathfrak{S}$ of S is equal to the left socle $\mathfrak{S}_1$ of S.*

(b) *The semi-homogeneous left components of $\mathfrak{S}$ are equal to the semi-homogeneous components as well as to the homogeneous left components and to the homogeneous components; thus they are completely simple.*

(c) *If S has neither right nor left zeros, then it has only one homogeneous component.*

14.9. THEOREM. *The socle of every* [0-]*primitive semigroup S with both* [0-]*minimal right ideals and* [0-]*minimal left ideals is equal to the left socle $\mathfrak{S}_1$ of S; moreover it is completely simple.*

Proof. By Theorem 13.3, the socle $\mathfrak{S}$ is a simple semigroup containing irreducible right ideals. Let $T = \bigcup L$ be the sum of all the [0-]minimal left ideals of S. By Lemma 12.4, T is an ideal of S. Then $|\mathfrak{S}L| > 1$. For otherwise, S would contain a right zero and hence a zero-element 0. Since $|\mathfrak{S}L| = \{0\}$, L is contained in the ideal

$$\mathfrak{T} = \{x | x \in S, \mathfrak{S}x = \{0\}\}.$$

$\mathfrak{T}$ satisfies $\mathfrak{S}\mathfrak{T} = \{0\}$. Since $\{0\}$ is prime in S and $\mathfrak{S} \neq \{0\}$, we find that $\mathfrak{T} = \{0\}$, a contradiction to $\{0\} \neq L \subset \mathfrak{T}$.

$|\mathfrak{S}L| > 1$ and $\mathfrak{S}L \subset L$, together with the fact that L is a [0-]minimal left ideal, yield $\mathfrak{S}L = L$. Therefore, L is an irreducible left ideal of S and $L = \mathfrak{S}L \subset \mathfrak{S}$. Hence, $\mathfrak{S}_1 = T \subset \mathfrak{S}$. The simplicity of $\mathfrak{S}$ implies that $\mathfrak{S}_1 = \mathfrak{S}$. Since $\mathfrak{S}_1$ contains at least one irreducible left ideal of S, $\mathfrak{S}$ is a simple semigroup containing both [0-]minimal right and left ideals; hence $\mathfrak{S}$ is completely simple.

15. Dual vector sets. In this section we study concepts analogous to those occurring in the theory of dual vector spaces (**7**, pp. 68–74). In this way we develop a structure theorem for primitive semigroups with irreducible right ideals that are generated by an idempotent.

Let M′ be a right vector set over Δ. A mapping f of the product set $M \times M'$ into Δ is called a *bilinear form* on M and M' if

$$f(\alpha x, x') = \alpha f(x, x') \quad \text{and} \quad f(x, x'\alpha) = f(x, x')\alpha$$

for all $x \in M$, $x' \in M'$, and $\alpha \in \Delta$. The bilinear form f is said to be *non-degenerate* if

$$(15.1) \qquad f(x, x') = f(y, x') \quad \text{for all } x' \in M' \Rightarrow x = y$$

and

$$(15.2) \qquad f(x, x') = f(x, y') \quad \text{for all } x \in M \Rightarrow x' = y'.$$

We further consider the following conditions:

(15.3) If $0 \in \Delta$ and $f(x, x') = 0$ for all $x' \in M'$, then $x \in FM$,

(15.4) If $0 \in \Delta$ and $f(x, x') = 0$ for all $x \in M$, then $x' \in FM'$.

(M, M') is called a *pair of dual vector sets over* Δ if there exists a bilinear form f on M and M' that satisfies the conditions (15.3) and (15.4). For instance, every non-degenerate bilinear form f satisfies (15.3) and (15.4). If there exists a non-degenerate bilinear form on M and M', the pair (M, M') is said to be *non-degenerate*. It is convenient to use the symbol (x, x') for a fixed bilinear form on M and M'.

Let (M, M') be a pair of dual vector sets over Δ. Let $\mathfrak{L}(M)$ be the semigroup of all endomorphisms of M (i.e., the set of all homomorphisms of the Δ-system M into itself). A mapping s' of M' into itself is called an *adjoint in M' of the element $s \in \mathfrak{L}(M)$ relative to the bilinear form* (x, x') if

$$(xs, x') = (x, x's')$$

for all $x \in M$ and $x' \in M'$. If (x, x') satisfies condition (15.2), then s' is uniquely determined by s and belongs to $\mathfrak{L}(M')$. The following lemma can be verified directly.

15.5. LEMMA. *If $s_i \in \mathfrak{L}(M)$ has the adjoint s'_i $(i = 1, 2)$, then $s'_2 s'_1$ is an adjoint of $s_1 s_2$.*

If (M, M') is a pair of dual vector sets over Δ, then by Lemma 15.5 the set $\mathfrak{L}_{M'}(M)$ of all those elements of $\mathfrak{L}(M)$ that have an adjoint in M' is a subsemigroup of $\mathfrak{L}(M)$.

Note that every Δ-subsystem of a vector set over Δ is either again a vector set over Δ or it contains only one element (which then is the unique fixed "zero" of M). Let $\mathfrak{F}(M)$ denote the set of all the endomorphisms $s \in \mathfrak{L}(M)$ such that the image Ms is either a vector subset of dimension 1 or contains only the zero-element. Let

$$\mathfrak{F}_{M'}(M) = \mathfrak{F}(M) \cap \mathfrak{L}_{M'}(M).$$

The set $\mathfrak{F}_{M'}(M)$ is an ideal of $\mathfrak{L}_{M'}(M)$.

15.6. LEMMA. *Let (M, M') be a pair of dual vector sets over Δ. An element s of $\mathfrak{L}(M)$ belongs to $\mathfrak{F}_{M'}(M)$ if and only if it has the form $x \to (x, y')u$ where $y' \in M'$ and $u \in M$.*

Proof. Let $s \in \mathfrak{F}(M)$; then for every $x \in M$, we have $xs = \sigma(x)u$ with a suitable $u \in M$. If $0 \in \Delta$ and $u \in FM = 0M$, we set $\sigma(x) = 0$. Then $\sigma(x) \in \Delta$ is uniquely determined by x. Evidently, $x \to \sigma(x)$ is a linear form on M. If $u \in FM$, then obviously $\sigma(x)u = (x, y')u = u$ for each $y' \in M'$. Let $u \notin FM$. If s has an adjoint s', then $(xs, x') = (x, x's')$. This implies that

$$\sigma(x)(u, x') = (x, x's').$$

Choose $x' \in M'$ such that $(u, x') = \alpha \neq 0$. (Obviously this condition only occurs in the case of $0 \in \Delta$.) Then $\sigma(x) = (x, y')$ and $y' = x's'\alpha^{-1}$. Conversely, if $s \in \mathfrak{L}(M)$ has the form $x \to (x, y')u (u \in M, y' \in M')$, define $x's' = y'(u, x')$. Then

$$(xs, x') = (x, y')(u, x') = (x, x's').$$

Thus s' is an adjoint of s.

15.7. THEOREM. *The following two conditions are equivalent:*

(a) *S is a primitive semigroup with an irreducible right ideal generated by an idempotent.*

(b) *There exists a pair of dual vector sets (M, M') (where $|M| \neq 1$) over Δ such that S is isomorphic to a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$.*

If S is isomorphically represented as in (b), *then its socle is $\mathfrak{F}_{M'}(M)$.*

Proof. (b) $\Rightarrow$ (a). Let S be a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$. Let y' be any element of M' and let $R_{y'}$ be the set of all the mappings of M into itself of the form r: $x \to xr = (x, y')u$, $u \in M$. Since $x(rs) = (x, y')us$, $R_{y'}$ is a right ideal. Let u_1 be any element of M. Suppose that $u_1 \notin FM (=0M)$ if $0 \in \Delta$. Choose $y'_1 \in M'$ either arbitrarily (if $0 \notin \Delta$) or such that

$$\alpha = (u_1, y'_1) \neq 0 \qquad (\text{if } 0 \in \Delta).$$

Define

$$r_1 : x \to xr_1 = (x, y')u_1 \quad \text{and} \quad s : x \to xs = (x, y'_1)\alpha^{-1}u$$

where u is any element of M. Then $u_1 s = u$ and $xr_1 s = (x, y')u = xr$, i.e., $r_1 S = R_{y'}$. Suppose that $y' \notin FM'$ $(=M'0)$ if $0 \in \Delta$. Then in any case $R_{y'}$ has at least two elements. The only element of R_y that does not strictly generate $R_{y'}$ is $r_0 : x \to xr_0 = (x, y')u_0$ where $u_0 \in FM$ $(=0M)$. This can occur only when $0 \in \Delta$. (If r_0 does not strictly generate $R_{y'}$, then $r_0 \in F_S R_{y'}$; for if $r_0 S \neq \{r_0\}$ and $r \in r_0 S$, $r \neq r_0$, then

$$R_{y'} = rS \subset r_0 SS \subset r_0 S \subset R_y',$$

i.e., $r_0 S = R_{y'}$, a contradiction.) Hence $R_{y'}$ is an irreducible right ideal of S.

Let $rs = rt$ $(s, t \in S)$ for all $r \in R_{y'}$ or equivalently $(x, y')us = (x, y')ut$ for all x, $u \in M$. Then $us = ut$ for all $u \in M$, i.e., $s = t$. Therefore S is a primitive semigroup with irreducible right ideals. Choose $v \in M$ such that $(v, y') = \beta \neq 0$. Put

$$e : x \to xe = (x, y') \ (\beta^{-1}v).$$

Then e is an idempotent of $R_{y'}$ not contained in $F_S R_{y'}$. Thus $R_{y'} = eS$. Since S is a primitive semigroup with irreducible right ideals, Lemma 11.4 (a) and Lemma 13.2 imply that SeS is the socle of S. Therefore $\mathfrak{F}_{M'}(M)$ is contained in SeS. On the other hand, since sr is an element of $SeS = SR_{y'}$, we find that

$$x \to xsr = (xs, y')u = (x, y's')u$$

where s' is an adjoint of s in M'. Hence equality holds.

(a) $\Rightarrow$ (b). We translate to semigroups an idea that I. Kaplansky has applied to prove the analogous statement on rings; cf. **(7**, pp. 77**)**. Let $M = eS$ be an irreducible right ideal of S generated by an idempotent e. Then by Lemma 13.2, M is a faithful irreducible S-system. The centralizer $\Delta = eSe$ of M is a group or a group with zero (cf. Theorem 9.4) and M is a vector set over Δ (cf. Theorem 10.9 (c)). Interpret $M' = Se$ as a Δ-system relative to the right multiplication as Δ-multiplication. M' is a right vector set over Δ. This is trivial if $|\Delta| = 1$. In order to prove it if $|\Delta| \neq 1$, we need the following lemma.

15.8. LEMMA. *Let S be a semigroup and let $O_r(S)$ be the set of the right zeros of S. If $|O_r(S)| > 1$, then $O_r(S)$ is an irreducible right ideal of S and the centralizer $\Gamma_{O_r(S)}$ consists of a single element.*

Proof. $O_r(S)$ is either void or the intersection of all the right ideals of S. Let $e \in O_r(S)$. Then $e^2 = e$ and $O_r(S) \subset eS \subset O_r(S)$. Hence $O_r(S) = eS$ and

$$\Gamma_{O_r(S)} = eSe = \{e\}.$$

We now proceed to prove (a) $\Rightarrow$ (b). If $|\Delta| \neq 1$, then S contains either a zero-element, or it has neither right nor left zeros. Indeed, assume that $|O_r(S)| > 1$. Then $O(S) = \emptyset$. If $O_r(S) = fs$ and $f = f^2$, then obviously $eSf \not\subset O(S)$. Hence by Lemma 9.8 (a), $eS \simeq fS$, and by Lemma 9.8 (b), $eSe \simeq fSf$; this contradicts Lemma 15.8. Thus we can apply Corollary 9.6 to show that $Se = M'$ is an irreducible left ideal of S. By Theorem 10.9 (c), M' is a right vector set over Δ. We define a bilinear form on M and M' by $(ex, y'e) = exy'e$ for $x, y' \in S$. The equation

$$((ex)a)(y'e) = (ex)(a(y'e))$$

shows that the right multiplication ρ_a in M has the left multiplication λ_a in M' as an adjoint, whence $\rho_a \in \mathfrak{L}_{M'}(M)$. Finally, we have to show that each element of $\mathfrak{F}_{M'}(M)$ is of the form ρ_a for some $a \in S$. The mapping

$$ex \to (ex, y'e)ey = exy'ey = ex\rho_a \qquad (a = y'ey)$$

indicates that this is the case.

REMARK. The S-system R_y used in the proof of (b) $\Rightarrow$ (a) of Theorem 15.7 is isomorphic to M under the mapping $u \to r_u$ $(u \in M)$ where

$$r_u : x \to xr_u = (x, y')u.$$

The mapping $u \to r_u$ is also an isomorphism with respect to Δ if we define $\delta r_u = r_{\delta u}$ for all $\delta \in \Delta$.

15.9. THEOREM. *The following two conditions are equivalent:*

(a) S is a primitive semigroup that has an irreducible right ideal generated by an idempotent; in addition, S is left primitive.

(b) *There exists a non-degenerate pair of dual vector sets* (M, M') *over* Δ *where* $|\Delta| \neq 1$ *such that* S *is isomorphic to a subsemigroup of* $\mathfrak{L}_{M'}(M)$ *which contains* $\mathfrak{F}_{M'}(M)$.

Proof. (b) $\Rightarrow$ (a). Since $|\Delta| \neq 1$, $(10.2)_1$ and $(10.2)_r$ imply that $|M| \neq 1$ and $|M'| \neq 1$. Let S be a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$. Then, by Theorem 15.7, S is a primitive semigroup with an irreducible right ideal generated by an idempotent. If we regard M as a right vector set over Δ' and M' as a left vector set over Δ', such that Δ' is anti-isomorphic to Δ, then (M', M) is a non-degenerate pair of dual vector sets over Δ' relative to the bilinear form $(x', x)' = (x, x')$. Since (x, x') is non-degenerate, the mapping $s \rightarrow s'$ (where s' is the adjoint of $s \in \mathfrak{L}_{M'}(M)$ in M') is an anti-isomorphism of $\mathfrak{L}_{M'}(M)$ onto $\mathfrak{L}_M(M')$. This anti-isomorphism maps $\mathfrak{F}_{M'}(M)$ onto $\mathfrak{F}_M(M')$. Indeed, if $s \in \mathfrak{F}_{M'}(M)$, i.e.,

$$s : x \rightarrow (x, y')u = u(y', x)',$$

then

$$s' : x' \rightarrow y'(u, x') = (x', u)'y'.$$

Hence $s' \in \mathfrak{F}_M(M')$, and conversely. The anti-isomorphism of $\mathfrak{L}_{M'}(M)$ onto $\mathfrak{L}_M(M')$ induces an anti-isomorphism of S onto a subsemigroup S' of $\mathfrak{L}_M(M')$. Since $\mathfrak{F}_{M'}(M) \subset S$, we have $\mathfrak{F}_M(M') \subset S'$. Thus S' is a primitive semigroup that contains an irreducible right ideal generated by an idempotent. Hence S is a left primitive semigroup with an irreducible left ideal generated by an idempotent.

(a) $\Rightarrow$ (b). Let S be a primitive semigroup with an irreducible right ideal eS generated by the idempotent e. In addition, let S be left primitive. Then $|O_r(S)| \leqslant 1$. Hence S either contains a zero-element or has neither right nor left zeros. Consequently Se is left irreducible. As in the proof of (a) $\Rightarrow$ (b) of Theorem 15.7, set $M = eS$ and $M' = Se$. By Lemma 13.2 (a), M and M' are faithful. Hence the bilinear form $(ex, y'e) = exy'e$ is non-degenerate. Therefore (M, M') is a non-degenerate dual pair over $\Delta = eSe$. Moreover, $|\Delta| \neq 1$. By Theorem 9.4 (a), this is clear if $0 \in S$. In the general case, let xe and ye be any two different elements of Se. Since eS is faithful, $esxe = esye$ fails to hold for all $s \in S$.

16. Choice of special dual pairs. As we shall see in this section, the investigation of $\mathfrak{L}_{M'}(M)$ and $\mathfrak{F}_{M'}(M)$ can be reduced to the study of dual pairs of a special kind. Let M^* be the conjugate set of the vector set M over Δ. A vector subset M' of M^* is called *total* if

(16.1) $\quad 0 \in \Delta$, $x \in M$, and $xf = 0$ for all $f \in M'$ implies $x \in FM$

and *non-degenerately total* if

(16.2) $$x, y \in M \text{ and } xf = yf \text{ for all } f \in M' \Rightarrow x = y.$$

If $0 \notin \Delta$, in particular if $|\Delta| = 1$, then by definition, M^* as well as each vector subset of M^* is total. If $|\Delta| \neq 1$, then M^* itself is non-degenerately total. Any vector subset of M^* that contains a [non-degenerately] total vector subset of M^* is again [non-degenerately] total.

Let M' be a vector subset of M^*. Then

$$(x, f) = xf \qquad (x \in M, f \in M')$$

is a bilinear form on M and M'. If M' is [non-degenerately] total, then (M, M') is a [non-degenerate] pair of dual vector sets over Δ. Conversely, if (M, M') is an arbitrary pair of dual vector sets over Δ, then there is a natural homomorphism (with the kernel zero if $0 \in \Delta$) of M' onto a total vector subset

$$\overline{M'} \subset M^*.$$

Indeed, if y' is any element of M', then

$$\overline{y'} : x \to (x, y')$$

is a linear form on M and

$$y' \to \overline{y'}$$

is the stated homomorphism of M' (with kernel $FM' = M'0$ if $0 \in \Delta$). If the pair (M, M') is non-degenerate, then the natural homomorphism of M' is an isomorphism onto the non-degenerately total vector subset $\overline{M'}$. Hence if (M, M') is a dual [non-degenerate] pair, then the same is true for $(M, \overline{M'})$.

For any vector set M over Δ, the pair (M, M^*) is dual with respect to

$$(x, f) = xf \qquad (x \in M, f \in M^*).$$

If $s \in \mathfrak{L}(M)$, then for each $f \in M^*$, $x \to xsf$ is a linear form f^* on M, and the mapping $f \to f^*$ is the adjoint of s in M^* which is uniquely determined by s. Hence, $\mathfrak{L}_{M^*}(M) = \mathfrak{L}(M)$. More generally the following theorem holds.

16.3. THEOREM. *Let (M, M') be a pair of dual vector sets over Δ, and let $\overline{M'}$ be the image of M' under the natural homomorphism. Then an element $s \in \mathfrak{L}(M)$ belongs to $\mathfrak{L}_{M'}(M)$ if and only if its adjoint in M^* maps $\overline{M'}$ into itself.*

Proof. $s \in \mathfrak{L}_{M'}(M)$ means that there is a mapping s' of M' into itself such that $(xs, x') = (x, x's')$ for $x \in M$ and $x' \in M'$. By setting

$$\overline{x'} : x \to (x, x'),$$

we can write this condition as

$$(xs)\overline{x'} = x\overline{x's'} \qquad \text{for } x \in M \text{ and } x' \in M'.$$

If s^* is the adjoint of s in M^*, then by definition, $x(fs^*) = (xs)f$ for $x \in M$ and $f \in M^*$. If s' exists, then

$$x(\overline{x'}s^*) = x\overline{x's'} \ (x \in M),$$

i.e.,

$$\overline{x'}s^* = \overline{x's'} \qquad \text{for all } x' \in M'.$$

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Hence s^* maps $\overline{M'}$ into itself. Conversely, if

$$\overline{M'}s^* \subset \overline{M'},$$

then we define the mapping $\overline{s'}$ to be the restriction of s^* to $\overline{M'}$. Then s' as a mapping of M' into itself (even when

$$x' \to \overline{x'}$$

is no isomorphism of M' onto $\overline{M'}$) can be chosen such that

$$\overline{x's'} = \overline{x'}\,\overline{s'}.$$

16.4. COROLLARY. *Let (M, M') be a pair of dual vector sets over Δ, and let $\overline{M'}$ be the image of M' under the natural homomorphism. Then $\mathfrak{L}_{M'}(M) = \mathfrak{L}_{\overline{M'}}(M)$ and $\mathfrak{F}_{M'}(M) = \mathfrak{F}_{M'}(M)$.*

REMARK. If (M, M') is a dual pair over Δ satisfying (15.1), then the corresponding dual pair $(M, \overline{M'})$ over Δ is non-degenerate. Hence by Corollary 16.4, Condition (b) of Theorem 15.9 is equivalent to the following condition:

(c) *There exists a dual pair (M, M') over Δ, where $|\Delta| \neq 1$, that satisfies* (15.1) *and S is isomorphic to a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$.*

17. Isomorphism theorems. Theorem 15.7 associates with every primitive semigroup S containing irreducible right ideals generated by idempotents a pair of dual vector sets (M, M') such that S is isomorphic to a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$. This raises the question: How is (M, M') determined by S.? As we shall see below, there exist conditions under which the corresponding isomorphism theorem of (**7**, p. 79) and its corollaries 1, 2, and 3 become immediately valid for semigroups.

A mapping s of a vector set M_1 over Δ_1 into a vector set M_2 over Δ_2 is called a *semi-linear transformation* if there exists an isomorphism $\sigma : \delta_1 \to \delta_1^{\sigma}$ of Δ_1 onto Δ_2 such that for all $x_1 \in M_1$ and $\delta_1 \in \Delta_1$,

$$(\delta_1 x_1)s = \delta_1^{\sigma}(x_1 s).$$

When the isomorphism σ is indicated explicitly, the semi-linear transformation s is written as (s, σ). The isomorphism σ is uniquely determined by s unless $0 \in \Delta_2$ and s is the "zero-mapping" of M_1 onto the fixed element of M_2 with respect to Δ_2. When (s, σ) is a 1–1 semi-linear transformation of M_1 onto M_2, the inverse mapping is the semi-linear transformation (s^{-1}, σ^{-1}).

For $i = 1, 2$, let (M_i, M'_i) be a pair of dual vector sets over Δ_i and let $(x_i, y'_i)_i$ be the associated bilinear forms. We generalize the definition of adjoint given in §15. A mapping s' of M'_2 into M'_1, as well as the pair (s', σ^{-1}), is called an *adjoint* of the semi-linear transformation (s, σ) relative to the bilinear forms $(x_i, y'_i)_i$, $i = 1, 2$, if

$$(x_1 s, y'_2)_2{}^{\sigma^{-1}} = (x_1, y'_2 s')_1$$

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for all $x_1 \in M_1$ and $y'_2 \in M'_2$. If condition (15.2) is valid for $(x_1, y'_1)_1$, then s' is uniquely determined by s and is a semi-linear transformation of M'_2 over Δ_2 into M'_1 over Δ_1 with the associated isomorphism σ^{-1}. The following lemma is obvious.

17.1. LEMMA. *Let (M_i, M'_i) be a pair of dual vector sets over Δ_i, $i = 1, 2, 3$. If (s, σ) is a semi-linear transformation of M_1 into M_2 with the adjoint (s', σ^{-1}) and (t, τ) is a semi-linear transformation of M_2 into M_3 with the adjoint (t', τ^{-1}), then $(t's', \tau^{-1}\sigma^{-1})$ is an adjoint of the semi-linear transformation $(st, \sigma\tau)$ of M_1 into M_3.*

The dual pairs (M_1, M'_1) and (M_2, M'_2) are said to be (*algebraically*) *equivalent* if there exists a 1–1 semi-linear transformation (s, σ) of M_1 onto M_2 with the following property:

$$\left\{\begin{array}{l}(s, \sigma) \text{ has an adjoint } (s', \sigma^{-1}) \text{ and}\\ (s^{-1}, \sigma^{-1}) \text{ has an adjoint } (s'', \sigma).\end{array}\right. \tag{17.2}$$

For $i = 1, 2$, let (M_i, M'_i) be a pair of dual vector sets over Δ_i, let s be any 1–1 semi-linear transformation of M_1 onto M_2, and let $a_1 \in \mathfrak{L}(M_1)$. Then $s^{-1}a_1 s \in \mathfrak{L}(M_2)$. If $a_1 \in \mathfrak{F}(M_1)$, then $s^{-1}a_1 s \in \mathfrak{F}(M_2)$. Hence if s and s^{-1} have adjoints, then the mapping $a_1 \rightarrow s^{-1}a_1 s$ is an isomorphism of $\mathfrak{L}_{M'_1}(M_1)$ onto $\mathfrak{L}_{M'_2}(M_2)$ which maps $\mathfrak{F}_{M'_1}(M_1)$ onto $\mathfrak{F}_{M'_2}(M_2)$.

To every $\delta_i \in \Delta_i$, there corresponds the *scalar multiplication*

$$(\delta_i)_1 : x_i \rightarrow \delta_i x_i$$

in M_i, and $\delta_i \rightarrow (\delta_i)_1$ is an anti-isomorphism of Δ_i. Hence Δ_i can be regarded as a subset of the centralizer of the S_i-system M_i for any subsemigroup S_i of $\mathfrak{L}(M_i)$.

17.3. THEOREM. *Let (M_i, M'_i) be a pair of dual vector sets over Δ_i and let S_i be a subsemigroup of $\mathfrak{L}_{M'_i}(M_i)$ containing $\mathfrak{F}_{M'_i}(M_i)$, $i = 1, 2$. If the centralizer of the S_i-system M_i is equal to Δ_i, $i = 1, 2$, then every isomorphism $\iota : a_1 \rightarrow a_1{}^\iota$ of S_1 onto S_2 has the form $a_1{}^\iota = s^{-1}a_1 s$ where a_1 runs over S_1 and (s, σ) is a* 1–1 *semi-linear transformation of M_1 onto M_2 with the property* (17.2).

Proof. Let S be an abstract semigroup such that $a \rightarrow a_1$ is an isomorphism of S onto S_1. Then $a \rightarrow a_1{}^\iota$ is an isomorphism of S onto S_2, and we may regard M_1 as well as M_2 as a faithful irreducible S-system. We noted in the Remark after the proof of Theorem 15.7 that the S_i-system M_i is isomorphic to an irreducible right ideal of S_i generated by an idempotent e_i. Hence the S-system M_i is isomorphic to an irreducible right ideal $e_i S$ of S. Since S is primitive, $e_1 S \simeq e_2 S$. Hence there is an isomorphism s of the S-system M_1 onto the S-system M_2. The relation

$$x_1(a_1 s) = (x_1 a_1)s = (x_1 a)s = (x_1 s)a = (x_1 s)a_1{}^\iota = x_1(sa_1{}^\iota)$$

holds for all $x_1 \in M_1$ and $a_1 \in S_1$. It implies that $a_1{}^\iota = s^{-1}a_1 s$. The scalar multiplication $(\delta_1)_1$ $(\delta_1 \in \Delta_1)$ commutes with every element $a_1 \in S$. Hence $s^{-1}(\delta_1)_1 s$ commutes with every element $s^{-1}a_1 s = a_1{}^\iota \in S_2$. Since the centralizer of the S_2-system M_2 coincides with Δ_2, there is a scalar multiplication

$$(\delta_1{}^\sigma)_1 : x_2 \to \delta_1{}^\sigma x_2$$

for $\delta_1{}^\sigma \in \Delta_2$ such that $s^{-1}(\delta_1)_1 s = (\delta_1{}^\sigma)_1$. We verify directly that $\sigma : \delta \to \delta^\sigma$ is an isomorphism of Δ_1 into Δ_2. If δ_2 is any element of Δ_2, then in a similar manner we deduce that $s(\delta_2)_1 s^{-1} = (\delta_1)_1$ for some $\delta_1 \in \Delta_1$. Thus $\delta_2 = \delta_1{}^\sigma$ and σ is an isomorphism onto Δ_2. Since $(\delta_1)_1 s = s(\delta_1{}^\sigma)_1$, we obtain

$$(\delta_1 x_1)s = x_1(\delta_1)_1 s = x_1 s(\delta_1{}^\sigma)_1 = \delta_1{}^\sigma(x_1 s),$$

whence (s, σ) is a semi-linear transformation.

We next show that (s, σ) and (s^{-1}, σ^{-1}) have adjoints. Let

$$r_1 : x_1 \to (x_1, y'_1)_1 u_1$$

be any element of $\mathfrak{F}_{M'_1}(M_1)$ such that $u_1 \notin FM_1(=0M_1)$ when $0 \in \Delta_1$. The mapping

$$f : x_2 \to (x_2 s^{-1}, y'_1)_1{}^\sigma$$

is a linear form on M_2. Since $s^{-1}r_1 s \in \mathfrak{L}_{M'_2}(M_2)$ and

$$x_2(s^{-1}r_1 s) = (x_2 f)(u_1 s) \in \Delta_2(u_1 s),$$

we have $s^{-1}r_1 s \in \mathfrak{F}_{M'_2}(M_2)$. Hence

$$(x_2 f)(u_1 s) = (x_2, y'_2)_2 u_2$$

for suitable $y'_2 \in M'_2$ and $u_2 \in M_2$. This relation, together with

$$u_1 s \notin FM_2 \ (=0M_2)$$

in the case of $0 \in \Delta_2$, yields the equation

$$x_2 f = (x_2 s^{-1}, y'_1)_1{}^\sigma = (x_2, z'_2)$$

for a certain $z'_2 \in M'_2$. Therefore $y'_1 \to z'_2$ is an adjoint of (s^{-1}, σ^{-1}). By symmetry, (s, σ) also has an adjoint.

17.4. Corollary. *Let (M_i, M'_i) be a pair of dual vector sets over Δ_i, $i = 1, 2$. If (M_1, M'_1) and (M_2, M'_2) are equivalent, then*

$$\mathfrak{L}_{M'_1}(M_1) \simeq \mathfrak{L}_{M'_2}(M_2).$$

Conversely, let S_i be a subsemigroup of $\mathfrak{L}_{M'_i}(M_i)$ containing $\mathfrak{F}_{M'_i}(M_i)$ such that the centralizer of the S_i-system M_i coincides with Δ_i, $i = 1, 2$, and let S_1 be isomorphic to S_2. Then (M_1, M'_1) and (M_2, M'_2) are equivalent.

17.5. Corollary. *Let (M, M') be a pair of dual vector sets over Δ. If (s, σ) is a 1–1 semi-linear transformation of M that satisfies (17.2), then $a \to s^{-1}as$ is an automorphism of $\mathfrak{L}_{M'}(M)$ and of $\mathfrak{F}_{M'}(M)$. Conversely, if S is a subsemigroup*

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of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$ such that the centralizer of the S-system M coincides with Δ, then every automorphism of S has the form $a \to s^{-1}as$ where s is a 1–1 *semi-linear transformation of M onto itself that satisfies* (17.2).

17.6. COROLLARY. *For $i = 1, 2$ let (M_i, M'_i) be a pair of dual vector sets over Δ_i. If S_i is a subsemigroup of $\mathfrak{L}_{M'_i}(M_i)$ containing $\mathfrak{F}_{M'_i}(M_i)$ such that the centralizer of the S_i-system M_i coincides with Δ_i, then every isomorphism of S_1 onto S_2 can be extended to an isomorphism of $\mathfrak{L}_{M'_1}(M_1)$ onto $\mathfrak{L}_{M'_2}(M_2)$.*

This result raises the question: When does the centralizer of the S_i-system M_i coincide with Δ_i? A sufficient condition is contained in the following theorem.

17.7. THEOREM. *Let M be a vector set over Δ, and let S be any subsemigroup of $\mathfrak{L}(M)$ that satisfies the following two properties:*

(a) *M is an irreducible S-system.*

(b) *If u and v are any two elements of M such that $u \notin F_S M$ and $v \notin \Delta u$, then there exist two elements a and b of S for which*

$$ua = ub \text{ and } va \neq vb.$$

Then the centralizer of the S-system M coincides with Δ.

Proof. Let c be a mapping of M into itself such that $ac = ca$ for all $a \in S$. Let u be any element of $M - F_S M$ where $F_S M$ either consists of the fixed element of M with respect to S or is void. Then $uc \in \Delta u$. For otherwise, S would contain elements a and b such that $ua = ub$ and $uca \neq ucb$, contrary to

$$uca = uac = ubc = ucb.$$

Thus $uc = \delta u$ for some $\delta \in \Delta$. If $x = ua (a \in S)$ is an arbitrary element of M, then

$$xc = (ua)c = (uc)a = (\delta u)a = \delta(ua) = \delta x,$$

i.e., c coincides with the left multiplication $(\delta)_l : x \to \delta x$.

17.8. THEOREM. *Let (M, M') be a pair of dual vector sets over Δ. Then the centralizer of the $\mathfrak{L}_{M'}(M)$-system M coincides with Δ.*

Proof. Put $S = \mathfrak{L}_{M'}(M)$. Take $u, v \in M$ such that $u \notin F_S M$ and $v \notin \Delta u$. If $0 \in \Delta$, then clearly $u \notin F_\Delta M = 0M$. For otherwise,

$$u = 0u, \qquad ua = (0u)a = 0(ua) \in 0M, \qquad \text{and } |0M| = 1$$

would imply that $ua = u$ for all $a \in S$, a contradiction. Let a be an element of $\mathfrak{F}_{M'}(M)$ such that $ua = u$ and let b be the identical mapping of M. Then $ua = ub = u$, $va \in \Delta u$, $vb = v$, and hence $va \neq v$. Therefore, we can apply Theorem 17.7.

17.9. THEOREM. *Let (M, M') be a pair of dual vector sets over Δ and let S be any subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$. If the associated bilinear form*

(x, x') *of* (M, M') *satisfies* (15.1), *the centralizer of the S-system M coincides with* Δ.

Proof. The centralizer of the $\mathfrak{F}_{M'}(M)$-system M is equal to Δ if and only if this is also true for every S with $\mathfrak{F}_{M'}(M) \subset S \subset \mathfrak{L}(M)$. Thus it is sufficient to consider the case $S = \mathfrak{F}_{M'}(M)$. We know that M is an irreducible S-system. Take $u, v \in M$ such that $w = ub \notin F_\Delta M(=0M)$ if $0 \in \Delta$. Then $vb = \delta w$ for some $\delta \in \Delta$. By (15.1), there is an element $y' \in M'$ for which $(v, y') \neq (\delta u, y')$. Assume first that $(u, y') \neq 0$ if $0 \in \Delta$. Setting

$$a : x \to (x, y')(u, y')^{-1}w,$$

we obtain

$$ua = ub = w, \qquad va = (v, y')(u, y')^{-1}w \neq \delta w = vb.$$

Next let $0 \in \Delta$ and $(u, y') = 0$. Set $a_1 : x \to (x, y')w$ and $b_1 : M \to 0M$. Then $ua_1 = ub_1$ and, since $(v, y') \neq 0$, $va_1 \neq vb_1$. Hence Theorem 17.7 can be applied in either case.

Remark. Let S be a primitive semigroup with an irreducible right ideal eS $(e^2 = e)$. As the proof of Theorem 15.7 shows, we can associate with S a dual pair (M, M') over Δ such that (i) S is isomorphic to a subsemigroup of $\mathfrak{L}_{M'}(M)$ containing $\mathfrak{F}_{M'}(M)$ and (ii) the centralizer of the S-system M coincides with Δ (e.g., put $M = eS$, $M' = Se$, $\Delta = eSe$). By Corollary 17.4, any such pair (M, M') is uniquely determined by S up to equivalence. In particular, Δ is uniquely determined by S up to isomorphism. Therefore, it is very natural to call Δ the *group, with or without zero, of S.*

A primitive semigroup S with an irreducible right ideal eS $(e^2 = e)$ is said to be *maximal* if S is not properly contained in a second primitive semigroup with the same socle $\mathfrak{S} = SeS$.

17.10. Theorem. *A primitive semigroup S with an irreducible right ideal* eS $(e^2 = e)$ *is maximal if and only if it is isomorphic to a semigroup* $\mathfrak{L}_{M'}(M)$.

Proof. Clearly, we may assume that

$$\mathfrak{S} = \mathfrak{F}_{M'}(M) \subset S \subset \mathfrak{L}_{M'}(M)$$

for a suitable dual pair (M, M'). If S is maximal, then necessarily $S = \mathfrak{L}_{M'}(M)$. Conversely, assume that $S = \mathfrak{L}_{M'}(M)$. If S is properly contained in a primitive semigroup T with the same socle $\mathfrak{S} = SeS$, then, since eS is an irreducible right ideal of T, there is a dual pair (M_1, M'_1) over Δ_1 such that T is isomorphic to a subsemigroup T_1 of $\mathfrak{L}_{M'_1}(M_1)$ that contains $\mathfrak{F}_{M'_1}(M_1) = S_1$. The isomorphism $T \simeq T_1$ induces an isomorphism $\mathfrak{S} \simeq \mathfrak{S}_1$ and an isomorphism $S \simeq S_1$ of S onto a subsemigroup S_1 of T_1 that contains $\mathfrak{F}_{M'_1}(M_1)$.

By Theorem 17.8 the centralizer of the S-system M coincides with Δ. We wish to show that the centralizer of the S_1-system M_1 coincides with Δ_1. Since $S \simeq S_1$, the semigroup S_1 has an identity e_1. More generally, let S_1 be any semigroup of $\mathfrak{L}(M_1)$ that contains both $\mathfrak{F}_{M'_1}(M_1)$ and an identity e_1. Since M_1 is an irreducible $\mathfrak{F}_{M'_1}(M_1)$-system, it is also an irreducible S_1-system. Any element $y_1 \in M_1$ can be written in the form $y_1 = x_1 a_1$ where $x_1 \in M_1$ and $a_1 \in S_1$. From

$$y_1 e_1 = (x_1 a_1) e_1 = x_1(a_1 e_1) = x_1 a_1 = y_1$$

it follows that e_1 is the identity mapping of M_1. Hence, the proof of Theorem 17.8 can be applied to verify that the centralizer of the S_1-system M_1 is equal to Δ_1. In our former special situation where $S \simeq S_1$, this implies that the isomorphism of S onto S_1 can be extended to an isomorphism

$$S = \mathfrak{L}_{M'}(M) \simeq \mathfrak{L}_{M'_1}(M_1),$$

contrary to $S_1 \neq \mathfrak{L}_{M'_1}(M_1)$.

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